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Around the Borromean link

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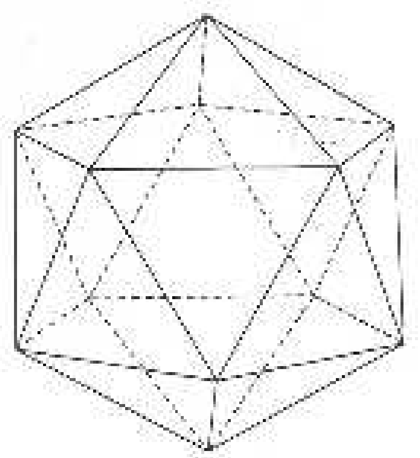
Abstract. This is a survey of some consequences of the fact that the fundamental group of the orbifold with singular set the Borromean link and isotropy cyclic of order 4 is a universal kleinian group.

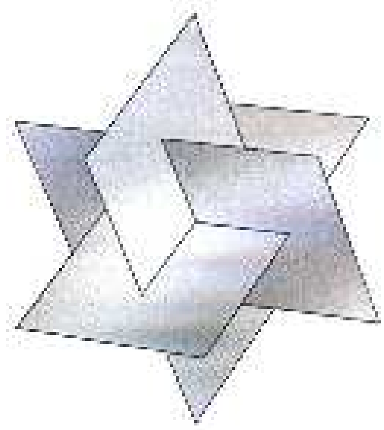
En torno al enlace de Borromeo

Resumen. Se presenta una panorámica de lo que se ha podido deducir hasta ahora del hecho de ser universal el grupo fundamental de los anillos de Borromeo con isotropía 4.

1 Introduction

Three golden ratio cards symmetrically intertwined produce the 12 vertexes of an icosahedron.





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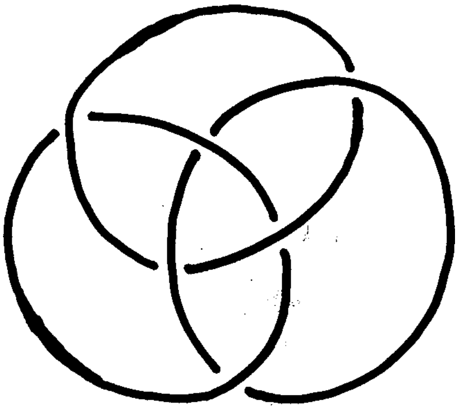
The boundaries of the three cards form the following link:



This is the famous *Borromean link,* so called for its relationship with the Borromeo family (see [2]). A cousin of the Borromean link appears in the house of Admiral Oquendo at Mount Ulia in San Sebastián (Spain). The link seems to be used here as a hieroglyphic of the word Oquendo.

O-QU-EN-(D)-O

The Borromean link (a true link made up of three unknots) appeared for the first time in a mathematical context in Tait's Table of Knots (1876). Ralph H. Fox referred to this link as the "Borromean rings" in his celebrated survey [3], but for cacophony reasons I will refer to it, along this expository article, as the Borromean link, denoted B. This is the link 6^3_2 in Rolfsen's Table of Knots and Links [24].

The Borromean link is important in relation with the topology and geometry of three-manifolds. I intend to give here a very short exposition of my investigations together with Maite Lozano and Hugh Hilden in these matters. I am very much indebted to them for many years of close colaboration. Thus I dedicate this paper to Maite Lozano and Hugh Hilden with affection and thanks.

Along the paper we will make frequent use of the geometry and topology of 2-manifolds (surfaces) to illustrate facts that generalize to dimension three.

For the basic definitions it would be of interest to some readers to consult Ratcliffe [23], Rolfsen [24] and Thurston [26].

2 Combinatorial level

Manifolds of dimensions 2 and 3 are triangulable (Moise). This is the starting point to represent these manifolds as branched coverings of the sphere.

Let us see in detail the case of surfaces.

Take an arbitrary unbounded, orientable surface Σ and a triangulation K of it. Subdivide barycentrically K to obtain another triangulation K'. The vertexes of K' fall naturally in three classes: barycenters of vertexes (resp. edges, faces) of K called, respectively, bary-vertexes, bary-edges or bary-faces. Accordingly, any face of K' has a natural orientation, namely the one given by the following ordering of its vertexes: bary-vertex, bary-edge, bary-face. Color this face white iff this natural orientation coincides with a fixed orientation of Σ. Otherwise color it black. Then we have obtained a check-board coloration of the faces of K' because two different faces sharing an edge get different colors.

