Completeness

Last time: family of hyp. structures on $S^3 \setminus K$
Q. Which are complete? Who cares?

Complete hyperbolic manifolds

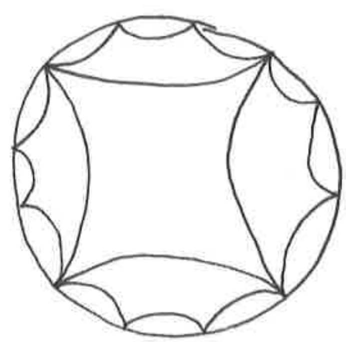
Thm. If M is a simply conn. complete hyp. n-man then M is isometric to $\mathbb{H}^n$.

Cor. The universal cover of a complete hyp. n-man is isometric to $\mathbb{H}^n$.

So we now have 3 ways to think about hyp mans:

① topological charts with $\mathrm{Isom}(\mathbb{H}^n)$ transitions
② locally isometric to $\mathbb{H}^n$
③ quotient of $\mathbb{H}^n$ by free, proper disc. action.

e.g.



Special case of Mostow Rigidity. If a hyp. n-man $(n \geq 3)$ has a hyp. metric that is complete and has finite volume, then the metric is unique.

Fig 8 Knot Complement as a complete manifold

Prop. M a metric space
S_t = family of compact subsets, $t \geq 0$
that cover M, and
$S_{t+a} \supseteq \mathrm{Nbd}(S_t, a)$
Then M is complete.

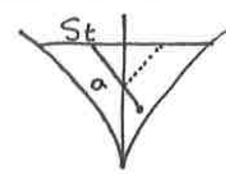
Pf. exercise.

Consider the hyp structure on $S^3 \setminus K$ given by two regular, ideal tetrahedra T_i. Put ideal vertices of one tetrahedron on vertices of regular ~~te~~ (Euclidean) tetrahedron. (ball model).

Let $S_t^{(i)}$ = intersection of T_i with $B(0,t)$

$S_t = S_t^{(1)} \cup S_t^{(2)}$

exercise: these S_t satisfy the Prop (use the fact that both tetrahedra are regular & that the pic is symmetric! Hint: at each ideal vertex have reflection:



)

Cor. K = fig 8 knot.
The universal cover of $S^3 \setminus K$ with above metric is $\mathbb{H}^3$.
In particular, the univ cover of $S^3 \setminus K$ is homeo to $\mathbb{R}^3$.

Other Consequences

1. A complete finite vol. hyp. man has infinite π_1. (must show $\mathrm{vol}(\mathbb{H}^n) = \infty$)
2. S^n has no hyp. structure, $n > 1$.
3. A compact hyp. man has no $\mathbb{Z}^2 < \pi_1$.

   so, e.g. T^n not hyperbolic

   more generally a closed, hyp. 3-man is atoroidal.

④ A complete hyp. 3-man is irred.

Pf of ③.: Step 1. Universal cover is $\mathbb{H}^n$ (by completeness)

Step 2. Deck trans are hyperbolic

- elliptics have fixed pts
- parabolics violate compactness (can find arbitrarily short loops)

Step 3. Commuting hyp. isometries have same axis

Step 4. Two translations of $\mathbb{R}$ either ① have a common power or ② have dense orbits.

Pf of ④. Let $S^2 \subseteq M$

Preimage in $\mathbb{H}^3$ is a collection of spheres. (using completeness here).

Alexander $\rightarrow$ each bounds a ball

Compactness $\Rightarrow \exists$ innermost lift of S^2, call it $\widetilde{S}^2$

$\rightsquigarrow$ ball in $\mathbb{H}^3$ with $\partial B = \widetilde{S}^2$

Translates of $\widetilde{S}^2$ all disjoint

$\Rightarrow$ B projects homeomorphically to closed ball $\bar{B}$ in M with $\partial \bar{B} = S^2$

Complete Structures on Surfaces

An example of an incomplete structure.

