Hyperbolic Manifolds

Goal: S_g has a hyp. structure $g \geq 2$
$S^3 \setminus \text{Fig}8$ has hyp. structure

A hyperbolic manifold is a topological manifold with a cover by open sets U_i and open maps $\varphi_i : U_i \to \mathbb{H}^n$ that are homeos onto their image and so for each component X of $U_i \cap U_j$,

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(X) \longrightarrow \varphi_j(X)$$

is the restriction of an elt of $\text{Isom}(\mathbb{H}^n)$.

Note: A hyp. man inherits a Riem. metric.

Prop. A Riem. manifold is a hyperbolic n-manifold iff each point has a nbd isometric to an open subset of $\mathbb{H}^n$.

Pf. $\Rightarrow$ by defn of inherited metric.

$\Leftarrow$ Take the local isometries as the charts $\varphi_i : U_i \to \mathbb{H}^n$
Let X = component of $U_i \cap U_j$
Then $\varphi_i \circ \varphi_j^{-1} \big|_{\varphi_j(X)}$ is an isometry $\varphi_j(X) \longrightarrow \varphi_i(X)$.
Want an elt of $\text{Isom}(\mathbb{H}^n)$ restricting to this.
But we can find an elt of $\text{Isom}(\mathbb{H}^n)$ that agrees with $\varphi_i \circ \varphi_j^{-1}$ at any $x \in \varphi_j(X)$.
This isometry then agrees on all of $\varphi_j(X)$. ▨

Polyhedra

Polyhedron: compact subset of $\mathbb{H}^n$, intersection of finitely many half-spaces.
Ideal polyhedron: intersection of finitely many half-spaces in $\mathbb{H}^n$, no vertices in $\mathbb{H}^n$, closure in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ is a finite set of pts.

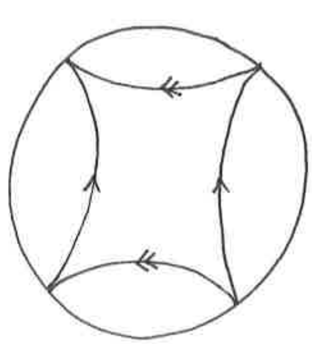
M = space obtained from a collection of (possibly ideal) hyp. polyhedra P_i by gluing codim 1 faces by isometries.
M° = image of $\bigcup \operatorname{int} P_i$.

<u>Thm</u>. M as above. Say each $x \in M$ has a nbd U_x and an open mapping $\varphi_x : U_x \longrightarrow B_{\varepsilon(x)}(0) \subseteq B^n$ (ball model) that is (1) a homeo onto its image (2) sends x to 0 and (3) restricts to isometry on each component of $U_x \cap M^\circ$. Then M is a hyperbolic manifold.

<u>Pf</u>. Need to check condition on overlaps.
This works because gluing maps are isometries (see Lackenby) ▨

↖ or use the Prop.

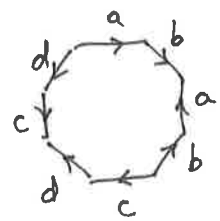
A First example:



SURFACES

Will show S_g has hyp. structure $g \geq 2$.

Fact 1. S_2 given by



and similar for $g > 2$.

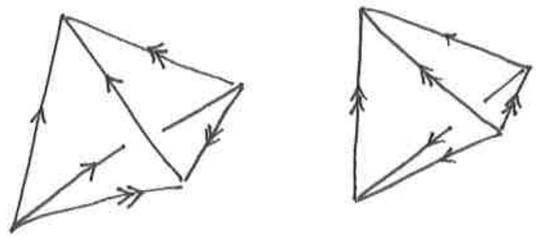
Fact 2. $\exists$ regular 4g-gon in $\mathbb{H}^2$ with angles $2\pi/4g$

Pf: IVT. Small 4g-gons are near Euclidean, angles $> 2\pi/4g$
Large 4g-gons are ideal, angle 0.

