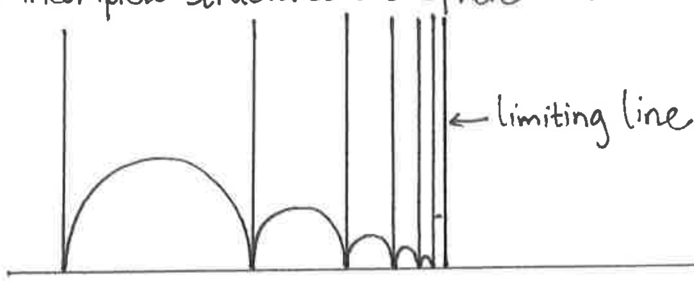
Completions

Surfaces

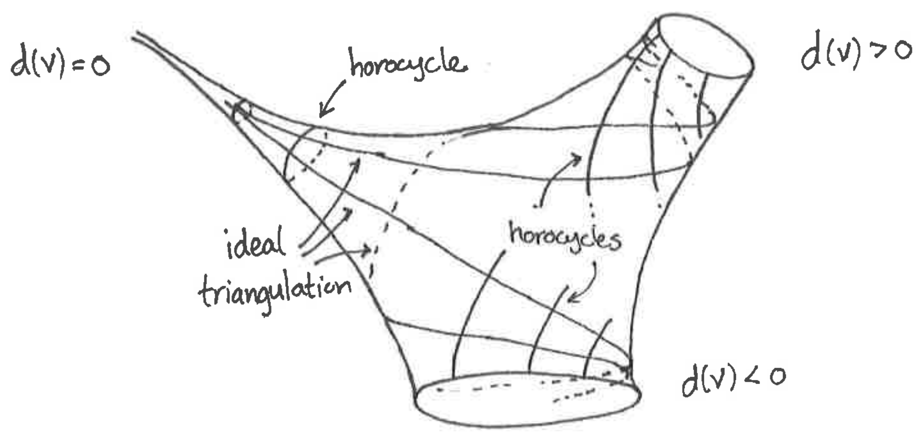
Recall incomplete structures on sphere with 3 punctures:



A horocycle converging to the limiting line gives a nonconvergent Cauchy seq.

Horocycles at (oriented) distance $d(v)$ are *identified*

$\rightsquigarrow$ need to adjoin a segment of length $d(v)$.



COMPLETIONS : 3-MANIFOLDS.

M = hyp. 3-man obtained by gluing polyhedra.
G = holonomy gp corresponding to cusp torus T
about ideal vertex v
M incomplete $\Rightarrow$ $G(\tilde{T}) = \mathbb{R}^2 \setminus pt$
$\Rightarrow$ $G(\tilde{M})$ misses a line L

Case 1. G has dense orbits in L
$\rightsquigarrow$ completion is 1 pt compactification, not a mnfld.

Case 2. G has discrete orbits in L.

Pts in each orbit have distance $d(v)$ apart.
$\rightsquigarrow$ completion obtained by adding geodesic circles of length $d(v)$.

What does the completion look like?
Any elt of $\ker(G \longrightarrow \mathrm{Isom}(L))$ acts by rotation by θ.
$\Rightarrow$ cross sections of completion are 2D hyp. cones.
Completion is a cone manifold.

When $\theta = 2\pi$, completion is a hyp. manifold. If we remove a nbd of completion pts, we recover M.

We say the completion is obtained by Dehn filling on M.

Hyperbolic Dehn Surgery Space

Next big goal: Which Dehn fillings of $S^3 \setminus K$ are hyperbolic?

M = orientably hyp 3-man of ideal tetrahedra
v = ideal vertex (assume only 1 for simplicity).
T = Link(v) torus
$\rightsquigarrow \pi_1(T) = \mathbb{Z}^2$

Dehn ~~Surgery~~ Filling

Choose coords on $\pi_1(T^2)$. The (p,q) Dehn filling of M, written $M_{(p,q)}$ is the mnfld obtained by gluing solid torus s.t. ∂ of meridian disk attaches to (p,q)-curve in T.

clasps the knot

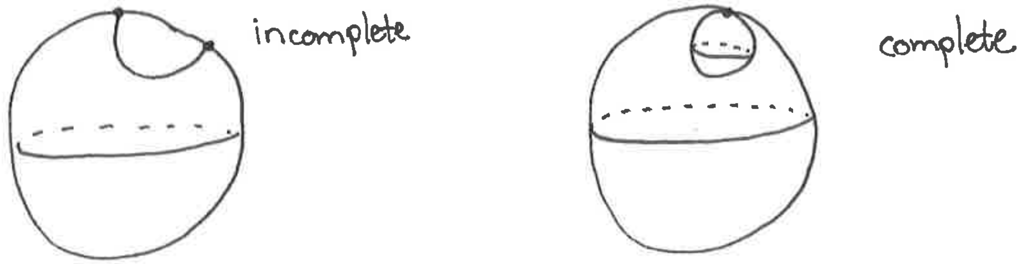
For $M = S^3 \setminus K$ there are canonical coords: meridian m is $1 \in H_1(M)$, longitude ℓ is 0.
$\searrow$ follows K.

