Mostow Rigidity via Gromov Norm

Thm. M, N complete, finite vol, hyp mans $n > 2$
Any isomorphism $\pi_1 M \to \pi_1 N$ is induced by
a unique isometry $M \to N$

Step 1. $\exists\ f: M \to N$ homotopy equiv. (uses completeness!)

Step 2. Lift to $\tilde{f}: \mathbb{H}^n \to \mathbb{H}^n$ quasi-isometry

Step 3. Extend to $\partial\tilde{f}: \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ continuous

Gromov Norm

Norm on real singular n-chains: $\| \Sigma t_i \sigma_i \| = \Sigma |t_i|$

$\rightsquigarrow$ pseudo-norm on $H_n(X; \mathbb{R})$:

$$\|\alpha\| = \inf_{[\Sigma t_n \sigma_n] = \alpha} \| \Sigma t_i \sigma_i \|$$ "Gromov norm"

Lemma. $f: X \to Y$ cont, $\alpha \in H_n(X; \mathbb{R})$
then $\|f_*(\alpha)\| \leq \|\alpha\|$

Cor. f a homot. equiv $\Rightarrow \|f_*(\alpha)\| = \|\alpha\|$.

For M closed, orientable: $\|M\| = \|[M]\|$

Fact. If M admits deg > 1 self-map then $\|M\| = 0$.

Step 4. Gromov norm vs. volume

<u>Thm</u> M = closed, hyp n-man

$$\|M\| = \mathrm{Vol}(M)/v_n$$

v_n = max vol of a simplex

<u>Cor.</u> ① M has no self-maps of deg > 1

② volume is an invariant.

Step 5. $\partial\tilde{f}$ preserves regular ideal tetrahedra ($n=3$).

Step 6. $\partial\tilde{f}$ is conformal (hence agrees with some isometry).

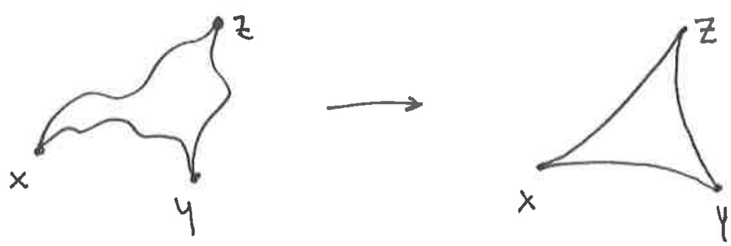
<u>Fact.</u> Let $n>2$, σ reg. ideal tet, τ = face.
$\exists!$ reg ideal tet σ' s.t. $\sigma \cap \sigma' = \tau$.

Let σ = any reg ideal tetrahedron.
Step 5 $\Rightarrow$ $\partial\tilde{f}_*(\sigma)$ regular
$\Rightarrow$ Up to postcomposing with conformal map can assume $\partial\tilde{f}_*(\sigma) = \sigma$.
Fact $\Rightarrow$ $\partial\tilde{f}_*$ fixes every simplex obtained from σ via the grp gen by reflections in faces of σ
But the vertices of these tetrahedra are dense in $\partial\mathbb{H}^3$
$\Rightarrow$ $\partial\tilde{f}_* = \mathrm{id}$, as desired. ∎

Gromov's Thm

Straightening simplices

In $\mathbb{H}^n$ an arbitrary singular simplex can be straightened:



This works for simplices in M (lift, straighten, project)

Note: ① Straightening takes cycles to cycles

② $\|\text{straight}(z)\| \leq \|z\|$ (some simplices might cancel/vanish).

Lower bound

Prop. $\|M\| \geq \text{vol}(M)/v_n$

Pf. Let $z = \sum t_i \sigma_i$ straight cycle with $[z] = [M]$

$$\text{vol}(M) = \int_M d\text{vol} = \sum t_i \int_{\Delta^n} \sigma_i^*(d\text{vol}) \leq \sum |t_i| v_n$$

$\Rightarrow \|z\| \geq \text{vol}(M)/v_n$ take inf. ∎

Upper bound

Prop. $\|M\| \leq \mathrm{vol}(M)/V_n$

Need chains σ_L with $[\sigma_L] = [M]$

and $\|\sigma_L\| \longrightarrow \mathrm{vol}(M)/V_n$ as $L \longrightarrow \infty$.

Smearing.

D = fund. dom. for M

σ = simplex in M

$\rightsquigarrow \tilde{\sigma}$ = simplex in $\tilde{M} = \mathbb{H}^n$

t = signed measure of simplices in $\mathbb{H}^n$ with vertices in same copies of D as $\tilde{\sigma}$ (sign means mult by -1 if τ reverses or.)

$\rightsquigarrow$ Smear$(\sigma) = t\sigma$

Defining σ_L.

Consider all regular straight simplices τ with side length L, zeroth vertex in D. Choose $x \in D$.

Let τ' be the straight simplex with vertices at corresponding translates of x.

$\sigma_L = \sum_{\tau} \mathrm{Smear}(\tau')$.

Check: ① volume of each such ∇ is $V_n - \varepsilon(L)$ $\qquad \lim_{L\to\infty} \varepsilon(L) = 0.$

② each such sum is finite, moreover

③ ∇_L is a cycle

In particular, some multiple of $[\nabla_L]$ is $[M]$.

Say this multiple is $z = \sum t_i \nabla_i$

$$\rightsquigarrow \|M\| \leq \sum t_i = \mathrm{vol}(M) / V_n - \varepsilon(L)$$

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Step 5. Regular ideal tetrahedra go to same.

If not, a definite fraction of ∇_L loses a definite amount of volume, violating Step 4.