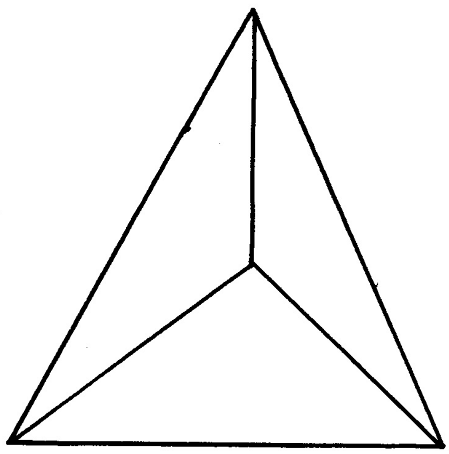
This simple argument of Ramirez has an important consequence. Namely, that *every unbounded, orientable surface is a covering of the sphere branched over three points* [22].

In fact, think of the sphere S as the result of pasting linearly together two triangles (one white; the other black) along their edges. Call the resulting vertexes 0, 1, 2. Then we map linearly white (black) triangles of the surface Σ to the white (black) triangle of S in such a way that bary-vertexes (resp. bary-edges, bary-faces) go to 0 (resp. 1, 2).

This argument works in fact for every triangulated unbounded, oriented n-manifold. Therefore, we have proved the following Theorem.

Theorem 1 ([22]) *Every unbounded, orientable 3-manifold is a covering of the sphere, branched over a graph G. This graph is the set of edges of a tetrahedron embedded in the sphere.*

Therefore the graph G is universal in the sense that every 3-manifold branches over it. But note that, while in the case of surfaces, the branching set is a manifold, this is not the case if the dimension of the manifold is greather or equal than 3.



The graph G

Problem 1 *Is there a universal branching set which is a manifold for every dimension?*

González-Acuña asked this question and W. Thurston found (in an unpublished paper) the first example of a (complicated) link in the 3-sphere S^3 that was universal. Thurston also asked if some familiar knots and links, (like the figure eight knot, Whitehead link or the Borromean rings) were in fact universal. This was answered positively in the papers [6] and [7] (see also [4, 5, 27, 28]), but the arguments are too complicated to be reproduced here. (It was also clear at the time that some knots and links, like the trefoil knot, could not be universal.)

Theorem 2 *The figure-eight knot and the Whitehead and Borromean links are universal branching sets for all closed, orientable 3-manifolds.*

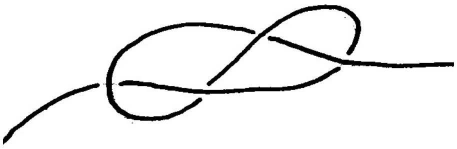
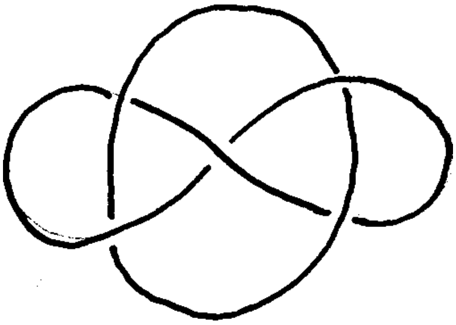


Figure eight knot



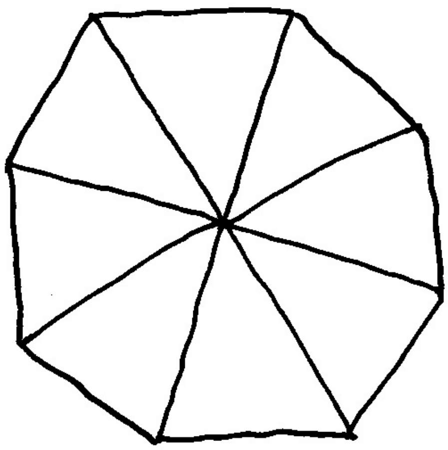
Whitehead link

3 Controling branching indexes

Coming back to surfaces, remember that we have proved that for every unbounded, orientable surface Σ there is a covering $f\colon \Sigma \to S$ of the sphere S branched over three points v_0, v_1, v_2, marked, respectively, 0, 1, 2. But note that if $w \in f^{-1}(v_2)$ the branch index of w is 3 because 6 barycentric triangles of K' are mapped onto two triangles of S. Similarly, the branch index of $w \in f^{-1}(v_1)$ is 2. But there is absolutely no control on the branch index of points belonging to the fiber of v_0.

Problem 2 *Is it possible to find, for any Σ, a covering $f\colon \Sigma \to S$ of the sphere S, branched over three points v_0, v_1, v_2 with extrict control on the branching indexes?*

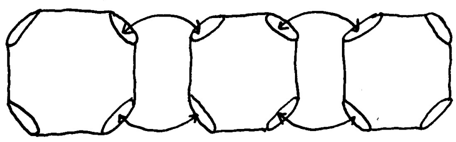
We will answer this question in the affirmative for compact, unbounded, orientable surfaces (closed surfaces).