Holonomy

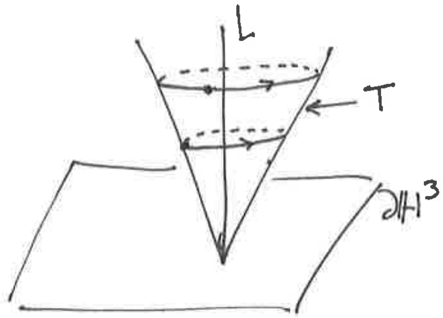
$\pi_1(T)$ abelian $\Rightarrow$ $\pi_1(T)$ fixes 1 or 2 pts of $\mathbb{H}^3$ (under holonomy)

Fixes 1 pt $\Rightarrow$ image of $\pi_1(T)$ parabolic $\Rightarrow$ M complete.
Fixes 2 pt $\Rightarrow$ image of $\pi_1(T)$ consists of hyp. isometries along single axis L. L is the pts missing from developing map of T in each horocycle. $\Rightarrow$ M incomplete.

Can see now why there is a 2D space of incomplete structures and one complete one:



Note: T is quotient of tube around L:



Complex Length

Any $\gamma \in \pi_1(T)$ translates L by d, rotates by $\theta \in \mathbb{R}$

to get a real number, need to keep track of the number of times it goes around L.

$\ell(\gamma) = d + i\theta$ "complex length"

$\rightsquigarrow \ell : H_1(T;\mathbb{Z}) \longrightarrow \mathbb{C}$ linear

$\rightsquigarrow \ell : H_1(T;\mathbb{R}) \longrightarrow \mathbb{C}$ linear

~~We are more interested in $\tilde{\ell} : H_1(T) \longrightarrow \tilde{\mathbb{C}}^*$ where you keep track of the number of times a loop circles L, not just angle.~~

Note: If we want a discrete action, $\pi_1(T) \longrightarrow \mathrm{Isom}(L)$ has nontrivial kernel.

Dehn Surgery Coefficients

In general $\exists!\ c \in H_1(T;\mathbb{R})$ s.t. $\mathcal{L}(c) = 2\pi i$

This is the Dehn surgery coeff of T.

If $c = (p,q)$ & $\gcd(p,q) = 1$ then c is a curve in T that bounds a hyp disk and $\overline{M} = M_{(p,q)}$ is hyperbolic.

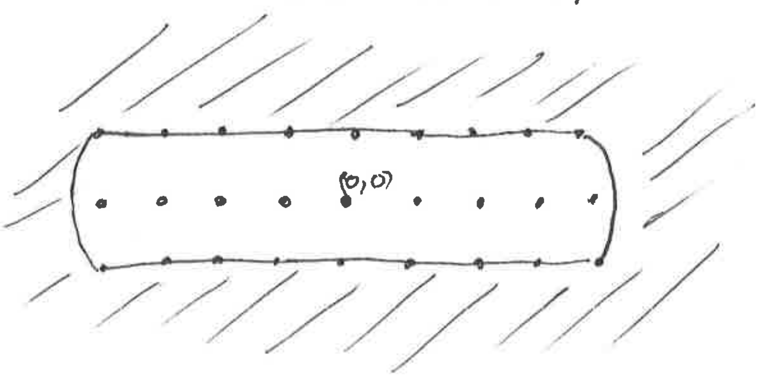
Thurston's Hyp. Dehn Surgery Thm

The hyp. Dehn surgery space for M is the set of all Dehn surgery coeffs, e.g. the Dehn fillings that give ~~topological~~ hyperbolic hyp. mans.

<u>Thm</u> (Thurston). The Dehn surgery space contains a nbd of ∞ in $\mathbb{C}$. Moreover $M_{(p_i,q_i)} \longrightarrow M_\infty$ as $(p_i,q_i) \longrightarrow \infty$.

(Analogous statement for multiple cusps: ~~exce~~ finitely many exceptional slopes on each ~~one~~ torus).

Example. $S^3 \setminus$ Fig 8 :



Idea: Explicitly analyze the map

$$\{\text{solutions to gluing eqns}\} \longrightarrow \{\text{Dehn surgery coeffs}\}$$

ie deform the triangles in T, then find the elements of $\pi_1(T)$ with complex length $2\pi i$.