HYPERBOLIC SPACE

Disk model

\[ B^n = \text{open unit ball in } \mathbb{R}^n, \quad dx^2 = \text{Euclidean metric} \]
\[ ds^2 = dx^2 \left( \frac{2}{1-r^2} \right)^2 \rightarrow H^n \]

Note: ① Since \( ds^2 \) is \( dx^2 \) scaled, hyp. angles = Euc. angles
② Distances large as \( r \rightarrow 1 \)
③ Inclusions \( D^1 \subset D^2 \subset \cdots \) induce isometries \( H^1 \subset H^2 \subset \cdots \)

\( \partial B^n \) is sphere at infinity, denoted \( \partial H^n \).

Upper half-space model

\[ U^n = \{ (x_1, \ldots, x^n) \in \mathbb{R}^n : x_n > 0 \} \]
\[ ds^2 = \frac{1}{x_n^2} dx^2 \]

Check: Inversion in sphere of rad \( R^2 \) centered at \( -e_n \) is an isometry \( B^n \rightarrow U^n \).
Here, \( \partial H^n \) is \( x_n=0 \) plane plus pt at \( \infty \).

Hyperboloid model

\[ \mathbb{R}^{n,1}, \text{ Lorentz metric, } x_1^2 + \cdots + x_n^2 - x_{n+1}^2 \]
Sphere of radius \( R^2 \) is hyperboloid
Upper sheet with induced metric is \( H^n \).
By defn, \( \text{Isom}_+ H^n = SO(n,1) \)
Isometry with \( B^n \) via stereographic proj from \( -e_n \)
Isometries of $\mathbb{H}^n$

Examples

1. Orthogonal maps of $\mathbb{R}^n$ restricted to $\mathbb{B}^n$
   $\rightarrow$ all possible rotations about $e_n$ in $\mathbb{U}^n$.
2. Translation of $\mathbb{U}^n$ by $v = (V_1, \ldots, V_{n-1}, 0)$
3. Dilation of $\mathbb{U}^n$ about $0$.
4. Rotation about $e_n$ axis.

Easy from defn of $ds^2$ that these are isometries.

Thm. The above isometries generate $\text{Isom}(\mathbb{H}^n)$.

Ps. Use: if two isometries of a Riem. manifold agree at a point, they are equal.

Consequences: Any isometry of $\mathbb{H}^n$

1. extends continuously to $\partial \mathbb{H}^n$
2. preserves $\{\text{spheres}\} \cup \{\text{planes}\}$
3. preserves angles between arcs in $\mathbb{H}^n$ and $\partial \mathbb{H}^n$.
4. In $\mathbb{U}^n$ model, each isometry of form $\lambda A x + b$ $\lambda > 0$, $A$ orthogonal & fixes $e_n$.

GEODESICS

Prop. In $\mathbb{U}^n$ $\exists!$ geodesic from $e_n$ to $x e_n$.

Ps. Given any path, its projection to $e_n$-axis is shorter.
Geodesics in $\mathbb{R}^n$ are unique.

Length is $\int_{x}^{\lambda} \frac{1}{y} \ dy = \ln \lambda$. 
Consequences:

1. $\mathbb{H}^n$ is a unique geodesic space. (use change of coords + Prop)
2. The geodesics in $\mathbb{H}^n$ are exactly the straight lines and circles $\perp$ to $\partial \mathbb{H}^n$.
3. Given a geodesic $L$ and $x \notin L$ there are infinitely many $L'$ with $x \in L'$, $L \cap L' = \emptyset$.
4. Between any pts of $\partial \mathbb{H}^n$ there is 1 geodesic (geodesic rays asymptotic $\iff$ endpts same).
5. Geodesics are infinitely long in both directions.

exercise: space of geodesics in $\mathbb{H}^n$ is homeo to Mabius strip.

**Classification of Isometries**

Via fixed pts:

1. Elliptic - fixes pt of $\mathbb{H}^n$
2. Parabolic - fixes 1 pt of $\partial \mathbb{H}^n$, no pt of $\mathbb{H}^n$
3. Hyperbolic - fixes 2 pts of $\partial \mathbb{H}^n$, no pt of $\mathbb{H}^n$

Thm. Each elt of $\operatorname{Isom}(\mathbb{H}^n)$ is one of these.

