Complex of Curves - overview

Main object of study: $MCG(S_g) = \pi_0 \, Homeo^+(S_g)$
$= Homeo^+(S_g)/\text{homotopy}$ "mapping class group"

Motivation: ① $MCG(S_g) \cong Out\, \pi_1(S_g)$ Dehn-Nielsen-Baer thm
$\rightsquigarrow$ $MCG(S_g)$ is analog of $GL_n\mathbb{Z} \cong Out\, \mathbb{Z}^n$

② $MCG(S_g) \cong \pi_1^{orb}(M_g)$ M_g = moduli space of hyp. surfs

③ $MCG(S_g)$ classifies S_g-bundles

S_g-bundles over B $\longleftrightarrow$ $\pi_1 B \longrightarrow MCG(S_g)$

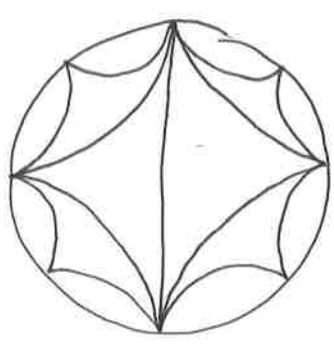
(already interesting for $B = S^1$).

Main tool: Complex of curves

$C(S_g)$ vertices: homotopy classes of essential simple closed curves in S_g
edges: disjoint representatives.

We'll see $C(S_g)$ is ① connected
② ∞-diam
③ hyperbolic
but... ④ locally infinite.

For $g=1$ we modify the definition: disjoint $\rightsquigarrow$ minimal



"Farey graph"

Hyperbolicity

A geodesic metric space is δ-hyperbolic if for any geodesic Δ, the δ-nbd of any two sides contains the third.

Facts. ① $\mathbb{E}^n$ is not δ-hyp

② $\mathbb{H}^n$ is $\ln(1+\sqrt{2})$-hyp

③ Trees are 0-hyp.

Will show $C(S_g)$ is 17-hyp (indep. of g!)

$\rightsquigarrow$ can import ideas from hyp manifolds to MCG, for instance:

Prop. M = closed hyp n-man

$g_1, g_2 \in \pi_1 M$

Then $\exists\, n_1, n_2$ s.t. $g_1^{n_1}, g_2^{n_2}$ either commute or generate F_2.

Ping Pong Lemma. X = set, $G \circlearrowright X$, $g_1, g_2 \in G$

$X_1, X_2 \neq \emptyset$, $X_1 \cap X_2 = \emptyset$

$g_1^k(X_2) \subseteq X_1 \quad g_2^k(X_1) \subseteq X_2 \quad \forall\, k \neq 0.$

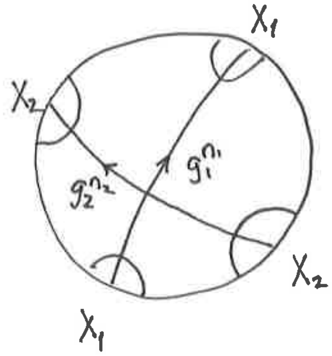
Then $\langle g_1, g_2 \rangle \cong F_2$

Pf. w = freely red word in g_1, g_2

Say $w = g_1^7 g_2^5 g_1^{-3} g_2 g_1$

Let $x \in g_2$. Note $w(x) \in X_1 \Rightarrow w(x) \neq x \Rightarrow w \neq \mathrm{id}$. $\blacksquare$

Pf of Prop. Apply PPL to:



∎

This entire approach will generalize to $MCG(S_g) \curvearrowright C(S_g)$.

Curves in Surfaces

Q. How can we tell if two vertices of $C(S_g)$ have disjoint reps?

simple closed curve

Prop (Bigon Criterion) Two transverse scc in S_g are in minimal position iff they do not form a bigon:

(minimal posn means smallest intersection number in homotopy classes).

Note: $\Rightarrow$ is easy:

Lemma. If two scc do not form a bigon then a pair of lifts to $\mathbb{H}^2$ can intersect in at most one pt.

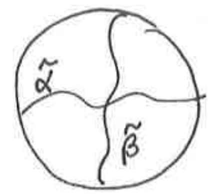
Pf. If not, an (innermost) bigon in $\mathbb{H}^2$ projects to a bigon in S_g ∎

Pf of Bigon Criterion (sketch). Assume $\alpha, \beta \subseteq S_g$ form no bigons

Lemma $\Rightarrow$ lifts can only intersect in 1 pt.

Can argue these lifts must have distinct endpts

So:



But isotopies in S_g do not move pts at ∞

So no isotopy can reduce intersection. ∎

Geodesics

Prop. Every scc in S_g $(g \geq 2)$ is homotopic to a unique geodesic

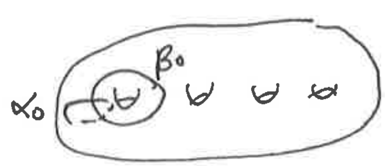
Prop. Geodesics in S_g are in minimal pos.

Change of Coordinates Principle

Configurations of curves can often be put into a standard picture via homeo of S_g.

examples ① If $\alpha \subseteq S_g$ is a nonsep scc in S_g, $\exists\ h \in \mathrm{Homeo}(S_g)$ s.t $h(\alpha) = \alpha_0$

② If $\alpha, \beta \subseteq S_g$ have $i(\alpha, \beta) = 1$ (geometric int num) then $\exists\ h \in \mathrm{Homeo}(S_g)$ s.t. $h(\alpha, \beta) = (\alpha_0, \beta_0)$



Proofs use classification of surfaces.