Let $B = \{(x,y) \in U^2 : 1 \le x \le 2\}$

Glue sides of B by $z \mapsto 2z$.

Result is incomplete: let $z_i = (1, 2^i) \sim (2, 2^{i+1})$

$$d(z_i, z_{i+1}) \le d_{\mathbb{H}^2}((2,2^{i+1}),(1,2^{i+1})) < 1/2^{i+1}$$

$\rightsquigarrow$ z_i Cauchy, does not converge since y-values $\longrightarrow \infty$.

More generally.

M = oriented hyp. surf. obtained by gluing ideal polygons

v = ideal vertex of M

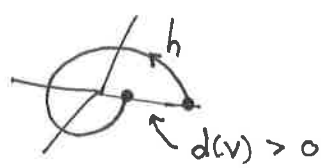
h = counterclockwise horocycle centered at v on one of the polygons P incident to v.

h meets ∂P in right angles

$\rightsquigarrow$ can continue h into next polygon.

$\rightsquigarrow$ eventually return to P.

$\rightsquigarrow$ $d(v)$ = resulting signed distance along ∂P (oriented to v).



exercise: $d(v)$ well defined.

Prop. M complete $\iff$ $d(v) = 0$ $\forall v$.

Pf. $d(v) \neq 0$ some v $\rightsquigarrow$ find nonconvergent Cauchy seq. as above.

$d(v) = 0$ $\forall v$ $\rightsquigarrow$ can make horocycles around each v.

S_t = subset of M obtained by deleting interior of horoballs bounded by horocycles distance t from originals.

Apply Prop. ▨

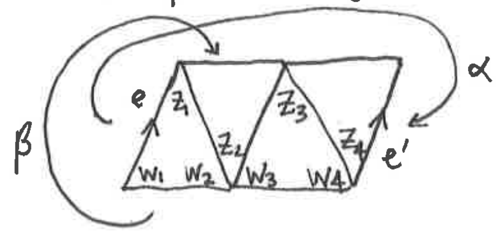
Complete Hyperbolic 3-Manifolds

Overview

M = orientable hyp. 3-man obtained by gluing ideal tetrahedra
The link of any ideal vertex is a torus.
The intersection of any such torus with a tetrahedron is
a triangle (or more than one) cf. $S^3 \setminus K$ example.
$\rightsquigarrow$ triangulation of the torus into Euclidean triangles.
Will show: M complete $\Leftrightarrow$ each such torus is Euclidean
(angle 2π around each vertex).
The two sides are related by the developing map.

Completeness Equations

M as above. Say the triangulation of some torus link is



Choose two gluing maps α, β so the surface obtained by doing both gluings is a torus (possibly with holes).

Consider α. Say it glues e to e'.
Choose a path from e to e' in 1-skeleton.

$\rightsquigarrow$ sequence of edges $e = e_0, \ldots, e_k = e'$

$\rightsquigarrow$ sequence of edge invariants $z_1, \ldots, z_k$. (vertices of the Δs are edges in M)

Raise z_i to $+1$ power if $e_{i-1} \longrightarrow e_i$ is counterclockwise

-1 otherwise

$\rightsquigarrow$ product of $z_i^{\pm 1}$, call it H.

forgot: multiply by -1 if the seq of edge swings takes e to reverse of e'.

In above example: $H(\alpha) = z_1 z_2^{-1} z_3 z_4^{-1}$

or $H(\alpha) = w_1^{-1} w_2^{-1} z_2^{-1} w_3^{-1} w_4^{-1} z_4^{-1}$

exercise: $H(\alpha)$ is well defined.

Completeness Equations

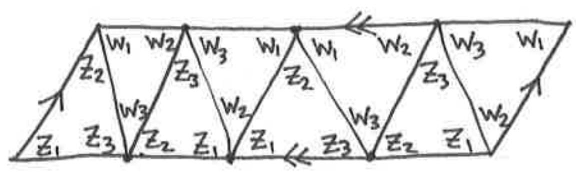
Proposition. The torus is Euclidean iff $H(\alpha) = H(\beta) = 1$.

