Mostow Rigidity

<u>Thm.</u> M, N complete, finite vol, hyp n-mans $n > 2$
Any isomorphism $\pi_1 M \to \pi_1 N$ is induced by a unique isometry $M \to N$

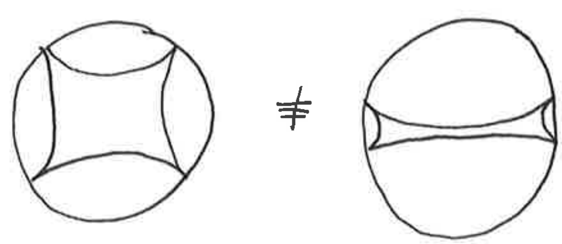
In particular: ① $\pi_1(M) \cong \pi_1(N) \Rightarrow M \equiv N$
② volume, diam, inj rad are invariants of M.

<u>Cor.</u> M closed, hyp n-man $n > 2$
$\mathrm{Isom}(M) \cong \mathrm{Out}(\pi_1 M)$
and these gps are finite.

Pf idea. Mostow $\Rightarrow$ $\mathrm{Isom}(M) \to \mathrm{Out}(\pi_1 M)$ is surjective.

<u>Non-rigidity</u>

① Mostow not true for $n = 2$:



$\rightsquigarrow$ Teichmüller space

② Mostow not true for non-hyp mans

$\pi_1 L(7,1) \cong \pi_1 L(7,2) \cong \mathbb{Z}/7$
but $L(7,1) \not\cong L(7,2)$ (Reidemeister)

Outline of Proof

Assume M, N compact.

Start with $F : \pi_1 M \xrightarrow{\cong} \pi_1 N$

Want to promote F to an isometry $M \longrightarrow N$

Step 1. Homotopy equivalence

M, N are $K(G,1)$ spaces since $\tilde{M} \cong \tilde{N} \cong \mathbb{H}^n$

$\Rightarrow \exists\ f : M \longrightarrow N$

$g : N \longrightarrow M$

s.t. $g \circ f \simeq \mathrm{id}$.

Step 2. Lift

$\rightsquigarrow \tilde{f} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ (lifting criterion)

Step 3. Extend

$\rightsquigarrow \partial\tilde{f} : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$

Step 4. Show $\partial\tilde{f}$ is conformal.

Step 5. Extend

$\rightsquigarrow \varphi : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ isometry

Step 6. φ descends to $\bar{\varphi} : M \longrightarrow N$.

Step 2. Properties of $\tilde{f}$

① $\tilde{f}$ is $\pi_1(M)$-equivariant:

$$\tilde{f}(g\cdot x) = f_*(g)\cdot \tilde{f}(x) \qquad \text{(exercise)}.$$

② $\tilde{f}$ is a quasi-isometry: $\exists$ K, C s.t.

$$\frac{1}{K} d(x,y) + C \le d(\tilde{f}(x), \tilde{f}(y)) \le K\, d(x,y) + C \qquad \left(\begin{array}{l}\text{and } \exists \\ \text{qi inverse}\end{array}\right)$$

pf of ②. Compactness + continuity $\Rightarrow$ $\tilde{f}, \tilde{g}$ Lipschitz, i.e. $\exists$ $K>0$ s.t.

$$d(\tilde{f}(x), \tilde{f}(y)) \le K\, d(x,y)$$

Other inequality. For $x, y \in \tilde{M}$ have

$$d(\tilde{g}\tilde{f}(x), \tilde{g}\tilde{f}(y)) \le K\, d(\tilde{f}(x), \tilde{f}(y))$$

But $\tilde{g}\tilde{f}$ equiv. homotopic to id & M compact

$\rightsquigarrow$ $d(\tilde{g}\tilde{f}(z), z) \le C$ for some C indep of z.

$$\Rightarrow d(\tilde{f}(x), \tilde{f}(y)) \ge 1/K\, d(\tilde{g}\tilde{f}(x), \tilde{g}\tilde{f}(y))$$
$$\ge 1/K\, (d(x,y) - 2C). \qquad \blacksquare$$

Step 3. Quasigeodesics and the boundary map

Thm. Any quasi-isometry $h: \mathbb{H}^n \to \mathbb{H}^n$ extends to a homeo $\partial\mathbb{H}^n \to \partial\mathbb{H}^n$

Note: h need not be continuous!

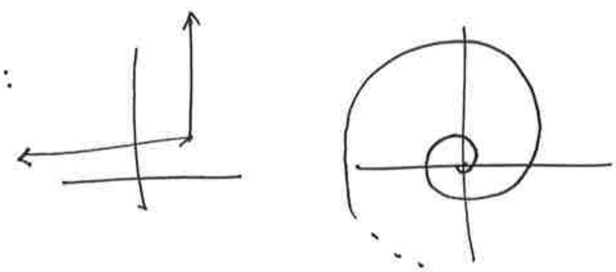
This works for $n = 2$.