Octogon

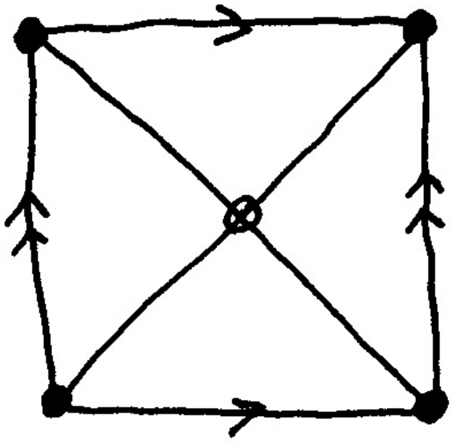
Take a regular octogon Ω and from its center draw segments to its vertices. This gives a triangulation of Ω by 8 triangles. Pasting together alternating sides of two of these triangulated octogons we obtain a sphere

$Soooo$ with 4 holes. Pasting the holes in pairs we get the orientable surface F_2 of genus 2. Note that F_2 is triangulated by 16 triangles so that each vertex belongs to 8 of them (has valence 8). If we paste n copies of $Soooo$ together we can obtain a surface $F_{n-1}oooo$ of genus $n-1$ with four holes. Pasting now the holes in pairs we get the orientable surface F_{n+1} of genus $n+1$. In this way we have proved the following result: every orientable, closed surface F_g of genus $g \geq 2$ is triangulated by $16(g-1)$ triangles with vertexes of valence 8.



What about the surfaces of genus less than 2?

We can see the torus F_1 as a square with opposite sides identified. We can divide it in 4 triangles by connecting its center to its vertexes. Thus the torus can be divided in triangles with one vertex of valence 4 and one vertex of valence 8.



On the other hand, the sphere F_0 can be triangulated with vertexes of valence 4 (octahedron).

If we apply Ramírez construction to these triangulations K we obtain the following Theorem.

Theorem 3 *Every closed, orientable surface is a covering of S^2 branched over three points A, B and C. The branching indexes on top of A (resp. B; C) are all 2 (resp. all 3; all 4 or 8).*

We can reformulate this theorem in orbifold terms. Let $S238$ denote the 2-orbifold with underlying space S and singular points A, B, C with isotropies cyclic of orders 2, 3, 8 respectively. Then a covering $f\colon \Sigma \to S$ branched over A, B and C such that the branching indexes on top of A (resp. B; C) are all 2 (resp. all 3; all 4 or 8) can be considered as an orbifold covering $f'\colon Q \to S238$, where Q is an orbifold with underlying space Σ and whose set of singular points is the set of points with branch index 4 under the branch covering f.

Define an *orbifold U* to be *universal* if and only if every closed, orientable manifold is the underlying space $|Q|$ of an orbifold Q covering U. Then we have proved the following Theorem.

Theorem 4 *The orbifold $S238$ is universal.*

On the other hand no euclidean orbifold can be universal.

Example 1 *The orbifold $S236$ is not universal.*

To show this, note that $S236$ is a euclidean orbifold. In fact $S236$ is the result of pasting together along their boundary two euclidean triangles of angles $30\,°$, $60\,°$ and $90\,°$. An orbifold Q covering $S236$ is euclidean except at some cone points with angles $\alpha < 2\pi$. These angles concentrate positive curvature $2\pi - \alpha$. Therefore the underlying surface $|Q|$ has a metric of non-negative curvature. Therefore $|Q|$ must have genus less or equal than 1. Thus $S236$ is not universal.

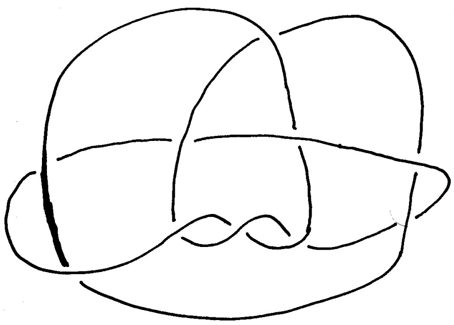
As we have seen, proving that $S238$ is universal is an almost trivial result. To obtain an analogous result for every dimension is difficult.

Think for instance in the dimension 3 case. We know that the set G of edges of a tetrahedron embedded in the 3-sphere is a universal branching set. We even have some control on the branch indexes. In fact, the interior points of an edge e are all covered by branch index 3 points; and the interior points of three edges meeting e are all covered by branch index 2 points. We do not have, in principle any control on the branch indexes for the points covering the remaining two edges. And I do not see any direct method to control this.

Thus the next Theorem ([8, 11]) is really surprising. To state this theorem is convenient to denote by (L, m) a 3-orbifold with underlying space S^3, singular set a knot or link L and isotropy cyclic of order m in every component of L:

Theorem 5 *The orbifolds (Figure-eight knot, 12), (Whitehead link, 12) and (Borromean link, 4) are universal orbifolds.*

It is also known [13] that *there is a universal orbifold* $(K, 2)$ *where* K *is a knot* though all known examples are extremely complicated (see also [17]). Hilden, Lozano and I have conjectured that the orbifold $(10_{161}, 2)$ is universal but so far we have been unable to prove this.



