HYPERBOLIC SPACE

Disk model

B^n = open unit ball in $\mathbb{R}^n$, dx^2 = Euclidean metric

$$ds^2 = dx^2 \left(\frac{2}{1-r^2}\right)^2 \rightsquigarrow \mathbb{H}^n$$

Note: ① Since ds^2 is dx^2 scaled, hyp. angles = Euc. angles

② Distances large as $r \to 1$

③ Inclusions $D^1 \subset D^2 \subset \cdots$ induce isometries $\mathbb{H}^1 \subset \mathbb{H}^2 \subset \cdots$

∂B^n is sphere at infinity, denoted $\partial \mathbb{H}^n$.

Upper half-space model

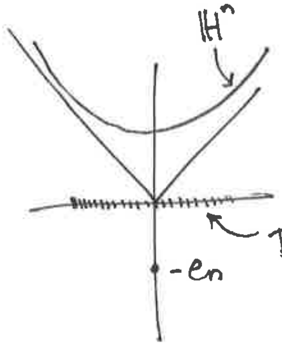
$U^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$

$$ds^2 = \frac{1}{x_n^2} dx^2$$

Check: Inversion in sphere of rad $\sqrt{2}$ centered at $-e_n$ is an isometry $B^n \to U^n$.

Here, $\partial \mathbb{H}^n$ is $x_n = 0$ plane plus pt at ∞.

Hyperboloid model



$\mathbb{R}^{n,1}$, Lorentz metric $x_1^2 + \cdots + x_n^2 - x_{n+1}^2$

Sphere of radius $\sqrt{-1}$ is hyperboloid

Upper sheet with induced metric is $\mathbb{H}^n$.

By defn, $\mathrm{Isom}^+ \mathbb{H}^n = SO(n,1)$

Isometry with B^n via stereographic proj from $-e_n$

Isometries of $\mathbb{H}^n$

Examples ① Orthogonal maps of $\mathbb{R}^n$ restricted to B^n
$\rightsquigarrow$ all possible rotations about e_n in U^n.
② Translation of U^n by $v = (v_1, \ldots, v_{n-1}, 0)$
③ Dilation of U^n about 0.
③' Rotation about e_n axis.

Easy from defn of ds^2 that these are isometries.

Thm. The above isometries generate $\mathrm{Isom}(\mathbb{H}^n)$

Pf. Use: if two isometries of a Riem. manifold agree at a point, they are equal.

Consequences: Any isometry of $\mathbb{H}^n$
① extends continuously to $\partial\mathbb{H}^n$
② preserves {spheres} ∪ {planes}
③ preserves angles between arcs in $\mathbb{H}^n$ and $\partial\mathbb{H}^n$.
consequence of pf of Thm. → ④ In U^n model, each isometry of form $\lambda Ax + b$ $\lambda > 0$, A orthogonal & fixes e_n
$b = (b_1, \ldots, b_{n-1}, 0)$

Geodesics

Prop. In U^n $\exists!$ geodesic from e_n to λe_n.
Pf. Given any path, its projection to e_n-axis is shorter.
Geodesics in $\mathbb{R}$ are unique.

Length is $\int_1^\lambda \frac{1}{y}\, dy = \ln \lambda$.

Consequences: ① $\mathbb{H}^n$ is a unique geodesic space (use change of coords + Prop)

② The geodesics in $\mathbb{H}^n$ are exactly the straight lines and circles $\perp$ to $\partial\mathbb{H}^n$.

③ Given a geodesic L and $x \notin L$ $\exists$ infinitely many L' with $x \in L'$, $L \cap L' = \emptyset$.

④ Between any pts of $\partial\mathbb{H}^n$ $\exists!$ geodesic

(geodesic rays asymp $\Leftrightarrow$ endpts same)

⑤ Geodesics are infinitely long in both directions.

exercise: space of geodesics in $\mathbb{H}^2$ is homeo to Möbius strip.

CLASSIFICATION OF ISOMETRIES

Via fixed pts: ① elliptic - fixes pt of $\mathbb{H}^n$

② parabolic - fixes 1 pt of $\partial\mathbb{H}^n$, no pt of $\mathbb{H}^n$

③ hyperbolic - fixes 2 pts of $\partial\mathbb{H}^n$, no pt of $\mathbb{H}^n$

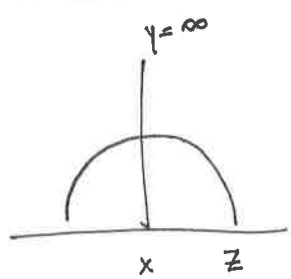
<u>Thm</u>. *Each elt of $\mathrm{Isom}(\mathbb{H}^n)$ is one of these.*

<u>Pf</u>. Brouwer $\Rightarrow$ at least one fixed pt.