Pf idea. $H(\alpha) = 1 \iff$ edges e, e' being glued are $\parallel$ and same length.

So $H(\alpha) = H(\beta) = 1 \iff$ corresponding deck trans are Euc. isometries. ▨

Figure 8 Example

Triangulation:



$\rightsquigarrow$ completeness eqns: $z_1^2 (w_2 w_3)^2 = (z/w)^2 = 1$

$w_1 / z_3 = w(1-z) = 1$.

first eqn $\rightsquigarrow$ $z = w$ (recall edge invariants have $\mathrm{Im} > 0$)

plugging into gluing eqn $\rightsquigarrow$ $(z(z-1))^2 = 1$

into second completeness eqn $\rightsquigarrow$ $z(z-1) = -1$

$\Rightarrow z = w = e^{i\pi/3}$ unique!

Developing Maps (Combinatorial Version)

M = hyperbolic (or Euclidean) manifold obtained by gluing (possibly ideal) polyhedra.

Will define $D : \tilde{M} \longrightarrow \mathbb{H}^n$ (or $\mathbb{E}^n$).

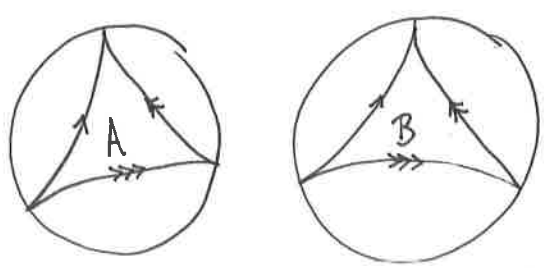
First, a description of $\tilde{M}$: glue polyhedra using same instructions as for M <u>except</u> each time we do a new gluing we take a new copy of the polyhedron

exercise: make sense of this and show the result is indeed $\tilde{M}$ (think of torus example).

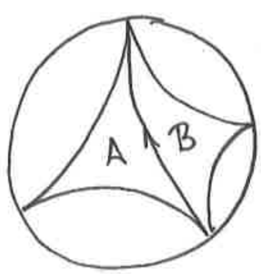
The map D is now evident: put the first polyhedron anywhere. Then glue in the rest of $\tilde{M}$ inductively.

The resulting map $\pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{H}^n)$ is called the <u>holonomy</u>.

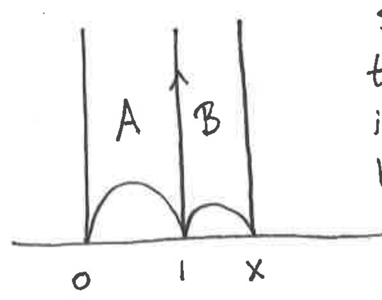
Example: sphere with punctures.



a gluing is prescribed by a picture like:

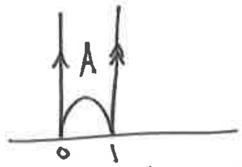


or

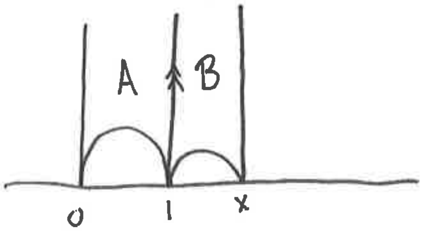


so a gluing of two ideal Δs is determined by $x > 1$.

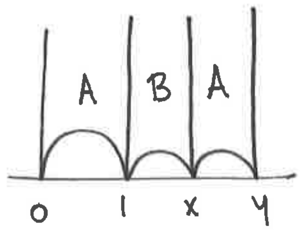
We first put A in $\mathbb{H}^2$:



Then put B in according to the prescribed gluing:



Then glue in A...