10_{161} knot

4 Geometric level

It follows from the previous section that if we are able to geometrize some universal orbifolds we will be able to geometrize also their coverings. That is, in a sense, all manifolds.

Note, for instance the case of the orbifold $S238$. This is a hyperbolic orbifold. In fact, take the hyperbolic triangle of angles $\pi/2,\ \pi/3,\ \pi/8$ and double it along its boundary. This is a sphere with a Riemannian metric of curvature, say, -1 and three cone-points $A,\ B,\ C$ of angles $2\pi/2,\ 2\pi/3,\ 2\pi/8$. That is, a hyperbolic orbifold. The orbifold coverings corresponding to branched coverings with all branch indexes on top of $A,\ B,\ C$ equal to 2, 3, 8, respectively, are in fact hyperbolic surfaces. Thus all surfaces of genus greather or equal than 2 are hyperbolic.

For the torus, the 12-fold branched covering $f\colon F_1 \to S$ has just one point of branch index 4 over C. Then the corresponding orbifold covering is

$$f'\colon (F_1, 2) \longrightarrow S238,$$

where the orbifold $(F_1, 2)$ has underlying space the torus F_1 and singular set a point of isotropy 2 (cone-angle π). The total curvature of $(F_1, 2)$ is the sum of the negative curvature lifted from $S238$ and the positive total curvature concentrated at the cone point. This last is

$$2\pi - \text{angle} = \pi.$$

The former is, using Gauss-Bonet,

$$12(2\pi\chi^{\circ}(S238)) = -\pi$$

because the Euler characteristic $\chi\circ(S238)$ of the orbifold $S238$ is

$$-1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{-1}{24}.$$

Therefore the total curvature of the cone-manifold $(F_1, 2)$ is zero, as expected.

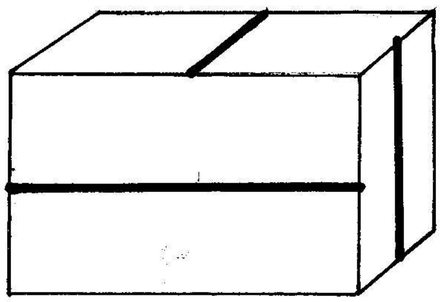
For the sphere S we have a 24-fold branched covering $f\colon S \to S$ with 6 points of branched index 4 on top of the point C of $S238$. Then f induces an orbifold covering $f'\colon S222222 \to S238$ from the hyperbolic orbifold $S222222$ with 6 singular points of isotropy of order 2 onto $S238$. Here the total curvature of the cone-manifold $S222222$ is

$$24(2\pi\chi^{o}(S238)) + 6(2\pi - \pi) = 4\pi$$

as it shoul be.

Now we consider the question in dimension three. Let B denote the Borromean link and let $(B, 4)$ the orbifold with underlying space S^3, singular set B and isotropy cyclic of order 4. We have stated before the theorem that $(B, 4)$ is a universal orbifold. That is, every closed, orientable 3-manifold is the underlying space of an orbifold, covering the orbifold $(B, 4)$.

Now the orbifold $(B, 4)$ is a hyperbolic orbifold. In fact, consider the following combinatorial dodecahedron:



Pasting faces in pairs, by reflection on the 6 thickened edges (there are 3 of them that are not visible in the picture, but the ones in opposite faces of the paralelepipedon are parallel) we get S^3. The boundary of the dodecahedron is sent to the three golden ratio cards, and the thickened edges go to the borromean link B. If we think on the above dodecahedron as a euclidean parallelepipedon, then S^3 inherits a euclidean structure with singular set B. Here the cone angle is π. Thus $(B, 2)$ is a euclidean orbifold.

But if we take a regular dodecahedron D inside a sphere S, both centered at the origen of $\mathbb{R}^3$, then the interior of S is the projective model of hyperbolic 3-space H^3. The dodecahedron D is also regular in H^3 but its dihedral angles depend on the radius of the sphere S. If the vertexes of D lie on S the dihedral angles are of 60° and when the radius of S tends to infinite then D tends to be euclidean with angles of approximately 116°. In between there is a radius for which the angles are of 90°. After the identifications, S^3 inherits a hyperbolic structure with singular set B. The cone angle is $\pi/2$. Thus $(B, 4)$ is a hyperbolic orbifold.

Then the universal orbifold covering of $(B,4)$ is H^3 and the group of automorphisms of this covering, say U, is a group of direct hyperbolic isometries acting on H^3 with compact quotient $H^3/U = S^3$, and defining the universal orbifold covering $p\colon H^3 \to (B,4)$.

Define a subgroup of direct isometries of H^3 to be a universal group if and only if given a closed, orientable 3-manifold M there is a finite index subgroup Γ of it such that H^3/Γ is homeomorphic to M.