Suppose f fixes $x, y, z \in \partial\mathbb{H}^n$

$\Rightarrow$ f ~~fixes~~ preserves $\overline{xy}$ and since $f(z) = z$, f fixes $\overline{xy}$ ptwise

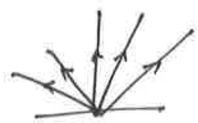
~~and does not rotate about xy~~. $\Rightarrow$ f elliptic.



Can give explicit descriptions of 3 types. Using change of coords, can assume a fixed pt in $\mathbb{H}^n$ is e_n and a fixed pt in $\partial \mathbb{H}^n = \infty$ in U_n model.

elliptic : rotation 

parabolic: $Ax+b$  A = orthogonal, preserves e_n
$b = (b_1, \dots, b_{n-1}, 0)$

hyperbolic: λAx  A as above
$\lambda \in \mathbb{R}_{>0}$

Via translation length $\quad \tau(f) = \inf \{ d(x, f(x)) : x \in \mathbb{H}^n \}$

<u>Prop</u>. Let $f \in \mathrm{Isom}(\mathbb{H}^n)$

① f elliptic $\Leftrightarrow$ $\tau(f) = 0$, realized

② f parabolic $\Leftrightarrow$ $\tau(f)$ not realized

③ f hyperbolic $\Leftrightarrow$ $\tau(f) > 0$, realized.

<u>Pf</u>. All $\Rightarrow$ follow from above descriptions.

First $\Leftarrow$ by defn

Second $\Leftarrow$ find x_n s.t. $d(x_n, f(x_n)) \longrightarrow \tau(f)$

note x_n leave every compact set

$\rightsquigarrow$ convergent seq $\rightsquigarrow$ limit $x \in \partial \mathbb{H}^n$.

Third $\Leftarrow$ If $d(x, f(x)) = \tau(f)$ then f preserves geodesic through $x, f(x), f^2(x), \dots$

$\rightsquigarrow$ 2 fixed pts in $\partial \mathbb{H}^n$.

Dimensions 2 & 3

Thm. $\text{Isom}^+(\mathbb{H}^2) \cong PSL_2\mathbb{R}$
$\text{Isom}^+(\mathbb{H}^3) \cong PSL_2\mathbb{C}$

Pf. $\mathbb{H}^3$ case first.

By above, there is:

$$\text{Isom}^+(\mathbb{H}^3) \longrightarrow \text{Homeo}(\partial\mathbb{H}^3) \cong \text{Homeo}(\hat{\mathbb{C}})$$ and this is injective.

By Möbius transformations

$$PSL_2\mathbb{C} \longrightarrow \text{Homeo}(\hat{\mathbb{C}}) \quad \text{injective.}$$

Suffices to show images are same.

First, $PSL_2\mathbb{C}$ gen. by $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$, $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

exercise: realize each by $\text{Isom}^+(\mathbb{H}^3)$.

For other dir, show each elt of $\text{Isom}^+(\mathbb{H}^3)$ fixes a pt in $\partial\mathbb{H}^3$. Change of coords: this pt is ∞.

By above, an isometry fixing ∞ is of form $z \mapsto \lambda A z + b$,
or $z \mapsto wz + b$, $w, b \in \mathbb{C}$

but this is Möbius.

$\mathbb{H}^2$ case. $PSL_2\mathbb{R}$ = subgp of $PSL_2\mathbb{C}$ preserving $\mathbb{R}$ with orientation.

$\Longrightarrow \text{Isom}^+(\mathbb{H}^2) \subseteq PSL_2\mathbb{R}$

For other inclusion, show every isometry of $\mathbb{H}^2$ extends to $\mathbb{H}^3$. (check on generators). ∎

Loose Ends

Intrinsic defn of $\partial \mathbb{H}^n$

$$\partial \mathbb{H}^n = \{\text{based geodesic rays in } \mathbb{H}^n\}/\sim$$

$\gamma \sim \gamma'$ if $\lim d_{\mathbb{H}^n}(\gamma(t), \gamma'(t)) = 0$.

topology: for open half-space $S \subseteq \mathbb{H}^n$

$V_S = \{[\gamma] : \gamma \text{ positively asymptotic into } S\}$

$\rightsquigarrow$ basis

(check this is same topology as before!)

This also gives topology on $\mathbb{H}^n \cup \partial \mathbb{H}^n$

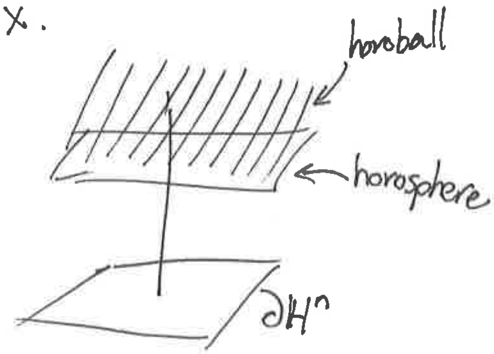
By defn, $\mathrm{Isom}(\mathbb{H}^n)$ acts continuously on the union.