Apply the theorem. When we glue, nothing to check on interiors of 1- and 2-cells. At 0-cells, angle condition is exactly what is needed.

Figure-Eight Knot Complement

$K =$ 

Consider 

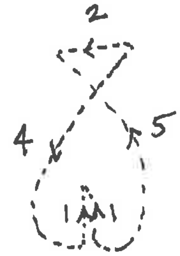
$\exists!$ way to glue faces so edges match up

$\rightsquigarrow$ cell complex M. with one vertex v.

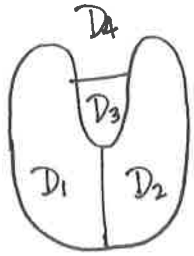
Will show: $M - v \cong S^3 \setminus K$

First note M is not a manifold. In fact, a neighborhood of v is a cone on T^2. To see this, the boundary of a nbd of v is a union of 8 triangles. Label the 24 edges, glue in pairs, result is T^2. (tedious but easy).

Γ = 2-complex in S^3 obtained by attaching 4 2-cells to

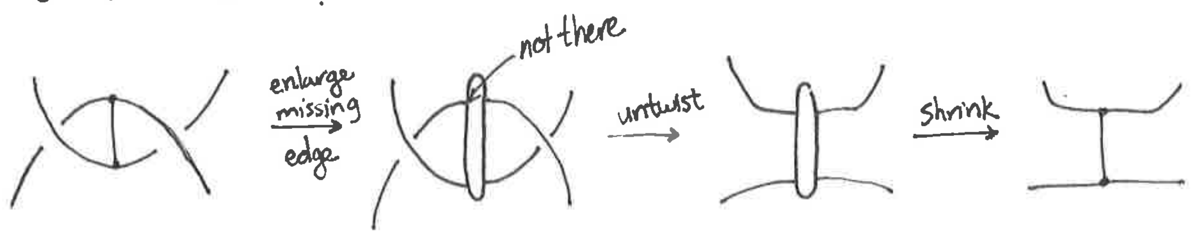
Sample 2-cell: 

(find the other three!)

Let $\Gamma' =$ 

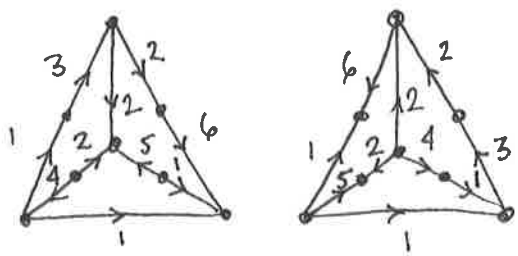
$\cong S^2$ Note. $S^3 - \Gamma' \cong \mathrm{int}(B^3) \sqcup \mathrm{int}(B^3)$

Claim. $S^3 - \Gamma \cong S^3 - \Gamma'$.

Pf. 

∎

Now go back to Γ picture. The claim tells us the 4 disks of Γ cover S^2. We can read off the gluing:



hard to see!

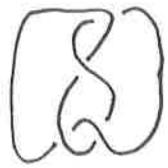
Note K is the union of the edges 3, 4, 5, 6.
So to remove K, can collapse these edges, then delete.
But this is $M \setminus v$!

THE HYPERBOLIC STRUCTURE

$M \setminus v$ has 2 edges, each with 6 dihedral angles around. So if we glue two <u>regular</u> ideal tetrahedra, get angle 2π around each edge. Thm $\Rightarrow$ result is hyperbolic.