Then

Theorem 6 ([8]) *U is a universal group.*

In fact, let M be a closed, orientable 3-manifold. Since $(B,4)$ is a universal orbifold, then M is the underlying space $|Q|$ of an orbifold Q covering $(B,4)$. Let $q\colon Q \to (B,4)$ be this orbifold covering. Since $p\colon H^3 \to (B,4)$ is the universal orbifold covering, it follows that there is an orbifold covering $r\colon H^3 \to Q$ such that $qr = p$. Therefore the regular orbifold covering r is the quotient of H^3 under the action of a subgroup Γ of U. Thus

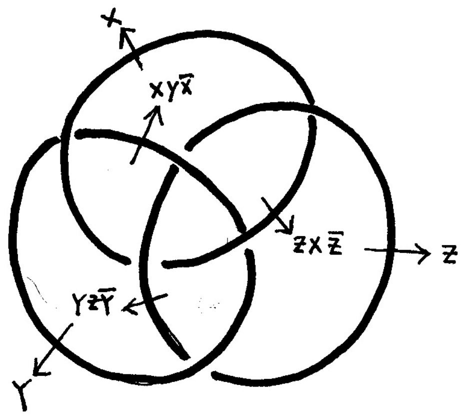
$$M = |Q| = H^3/\Gamma$$

and Γ has finite index in U because this index is the degree of q. This degree is finite because M is compact.

5 The universal group U

The universal group U is the group of automorphisms of the universal covering $p\colon H^3 \to (B,4)$. Then U is isomorphic to the fundamental group $\pi_1^o(B,4)$ of the orbifold $(B,4)$.

The group $\pi_1^o(B,4)$ comes from $\pi_1(S^3\backslash B)$ by killing the fourth powers of the meridians of B (see, for instance [20]). The group $\pi_1(S^3\backslash B)$ has a presentation with three generators $x,\ y,\ z$ (meridians of the components of B) and three relations (anyone of which is unnecessary) that declare the commutativity of each meridian with its corresponding longitud. Thus



Meridians

we have the following presentation for $U = \pi_1^o(B,4)$ is:

$$U = \left| x,y,z : \left[x,\left[z^{-1},y\right]\right] = \left[y,\left[x^{-1},z\right]\right] = \left[z,\left[y^{-1},x\right]\right] = x^4 = y^4 = z^4 = 1 \right|$$

Under the isomorphism from the group U of automorphisms of $p\colon H^3 \to (B,4)$ and the fundamental group $\pi_1^o(B,4)$ of the orbifold $(B,4)$ the meridians $x,\ y,\ z$ correspond to the 90° rotations around the three thickened edges of the dodecahedron. Thus the group U is generated by these three rotations (that we denote $x,\ y,\ z$) subject to the above relations.

The group U acts on H^3 and the regular dodecahedron D with 90° dihedral angles is the Voronoi domain of this action with respect to the center of D. Thus H^3 is tessellated by replicas of D. There are 4 replicas around every edge and 8 replicas around every vertex. The dual tessellation is formed by cubes with $2\pi/5$ dihedral angles.

6 Some consequences of U being universal

Now let M be an arbitrary closed, orientable manifold. Then there is some $\Gamma \leq U$ of finite index such that H^3/Γ is homeomorphic to M. In the course of proving that U is universal in [8] it was shown that Γ can always be supposed to contain a 90° rotation. We will assume this.

In the next diagram the horizontal lines are short exact sequences of groups and homomorphisms and the vertical arrows are natural inclusion homomorphisms.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & U & \longrightarrow & C_4 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \longrightarrow & N & \longrightarrow & \Gamma & \longrightarrow & C_4 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow\approx & & \\ 0 & \longrightarrow & S & \longrightarrow & t(\Gamma) & \longrightarrow & C_4 & \longrightarrow & 0 \end{array}$$

Here L is the kernel of the epimorphism sending $x,\ y,\ z$ to $1 \in C_4 = Z/4Z;\ N = L \cap \Gamma$ is a normal subgroup of $\Gamma;\ t(\Gamma)$ is the subgroup of Γ generated by rotations (it is a normal subgroup); and S is the subgroup $N \cap t(\Gamma)$. Since we are assuming that $t(\Gamma) \to C_4$ is onto, the vertical arrows in the third column are isomorphisms.

Since S is normal in Γ, it is normal in N. The group $N/S = N/(N \cap t(\Gamma))$ is isomorphic to $N + t(\Gamma)/t(\Gamma)$. But $N + t(\Gamma) = \Gamma$ because $t(\Gamma)$ contains a rotation γ of order 4 and

$$\Gamma = N \cup \gamma N \cup \gamma^2 N \cup \gamma^3 N.$$

Thus N/S is isomorphic to $\Gamma/t(\Gamma)$.