By Brouwer $\Rightarrow$ at least one fixed pt.

Suppose $f$ fixes $x, y, z \in \partial \mathbb{H}^n$

$\Rightarrow f$ fixes $xy$ and since $f(z) = z$, $f$ fixes $xy$ ptwise.

$\Rightarrow f$ elliptic.

\[ x \rightarrow \infty \]

\[ y \rightarrow \infty \]

\[ x \sim y \]

\[ x \sim z \]
Can give explicit descriptions of 3 types. Using change of coords, can assume a fixed pt in \( \mathbb{H}^n \) is \( \infty \) and a fixed pt in \( \mathbb{H}^n = \infty \) in Un model.

elliptic: rotation

parabolic: \( Ax + b \)
\[ A = \text{orthogonal, preserves } \mathbb{H}^n \]
\[ b = (b_1, \ldots, b_n, 0) \]

hyperbolic: \( \lambda Ax \)
\[ \lambda \text{ as above} \]
\[ x \in \mathbb{R}_{>0} \]

Via translation length \( \tau(f) = \inf \{ d(x, f(x)) : x \in \mathbb{H}^n \} \)

Prop. Let \( f \in \text{Isom}(\mathbb{H}^n) \)

1. \( f \) elliptic \( \iff \tau(f) = 0 \), realized
2. \( f \) parabolic \( \iff \tau(f) \) not realized
3. \( f \) hyperbolic \( \iff \tau(f) > 0 \), realized.

Proof. All \( \Rightarrow \) follow from above descriptions.

First \( \iff \) by defn

Second \( \iff \) find \( x_n \) s.t. \( d(x_n, f(x_n)) \to \tau(f) \)

note \( x_n \) leave every compact set

\( \to \) convergent seq \( \to \) limit \( x \in \mathbb{H}^n \).

Third \( \iff \) if \( d(x, f(x)) = \tau(f) \) then \( f \) preserves geodesic through \( x, f(x), f^2(x), \ldots \)

\( \to 2 \) fixed pts in \( \mathbb{H}^n \).
Thm. \[ \text{Isom}^+(\mathbb{H}^2) \cong \text{PSL}_2\mathbb{R} \]
\[ \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}_2\mathbb{C} \]

Pf. \[ \mathbb{H}^3 \text{ case first.} \]

By above, there is:
\[ \text{Isom}^+(\mathbb{H}^3) \rightarrow \text{Homeo}(\partial \mathbb{H}^3) = \text{Homeo}(\hat{\mathbb{C}}) \] and this is injective.

By Möbius transformations
\[ \text{PSL}_2\mathbb{C} \rightarrow \text{Homeo}(\hat{\mathbb{C}}) \] injective.

Suffices to show images are same.
First, \( \text{PSL}_2\mathbb{C} \) gen. by
\[ \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

exercise: realize each by \( \text{Isom}^+(\mathbb{H}^3) \).

For other dir, show each elt of \( \text{Isom}^+(\mathbb{H}^3) \) fixes a pt in \( \partial \mathbb{H}^3 \). Change of coords: this pt is \( \infty \).

By above, an isometry fixing \( \infty \) is of form \( z \mapsto \lambda Az + b \),
or \( z \mapsto wz + b \), \( w, b \in \mathbb{C} \)

but this is Möbius.

\[ \mathbb{H}^2 \text{ case.} \] \( \text{PSL}_2\mathbb{R} \) = subgp of \( \text{PSL}_2\mathbb{C} \) preserving \( \mathbb{R} \) with orientation.
\[ \implies \text{Isom}^+(\mathbb{H}^2) \subseteq \text{PSL}_2\mathbb{R} \]

For other inclusion, show every isometry of \( \mathbb{H}^2 \) extends to \( \mathbb{H}^3 \) (check on generators). \( \square \)
Loose Ends

Intrinsic defn of $\partial H^n$

$\partial H^n = \{ \text{based geodesic rays in } H^n \} / \sim$

$\sim$ if $\lim_{t \to \infty} d_{H^n}(\gamma(t), \gamma'(t)) = 0$.

topology: for open half-space $S \subset H^n$

$V_S = \{ [\gamma] : \gamma \text{ positively asymptotic into } S \}$

$\rightarrow$ basis

(check this is same topology as before!)