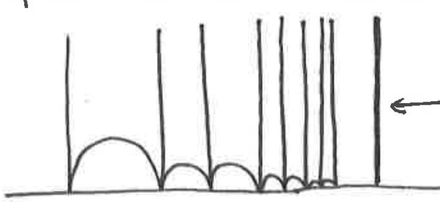
etc.

Recall our condition for completeness: horocycles obtained by extending the horocycle from one triangle should close up.

exercise: in our example this works iff $x=2, y=3$.*

$\Rightarrow$ exactly one complete structure on 

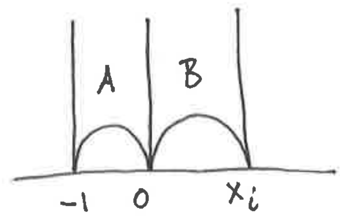
An incomplete example. If in the above construction we take $x = 3/2, y=2$ we get



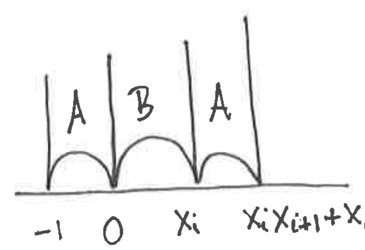
$\leftarrow$ limiting lines.

---

* More carefully: Say the 3 gluings are given by:



The condition for completeness at a single cusp is $x_i x_{i+1} = 1$ (indices mod 3). Indeed, this is equivalent to the two copies of A differing by horizontal translation. The three eqns together imply $x_1 = x_2 = x_3 = 1$.



Developing Maps and Completeness

Theorem. M = hyp. n-man.

M is complete iff $D : \tilde{M} \to \mathbb{H}^n$ is a covering map

(iff D is a homeo)

This works more generally for (G,X)-structures on manifolds.

Pf. $\boxed{\Rightarrow}$ Say M complete.

D is a local homeo, so suffices to show D has the path lifting property.

Let α_t = path in M

D a local homeo $\Rightarrow$ can lift α_t to path $\tilde{\alpha}_t$ in $\tilde{M}$ for $t \in [0, t_0)$ $t_0 > 0$.

$\tilde{M}$ complete $\Rightarrow$ $\tilde{\alpha}_t$ extends to $[0, t_0]$.

D local homeo $\Rightarrow$ $\tilde{\alpha}_t$ extends to $[0, t_0 + \varepsilon)$

So $\tilde{\alpha}_t$ extends to $[0,1]$.

Converse similar. $\blacksquare$

Compare with  example.

Prop. B = locally simply conn (any nbd of any pt contains a simply conn one)

$\tilde{B}$ = locally arcwise conn. (any nbd of any pt contains an arcwise conn. one)

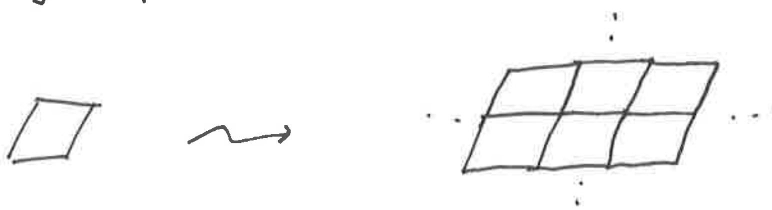
$\pi : \tilde{B} \to B$ local homeo s.t. every arc in B lifts to $\tilde{B}$.

Then π is a covering map.