The group $\Gamma/t(\Gamma)$ is isomorphic to $\pi_1(M)$ [1]. In fact, let $q\colon H^3 \to M$ denote the quotient under the group Γ. Then Γ is also the group of automorphisms of the regular (ordinary) covering

$$q|_{H\backslash A}\colon H^3\backslash A \to M\backslash q(A),$$

where A is the set of axes of Γ. Then Γ is isomorphic via some λ to

$$\pi_1(M\backslash q(A))/q_{\#}\pi_1(H^3\backslash A).$$

Here $q_{\#}\pi_1(H^3\backslash A)$ is generated by powers of meridians of $q(A)$ because H^3 is simply connected. Now $\lambda(t(\Gamma))$ is generated by the meridians of $q(A)$. Therefore $\Gamma/t(\Gamma)$ is isomorphic to

$$\pi_1(M\backslash q(A))/\lambda(t(\Gamma)) = \pi_1(M).$$

Hence

Theorem 7 ([8]) *M is simply connected if and only if Γ is generated by rotations, that is, if and only if $\Gamma = t(\Gamma)$.*

Note also that N/S is isomorphic to $\pi_1(M)$.

Consider the diagram of orbifold coverings associated to the above diagram of groups and homomorphisms (in what follows $\widetilde{A}$ denotes H^3/A):

$$\begin{array}{ccc} \widetilde{L} & \longrightarrow & (B,4) \\ \uparrow & & \uparrow \\ \widetilde{N} & \longrightarrow & \widetilde{\Gamma} = M \\ \uparrow & & \uparrow \\ \widetilde{S} & \longrightarrow & \widetilde{t(\Gamma)} \end{array}$$

Here the horizontal arrows are 4-fold cyclic orbifold coverings. The orbifold $\widetilde{L}$ (and therefore $\widetilde{N}$, $\widetilde{S}$) are hyperbolic manifolds (empty singular set). Then

Theorem 8 ([9]) *Every closed, orientable 3-manifold has a 4-fold cyclic branched covering which is a hyperbolic manifold. The cyclic action is by isometries.*

The orbifold coverings $\widetilde{S} \to \widetilde{N}$ is regular and the group of automorphisms is $N/S = \pi_1(M)$. Therefore,

Theorem 9 ([9]) *The fundamental group of a closed, orientable 3-manifold acts freely as a group of isometries of a hyperbolic manifold.*

Another consequence comes from the fact that the hyperbolic orbifold $(B, 4)$ is universal. Then

Theorem 10 *Every closed, orientable 3-manifold is the underlying space of a hyperbolic orbifold with singular set a link, and isotropy cyclic of orders 2 or 4.*

Since every closed, orientable 3-manifold is a branched covering of S^3 with branching set B and branching indexes divisors of 4 and $(B, 2)$ is euclidean, we deduce

Theorem 11 *Every closed, orientable 3-manifold has a euclidean cone manifold structure with a link as singular set. The cone angles are either π or 4π.*

Note that this singular riemannian structure can be desingularized by putting positive (resp. negative) curvature near the π (resp. 4π) singular set [15, 16].

7 Relating $(B, 4)$ with $(B, 2)$

As we have proved the orbifolds $(B, 4)$ and $(B, 2)$ are respectively hyperbolic and euclidean. In fact $(B, 2)$ is euclidean in many non isometric ways (change the parallelepiped defining it). Take as standard $(B, 2)$ the one defined by the cube C of side 1. Let $p\colon H^3 \to (B, 4)$, $q\colon E^3 \to (B, 2)$ be their universal orbifold coverings. Thus $\mathrm{Aut}(p) = U$ is the universal group and $\mathrm{Aut}(q) = \widehat{U}$ is a euclidean crystallographic group that appears in the international crystallographic tables under the notation $I2_12_12_1$. The group $I2_12_12_1 = \widehat{U}$ is generated by 180° rotations around the three thickened axes upon three of the faces of the cube C. As before:

$$\widehat{U} = \big|x, y, z : \big[x, \big[z^{-1}, y\big]\big] = \big[y, \big[x^{-1}, z\big]\big] = \big[z, \big[y^{-1}, x\big]\big] = x^2 = y^2 = z^2 = 1\big|$$

The map

$$\begin{array}{cccc} \omega\colon & \widehat{U} & \longrightarrow & \Sigma_4 \\ & x & \longmapsto & (12)(34) \\ & y & \longmapsto & (13)(24) \\ & z & \longmapsto & (14)(23) \end{array}$$

is a transitive homomorphism with image isomorphic to the Klein's group

$$S_{2,2,2} = |i, j, k\,;\; i^2 = j^2 = k^2 = ijk = e| \subset \Sigma_4.$$

The kernel K of this map is the subgroup of all translations of $\widehat{U}$. Then the orbifold covering map defined by K has the torus $T^3 = S^1 \times S^1 \times S^1$ as covering space. This orbifold covering

$$p\colon T^3 \longrightarrow (B, 2)$$

is a 4 fold regular covering. The group of automorphisms of p is $S_{2,2,2}$. It is not difficul to see that there exist a presentation of T^3 as trivial T^2-bundle over S^1 such that the fibers are transversal to $L = p^{-1}(B)$ in exactly 8 points.