Horospheres

B = Euclidean ball in ball model of $\mathbb{H}^n$ tangent to boundary sphere at x.

$\partial B \setminus x$ = horosphere

int B = horoball.



note: horosphere has Euclidean metric

Areas in $\mathbb{H}^2$

Circles. $f(t) = re^{it}$ circle in disk model, hyp. radius $s = \ln\left(\frac{1+r}{1-r}\right)$

$$C = \int_0^{2\pi} \frac{2}{1-r^2}\, r\, dt = \frac{4\pi r}{1-r^2} = \frac{4\pi \tanh s/2}{1-(\tanh s/2)^2} = \frac{4\pi \tanh s/2}{(\operatorname{sech} s/2)^2} = 2\pi \sinh s$$

$$\sim e^s$$

$$A = \int_0^s 2\pi \sinh t\, dt = 2\pi(\cosh s - 1) = 2\pi(2\sinh^2 s/2) = 4\pi \sinh s/2$$

Ideal triangles. All are isometric to: 

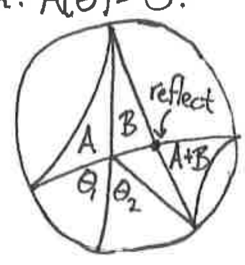
$$A = \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\, dx$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \pi$$

Polygons. Thm. $A(P) = (n-2)\pi$ - sum of int. angles

Step 1. 2/3 ideal Δ. $A(\theta)$ = area of Δ with angles $0, 0, \pi-\theta$.

Claim: $A(\theta) = \theta$.

Pf: 

A continuous

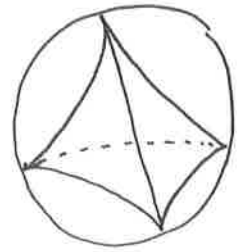
picture $\Rightarrow$ A linear

above $\Rightarrow A(\pi) = \pi$.

Step 2. Arbitrary Δ Hint: 

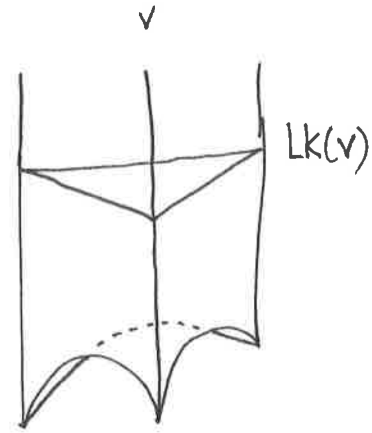
Step 3. Cut P into Δs.

IDEAL TETRAHEDRA



T = ideal tetrahedron in $\mathbb{H}^3$

S = horosphere based at ideal vertex v, disjoint from opp side



$Lk(v) = S \cap T$ = link of v in T

= Euclidean Δ, angles are dihedral angles of T, o.p. similarity class indep. of S.

Facts ① o.p. congruence class of (T, v) determined by $Lk(v)$

pf: similarities of $\mathbb{C}$ extend to isometries of $\mathbb{H}^3$

② If the dihedral angles corresp. to V are α, β, γ then $\alpha+\beta+\gamma = \pi$

pf: Euclid

③ The dihedral angles of opp. edges are equal

pf: 6 vars, 4 eqns

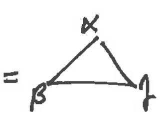
④ $Lk(v)$ same for all vertices of T

pf: ③

⑤ The o.p. ~~similarity~~ congruence class of T determ. by $Lk(v)$

pf: ① + ④

⑥ $\forall\ \alpha, \beta, \gamma$ s.t. $\alpha+\beta+\gamma = \pi$ $\exists\ T$ with $Lk(v) =$



pf: construct it. Notation $T_{\alpha,\beta,\gamma}$

⑦ Congruence class of T determ. by cross ratio of vertices.

pf: up to isometry, 3 vertices are $0, 1, \infty$.

<u>Thm.</u> $\mathrm{Vol}(T_{\alpha,\beta,\gamma}) = Л(\alpha) + Л(\beta) + Л(\gamma)$ $\qquad Л(\theta) = -\int_0^\theta \log|2\sin t|\,dt$

see Ratcliffe Thm 10.4.10 "Lobachevsky fn"

Consequences ① $\mathrm{Vol}(T_{\pi/3,\pi/3,\pi/3})$ maximal (easy calculus)

② it equals $3\,Л(\pi/3) \approx$ ~~2.0298832~~... 1.01....