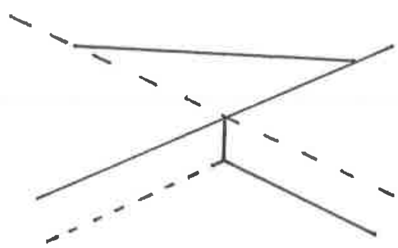
Hyperbolic volume ≈ 2.0298832 smallest among knot complements

Figure Eight Knot Complement – Reboot

K = 

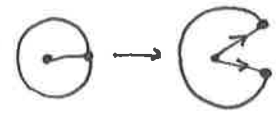
Idea: Simultaneously inflate balloons above and below. (3-cells). These press against each other in each planar region (2-cells). At crossings, the balloons compete:

see paper model on Purcell p. 11

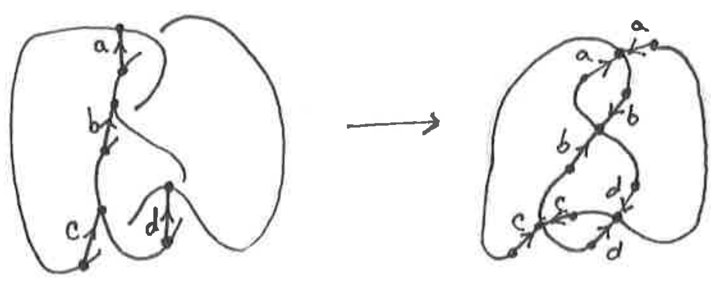


$\rightsquigarrow$ 1-cells

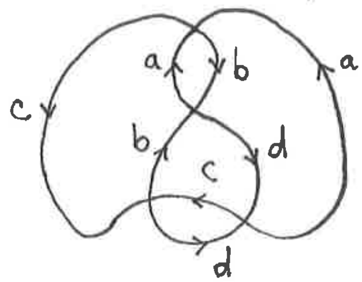
$\rightsquigarrow$ S^3 (with K) as a 3-complex. The 2-skeleton is a 2-sphere pinched near the crossings. To understand the attaching map we unpinch.

2D pic: 

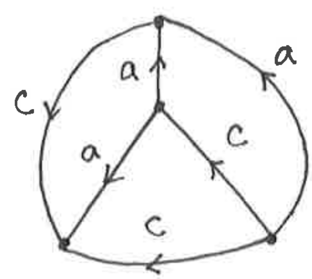
Unpinching from point of view of top ball:



Unlabeled edges make up K. To remove K, collapse each to a pt, think of as ideal vertices:

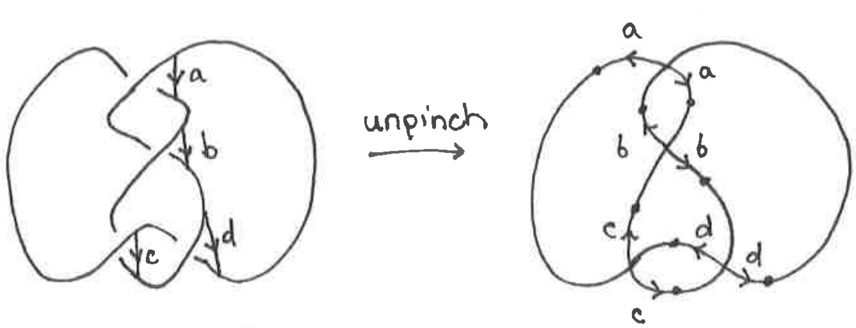


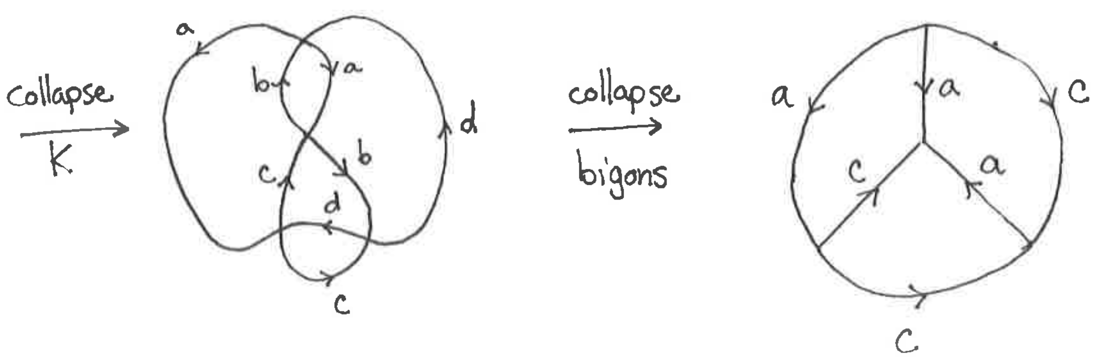
Next, gluing along a bigon is same as gluing along edge. Collapsing both bigons, we identify a with $\bar{b}$, c with $\bar{d}$ and get:



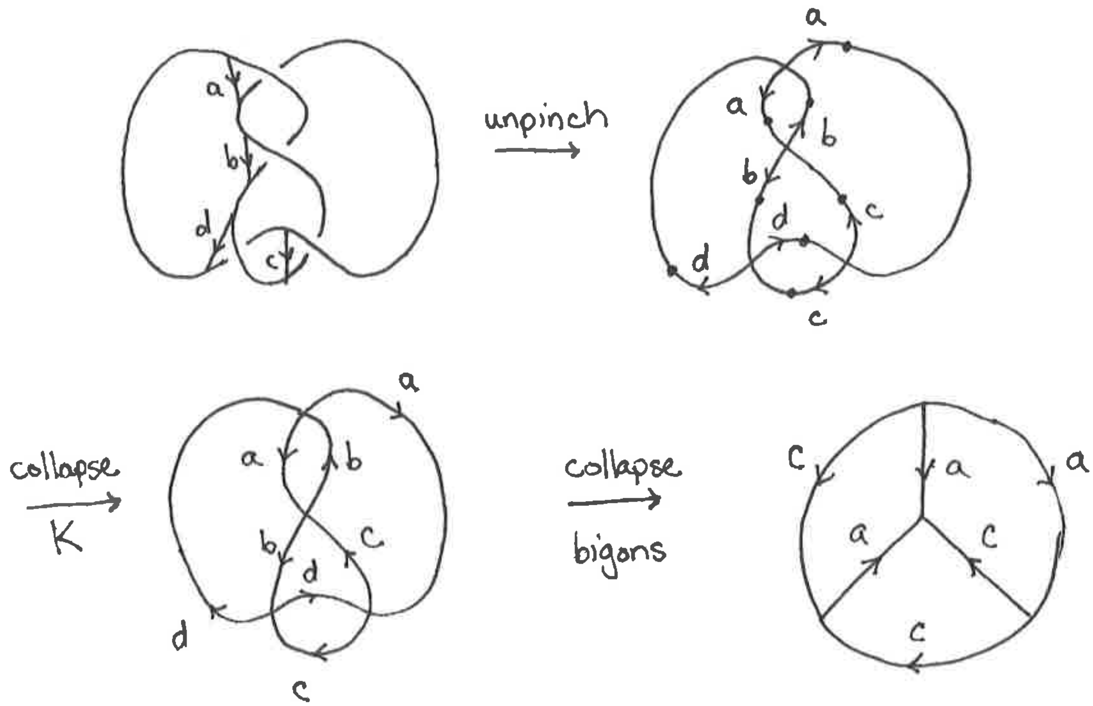
Doing same from the point of view of the bottom.

This is wrong! See next page.





Corrected bottom view:



Hyperbolic Structures on Ideal Triangulations

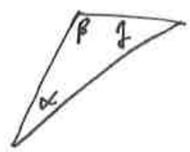
Say M = top. manifold obtained by gluing ideal simplices, e.g. $S^3 \setminus K$.

Q1. Which shapes of tetrahedra give hyp. structures?

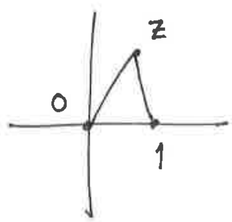
Q2. Which give <u>complete</u> hyp. structures? (Cauchys converge)

Again, by above thm, need angle 2π around each edge.