Now consider the universal group U. The map:

$$\begin{array}{rrcc} \mu\colon & U & \longrightarrow & \Sigma_8 \\ & x & \longmapsto & (1,2,5,6)(3,4,7,8) \\ & y & \longmapsto & (1,3,5,7)(2,8,6,4) \\ & z & \longmapsto & (1,8,5,4)(2,7,6,3) \end{array}$$

is a transitive homomorphism with image isomorphic to the quaternion group

$$\widetilde{S_{2,2,2}} = \mid i, j, k;\ i^2 = j^2 = k^2 = ijk \mid$$

The kernel K_μ of μ defines an 8-fold regular orbifold covering $q\colon M \to (B,4)$. Here M is a hyperbolic manifold. The group of automorphisms of q is isomorphic to the quaternionic group $\widetilde{S_{2,2,2}}$.

The map

$$p\colon T^3 \longrightarrow (B,2)$$

can also be understood as an orbifold covering $p'\colon (T^3, L) \to (B,4)$ from the hyperbolic orbifold (T^3, L) with singular set $L = p^{-1}(B)$ and isotropy 2. It corresponds to the kernel $K_{g\mu}$ of the composition $g \circ \mu$ where $g\colon \widetilde{S_{2,2,2}} \to S_{2,2,2}$ is defined by $g(i) = i,\ g(j) = j,\ g(k) = k$. The kernel of g is cyclic of order 2. Since $K_\mu \leq K_{g\mu}$, the covering q factors through p' defining a 2-fold cyclic orbifold covering $r\colon M \to (T^3, L)$.

The torus bundle structure of T^3 which cuts L in 8 points lifts to a F_g-bundle over S^1 structure on M, where the fiber F_g is the double covering of the fiber T^2 of T^3, branched over the 8 points of intersection with L. Then the fiber F_g is connected. The computation of the Euler characteristic of branched coverings of 2-dimensional orbifolds gives $g = 5$. Thus

Theorem 12 ([14]) *There exists a hyperbolic manifold which is a F_5-bundle over S^1, such that the quaternion group acts on it as a subgroup of isometries, giving the orbifold $(B,4)$ as quotient.*

Since, in fact, T^3 has infinitely many torus bundle structures with fiber transversal to L it follows that

Theorem 13 ([14]) *The manifold M has infinitely many different surface-bundle structures over S^1.*

8 The universal group as an arithmetic group

The universal group is an arithmetic group (see [10], [12] and [18]).

The Borromean link is the singular locus of a familly of hyperbolic structures parametrized by the cone angle. Between 0 and π the structure is hyperbolic. From π to 2π is spherical. For π is euclidean. For 2π the link degenerates into three geodesics of the round sphere of constant sectional curvature 1. Of course, these structures lift to all closed, orientable 3-manifolds providing analogous structures. Therefore, for each angle of the form $2\pi/n,\ n$ an integer, we have a hyperbolic orbifold (B,n) if $n \geq 3$. The volumes of these orbifolds can be computed.

In fact the angles in the three components of B can be different. Hilden, Lozano and I have investigated these structures and their degenerations (unpublished).

The problem of finding automorphic functions for the universal coverings of (B,n) is still open. The case (B,∞) has been solved by K. Matsumoto [19].

The paper by Toda [25] study the question of Thurston of whether or not all closed 3-manifolds admit a finite cover with positive first Betti number. It solves it partially by proving that if Γ is a subgroup of finite index in $B_{4,4,4}$ such that the torsion subgroup of Γ reduces modulo a congruence subgroup to a finite group Γ' containing no non-central normal subgroup, then the underlying space of the orbifold corresponding to Γ has a finite cover with positive first Betti number.

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References

[1] Armstrong, M. A., (1968). The fundamental group of the orbit space of a discontinuous group. *Proc. Cambridge Philos. Soc.*, **64**, 299–301.

[2] Cromwell, Peter, Beltrami, Elisabetta and Rampichini, Marta, (1998). The Borromean rings, *Math. Intelligencer*, **20**, 1, 53–62.

[3] Fox, R.-H., (1962). A quick trip through knot theory. *Topology of 3-manifolds and related topics* (Proc. The Univ. of Georgia Institute, 1961), 120–167, Prentice-Hall, Englewood Cliffs, N.J.55.20

[4] Hilden, Hugh M., Lozano, M. T. and Montesinos, José María, (1983). Universal knots. *Bull. Amer. Math. Soc.*, (N.S.), **8**, 3, 449–450.