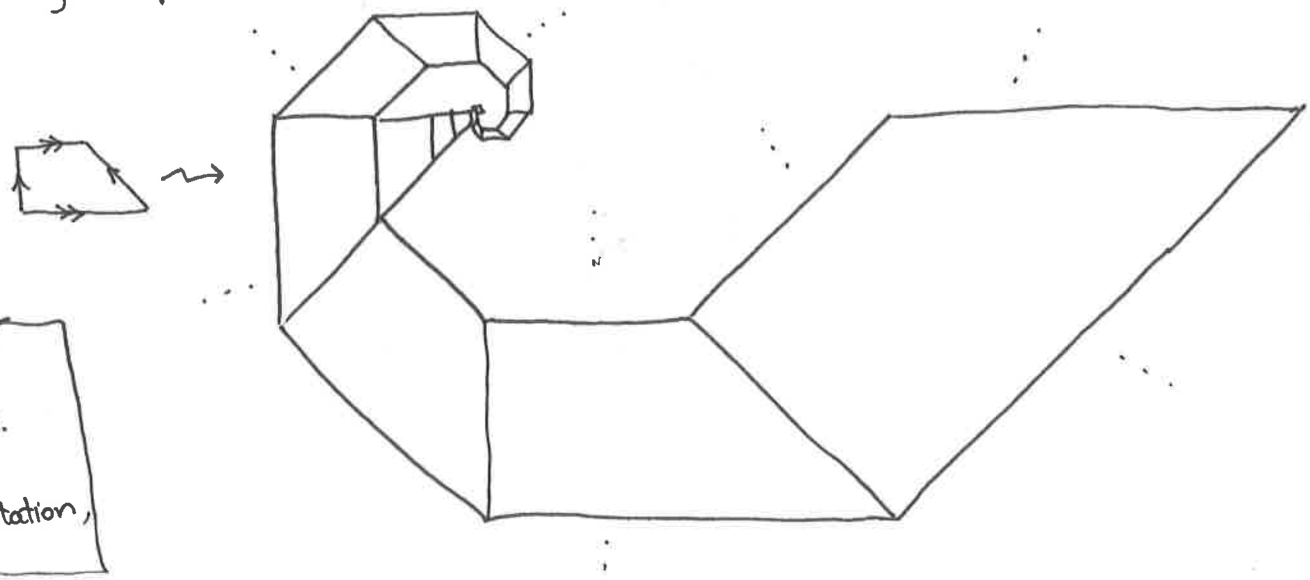
Pf. exercise (see baby do Carmo p. 383)

Affine Tori

Can do developing map with Euclidean tori:



Also makes sense with affine tori: arbitrary quadrilateral with gluing maps that are orient. pres. similarities of $\mathbb{E}^n$ instead of isometries.

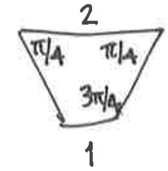


Classification of orientation pres. similarities: translation, rotation, spiral

If the quadrilateral is not a parallelogram, holonomy will have similarities that are not translations $\rightsquigarrow$ $\exists$ global fixed pt. (commuting similarities have same fixed pt).

~~To see that a similarity with nontrivial spiral scaling has a fixed pt, assume the scaling is < 1 (up to taking inverses). Iterate on a disk. It converges to a point.~~ Summarizing:

Good example



Prop. $D : \tilde{T} \to \mathbb{E}^2$ is surjective iff T Euclidean.

Can show: if not surjective, D misses exactly one pt.

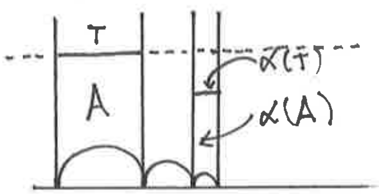
Complete Manifolds, Euclidean Cusps

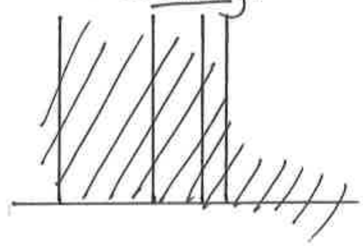
M = hyp 2- or 3-manifold obtained by gluing polyhedra.

v = ideal vertex

L = link of v (torus or circle)

L has a Euclidean similarity structure: under the developing map, simplices of L might change horocycles. To get any kind of Euclidean structure must project to a fixed horocycle. The cost of this is <u>scaling</u>.





<u>Thm</u>. M complete $\Leftrightarrow$ induced structure on each L is Euclidean.

<u>Pf</u>. M complete $\Leftrightarrow$ developing map preserves horocycles

$\Leftrightarrow$ L Euclidean. ∎