Recall: ideal determined by its link



This is congruent to



z = the complex parameter for the tetrahedron.

Note: z, $\frac{1}{1-z}$, $1-\frac{1}{z}$ all give congruent triangles.

But if we distinguish one vertex of the link (because it is on the edge we are focusing on) there is a unique complex param.

Let w_{ij} = complex param. for j^{th} tetrahedron around i^{th} edge.

<u>Thm.</u> M inherits a hyp. structure $\iff \prod_j w_{ij} = 1 \quad \forall i$.

~~Easier version: M inherits a hyp str~~ $\cancel{\iff \prod_j w_{ij} = 1}$ and $\sum_j \arg(w_{ij}) = 2\pi \ \forall i$.

"gluing equations"

Pf. Claim 1. M a man $\iff |\prod_j w_{ij}| = 1 \ \forall i$.

Claim 2. M has angle 2π around i^{th} edge $\iff \sum_j \arg(w_{ij}) = 2\pi$ and

~~Claim 3. $|\prod w_{ij}| = 1$ & $\sum \arg(w_{ij}) = 2\pi$ $\forall i$~~ $\prod_j w_{ij} = 1 \ \forall i$.

~~Note Claims 1,2 give easier version.~~

Pf of Claim 1. Let $e_1, \dots, e_k$ be the edges of ideal tets that get identified to i^{th} edge of M.

$\rightsquigarrow$ isometries $e_1 \to e_2 \to \cdots \to e_k \to e_1$

induced by face gluings.

$\rightsquigarrow e_1 \to e_1$ isometry

Subclaim. $e_1 \to e_1$ is id $\iff M$ a man.

pf. If $e_1 \to e_1$ is translation then each pt of i^{th} edge has ∞ many preimages

$\Rightarrow M$ not locally compact.

If $e_1 \to e_1$ is reflection, $\exists$ fixed pt

$\rightsquigarrow$ pt in M with link $\cong$ cone on $\mathbb{RP}^2$

Subclaim. $e_1 \to e_1$ is id $\iff |\prod_j w_{ij}| = 1$.

pf. place tetrahedra around i^{th} edge in U^3 around line from 0 to ∞.

and so first has vertices $0, \infty, 1, w_{i1}$

Then second has vertices $0, \infty, w_{i1}, w_{i1}w_{i2}$

Last face $0, \infty, \prod_j w_{ij}$ gets glued to first face $0, \infty, 1$ in a unique way by isometry.

The isometry fixes $0, \infty$ so it is dilation, which ~~So $e_1 \to e_1$ is id~~ is trivial iff $|\prod_j w_{ij}| = 1$.

Claim 2 now evident. $\square$

Gluing Eqns for Fig 8

If the 3 complex parameters for the link of a tetrahedron in $S^3 \setminus K$ are z_1, $z_2 = 1 - \frac{1}{z}$, $z_3 = \frac{1}{1-z}$ (first tet)

and w_1, $w_2 = 1 - \frac{1}{w}$, $w_3 = \frac{1}{1-w}$ (second)

then the two sets of gluing eqns are:

$$z_1^2 z_2 w_1^2 w_2 = 1$$

$$z_3^2 z_2 w_3^2 w_2 = 1$$

Set $z_1 = z$, $w_1 = w$. First eqn gives:

$$z^2\left(1-\frac{1}{z}\right) w^2 \left(1 - \frac{1}{w}\right) = 1$$

$$z(z-1)\, w(w-1) = 1$$

$$\rightsquigarrow \quad z = \frac{1 \pm \sqrt{1 + 4/(w(w-1))}}{2}$$

parameter space has one complex dim.

~~Need imag. parts of z, w to [illegible]~~.

Note $z = w = e^{i\pi/3}$ is a solution. But there are many others.

Will show this is the only solution giving a complete metric.