[5] Hilden, Hugh M., Lozano, María Teresa and Montesinos, José María, Universal knots. *Knot theory and manifolds* (Vancouver, B.C., 1983), 25–59, Lecture Notes in Math., 1144, Springer, Berlin, 1985.

[6] Hilden, Hugh M., Lozano, María Teresa and Montesinos, José María, (1983). The Whitehead link, the Borromean rings and the knot 9_{46} are universal. *Collect. Math.*, **34**, 1, 19–28.

[7] Hilden, Hugh M., Lozano, María Teresa and Montesinos, José María, (1985). On knots that are universal. *Topology*, **24**, 4, 499–504.

[8] Hilden, H. M., Lozano, M. T., Montesinos, J. M. and Whitten, W. C., (1987). On universal groups and three-manifolds, *Invent. Math.*, **87**, 3, 441–456.

[9] Hilden, Hugh M., Lozano, María Teresa and Montesinos, José María, (1988). On the universal group of the Borromean rings, *Differential topology* (Siegen, 1987), 1–13, Lecture Notes in Math., 1350, Springer, Berlin, 1988.

[10] Hilden, Hugh M., Lozano, María Teresa and Montesinos-Amilibia, José María, On the Borromean orbifolds: geometry and arithmetic. Topology ’90 (Columbus, OH, 1990), 133–167, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992. (Reviewer: B. N. Apanasov)

[11] Hilden, Hugh M., Lozano, María Teresa and Montesinos-Amilibia, José María, (1993). Universal 2-bridge knot and link orbifolds. *J. Knot Theory Ramifications*, **2**, 2, 141–148.

[12] Hilden, Hugh M., Lozano, María Teresa and Montesinos-Amilibia, José María, (2001). The arithmetic structure of a universal group. Dedicated to the memory of Professor M. Pezzana (Italian). *Atti Sem. Mat. Fis. Univ. Modena*, **49**, suppl., 1–14.

[13] Hilden, Hugh M., Lozano, María Teresa and Montesinos-Amilibia, José María, (2004). On 2-universal knots. *Bol. Soc. Mat. Mexicana (3)*, **10**, Special Issue, 239–253.

[14] Hilden, Hugh M., Lozano, María Teresa and Montesinos-Amilibia, José María, (2006). On hyperbolic 3-manifolds with an infinite number of fibrations over S^1, *Math. Proc. Cambridge Philos. Soc.*, **140**, 1, 79–93.

[15] Jones, Kerry N., (1994). The structure of closed non-positively curved Euclidean cone 3-manifolds. *Pacific J. Math.*, **163**, 2, 297–313.

[16] Jones, Kerry N., (1994). Geometric structures on branched covers over universal links. *Geometric topology* (Haifa, 1992), 47–58, Contemp. Math., 164, Amer. Math. Soc., Providence, RI.

[17] Lozano, María Teresa and Montesinos-Amilibia, José María, (1997). Geodesic flows on hyperbolic orbifolds, and universal orbifolds. *Pacific J. Math.*, **177**, 1, 109–147.

[18] Maclachlan, Colin and Reid, Alan W. *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, **219**. Springer-Verlag, New York, 2003. xiv+463

[19] Matsumoto, K., (2006). Automorphic functions with respect to the fundamental group of the complement of the Borromean rings, *J. Math. Sci. Univ. Tokyo*, **13**, 1, 1–11.

[20] Matsumoto, Yukio and Montesinos-Amilibia, José María, (1991). A proof of Thurston's uniformization theorem of geometric orbifolds. *Tokyo J. Math.*, **14**, 1, 181–196.

[21] Montesinos, José María, (1983). Representing 3-manifolds by a universal branching set. *Math. Proc. Cambridge Philos. Soc.*, **94**, 1, 109–123.

[22] Ramírez, Arturo, (1975). On a theorem of Alexander, (Spanish), *An. Inst. Mat. Univ. Nac. Autónoma México*, **15**, 1, 77–81.

[23] Ratcliffe, John G., (2006). *Foundations of hyperbolic manifolds*. Second edition. Graduate Texts in Mathematics, **149**, Springer, New York, xii+779.

[24] Rolfsen, Dale, (1990). *Knots and links*, Corrected reprint of the 1976 original. Mathematics Lecture Series, 7. Publish or Perish, Inc., Houston, TX, xiv+439 pp.

[25] Toda, Masahito, (2004). Representation of finite groups and the first Betti number of branched coverings of a universal Borromean orbifold. *Cent. Eur. J. Math.*, **2**, 2, 218–249 (electronic).

[26] Thurston, William P., *Three-dimensional geometry and topology*, Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, **35**. Princeton University Press, Princeton, NJ, 1997. x+311 pp.

[27] Uchida, Yoshiaki, (1992). Universal pretzel links. *Knots 90*, (Osaka, 1990), 241–270, de Gruyter, Berlin.

[28] Uchida, Yoshiaki, (1991). Universal chains, *Kobe J. Math.*, **8**, 1, 55–65.

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