Quasigeodesics

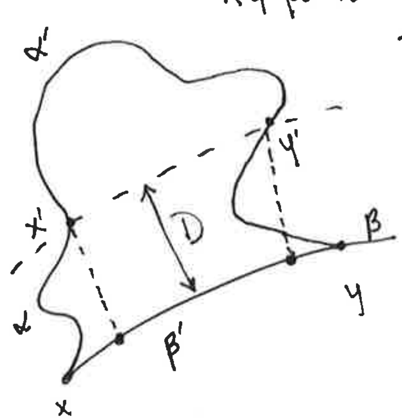
A geodesic in a metric space X is an isometric embedding $I \to X$.

A quasigeodesic is a quasi-isometric embedding $I \to X$.

examples in $\mathbb{R}^2$:



<u>Morse-Mostow Stability Lemma</u>. If $\alpha : \mathbb{R} \to \mathbb{H}^n$ is a quasigeod, $\exists!$ geod γ s.t. α lies in bdd nbd of γ.



Key point: Let $I = [a,b]$, $x = \alpha(a)$, $y = \alpha(b)$, β the geodesic from x to y.

Pick $D \gg K$ and suppose α does not stay within D of β.

Let x', y' be distinct pts of α at distance D from β.

Let β' be ~~projection~~ segment of β from projs of x' & y'.

Projections in $\mathbb{H}^n$ decrease length exponentially

$$\rightsquigarrow \ell(\alpha') \le K^2(\ell(\beta') + 2D) + CK$$
$$\le K^2(e^{-D} d(x',y') + 2D) + CK$$

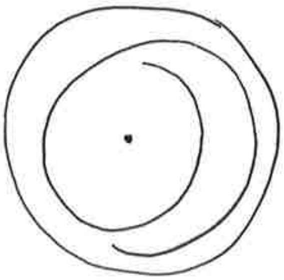
Now, $d(x',y') \le \ell(\alpha')$ and $D \gg K$

$$\Rightarrow \ell(\alpha') \le \frac{2DK^2 + CK}{1 - K^2 e^{-D}} \le 4D^2$$

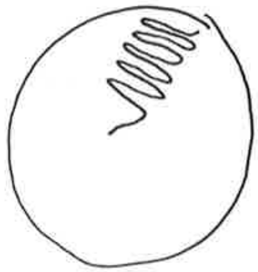
$\Rightarrow \alpha$ stays in $D + 4D^2$ nbd of β.

This only depends on K so works for any bdd interval $[a,b]$

Any quasigeodesic leaves every ball around O in $\mathbb{H}^n$, and this argument rules out spiralling:



or



The Extension

Recall $\partial\mathbb{H}^n = \{\text{geodesic rays}\}/\sim$

$\alpha \sim \beta$ if $d(\alpha(t), \beta(t))$ bounded

By the Lemma, h takes rays to rays (after straightening) and preserves $\sim$

$\rightsquigarrow \partial h : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$

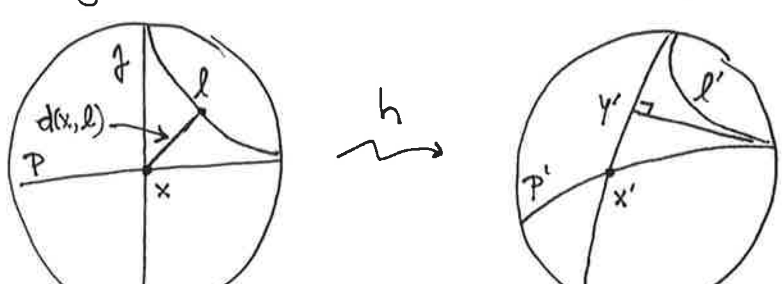
Check: ∂h is well def and 1-1.

Want to show ∂h is continuous.

"No tilting"

Lemma. $\exists\, D = D(K)$ s.t. for any hyperplane $P \subseteq \mathbb{H}^n$ and any geod $\gamma \perp P$ we have $\operatorname{diam} \operatorname{Proj}_\gamma(h(P)) \le D$.

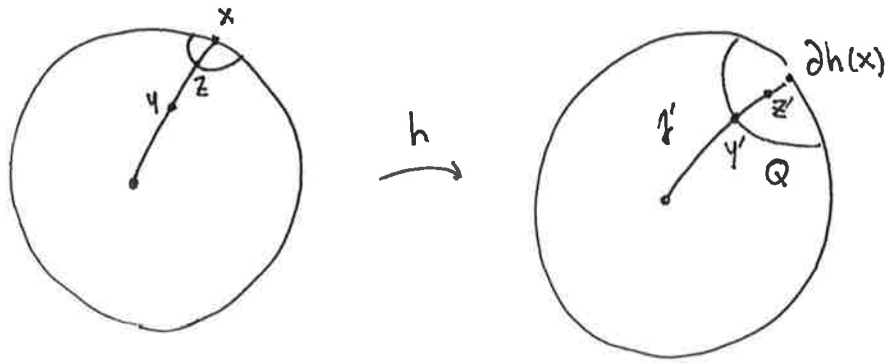
Pf.



prime ' means: apply h then straighten.

$$d(x', y') \le d(x', \ell') \le K\, d(x, \ell) + C.$$

Proof that $\partial\tilde{f}$ is continuous:



Open half-spaces $\perp$ to γ' form a nbd basis around $\partial h(x)$.

Pick such a half space Q.

Choose z on γ s.t. $d(z', \partial Q) > 100D$ ← as in lemma

Then the half-space $\perp$ to γ through z maps into Q. ▨