This also gives topology on $H^n \cup \partial H^n$

By defn, $\text{Isom}(H^n)$ acts continuously on the union.

Horospheres

$B = \text{Euclidean ball in ball model of } H^n$

tangent to boundary sphere at $x$.

$\partial B \setminus x = \text{horosphere}$

$\text{int } B = \text{horoball}$.

note: horosphere has Euclidean metric
Areas in $H^2$

Circles. \[ f(t) = r e^{i t} \] circle in disk model, hyp. radius $s = \ln\left(\frac{1+r}{1-r}\right) \]
\[
C = \int_0^{2\pi} \frac{2}{1-r^2} r \, dt = \frac{4\pi r}{1-r^2} = \frac{4\pi \tanh \frac{s}{2}}{1-(\tanh \frac{s}{2})^2} = \frac{4\pi \tanh \frac{s}{2}}{(\text{sech} \frac{s}{2})^2} = 2\pi \sinh s
\]
\[
\sim e^s
\]
\[ A = \int_0^s 2\pi \sinh t \, dt = 2\pi (\cosh s - 1) = 2\pi (2\sinh \frac{s}{2}) = 4\pi \sinh \frac{s}{2} \]

Ideal triangles. All are isometric to:

\[
A = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} \, dy \, dx
\]
\[
= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx = \pi
\]

Polygons. Thm. $A(P) = (n-2)\pi$ - sum of int. angles

Step 1. $2/3$ ideal $\Delta$. \[ A(\theta) = \text{area of $\Delta$ with angles } 0,0,\pi-\theta. \]

Claim: $A(\theta) = \theta$.

Proof:

A continuous picture $\Rightarrow$ A linear picture above $\Rightarrow A(\pi) = \pi$.

Step 2. Arbitrary $\Delta$. Hint:

Step 3. Cut $\mathcal{P}$ into $\Delta$s.
**Ideal Tetrahedra**

$T =$ ideal tetrahedron in $H^3$

$S =$ horosphere based at ideal vertex $v$, disjoint from opp side.

$Lk(v) = S \cap T =$ link of $v$ in $T$

$= \text{Euclidean } \Delta, \text{ angles are dihedral angles of } T, \text{ a.p. similarity class}$

$\text{indep. of } S.$

**Facts**

1. o.p. congruence class of $(T,v)$ determined by $Lk(v)$
   - pf: similarities of $C$ extend to isometries of $H^3$

2. If the dihedral angles corr. to $v$ are $\alpha, \beta, \gamma$ then $\alpha + \beta + \gamma = \pi$
   - pf: Euclid

3. The dihedral angles of opp. edges are equal
   - pf: 6 vars, 4 eqns

4. $Lk(v)$ same for all vertices of $T$
   - pf: (3)

5. The o.p. similarity congruence class of $T$ determ. by $Lk(v)$
   - pf: (1) + (4)

6. $\forall \alpha, \beta, \gamma \ s.t. \ \alpha + \beta + \gamma = \pi \ \exists \ T \text{ with } Lk(v) = \Delta$
   - pf: construct it.
   - Notation $T_{\alpha, \beta, \gamma}$

7. Congruence class of $T$ determ. by cross ratio of vertices.
   - pf: up to isometry, 3 vertices are $0, 1, \infty$

**Thm.** \[ \text{Vol}(T_{\alpha, \beta, \gamma}) = J_1(\alpha) + J_1(\beta) + J_1(\gamma) \]

\[ J_1(\theta) = -\int_0^\theta \log|2\sin t| \, dt \]

see Ratcliffe Thm 10.4.10

"Lobachevsky fn"

**Consequences**

1. \[ \text{Vol}(T_{\pi/3, \pi/3, \pi/3}) \] maximal (easy calculus)

2. \[ \text{it equals } 3 \cdot J_1(\pi/3) \approx 2.01565 \ldots < 1.01 \ldots \]