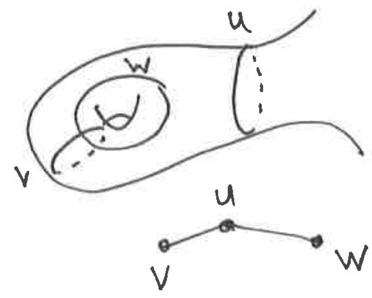
CONNECTIVITY

<u>Thm</u> $\mathcal{C}(S_g)$ is connected, $g \geq 2$.

<u>Pf.</u> Induction on $i(v,w)$.

For $i(v,w)=0$, nothing to do.

For $i(v,w)=1$, use change of coords:



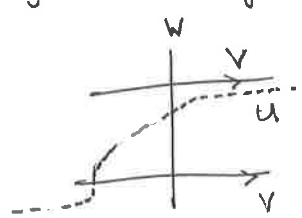
Now assume $i(v,w) \geq 2$.

Orient the curves v, w. and assume minimal pos.

Look at two consecutive intersections along w.

Orientations can agree or disagree.

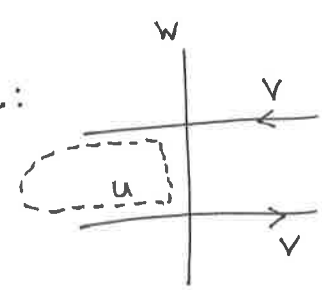
If they agree:



Note u is essential since $i(u,v)=1$.

By induction u connected to v and w.

If they don't agree:



u is essential because otherwise v, w not in min pos.

By induction u conn. to v, w.

$\boxtimes$

HYPERBOLICITY

<u>Thm</u> (Masur-Minsky). $C(S_g)$ is δ-hyp.

We'll show δ can be taken indep of g (Hensel-Przytycki-Webb and others)
Proof from Sisto's blog.

<u>Guessing geodesics lemma</u> (Masur-Schleimer) X = metric graph.
X is δ-hyp iff $\exists\, D$ and $\forall\, x,y \in X^{(0)}$ $\exists$ connected subgraph $A(x,y)$ s.t.

① $d(x,y) \le 1 \Rightarrow \operatorname{diam} A(x,y) \le D$.

② $A(x,y) \subseteq N_D\big(A(x,z) \cup A(z,y)\big)$ $\forall\, x,y,z$.

<u>Note</u>. $\Rightarrow$ easy: $A(x,y)$ is any geodesic
$D = \max(\delta, 1)$.

We will replace $C(S_g)$ with $C'(S_g)$. The latter has extra edges, namely, add edges between vertices a,b with $i(a,b) = 1$.

To check: ① $C'(S_g)$ is quasi-isometric to $C(S_g)$
(and constants do not depend on g)

② If X is δ-hyp, Y qi to X then Y is δ'-hyp
(δ' depends only on δ & qi constants).

Note: We need the guessing geodesics lemma precisely because we don't know how to find geodesics. And so it is hard to check δ-hyp'ity directly.

Thm. $C'(S_g)$ is δ-hyp.

Pf. First: $A(a,b) = \{$vertices of $C'(S_g)$ formed from one arc of a, one arc of $b\} \cup \{a,b\}$

↖ each arcs should have distinct endpts

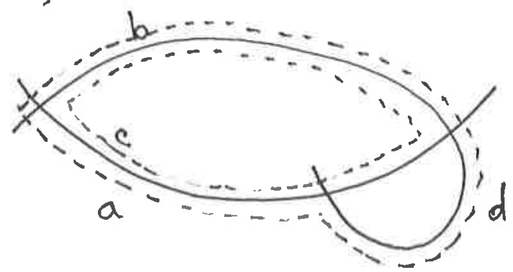
Claim. $A(a,b)$ connected

Pf. Define a partial order $c < d$ if b-arc of d contains the b-arc of c (so d is closer to being b)

Want for all $c \in A(a,b)$ a $d \in A(a,b)$ s.t. $d > c$ and $\overset{c}{\bullet}\!\!-\!\!\!-\!\!\overset{d}{\bullet}$

this isn't quite a partial order as stated since two curves can have same b-arc but opposite a-arcs

To find d, prolong one side of the b-arc of c until it hits a again, shorten the a-arc of c:



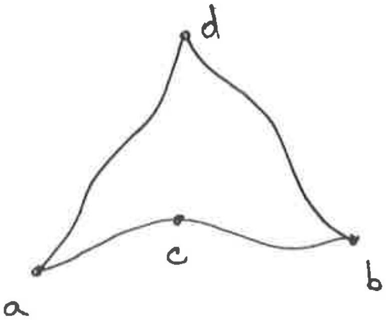
By defn, $d > c$. To see $i(c,d) \leq 1$ note the worst that can happen is the prolonged arc ends up on the wrong side of c.

Notice the $A(a,b)$ satisfy ① since $A(a,b) = \{a,b\}$ when $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}$

Claim. The $A(a,b)$ form thin triangles as in ②

Pf. Fix a,b and $c \in A(a,b)$ and d.

Need $e \in A(a,d) \cup A(d,b)$ close to c.

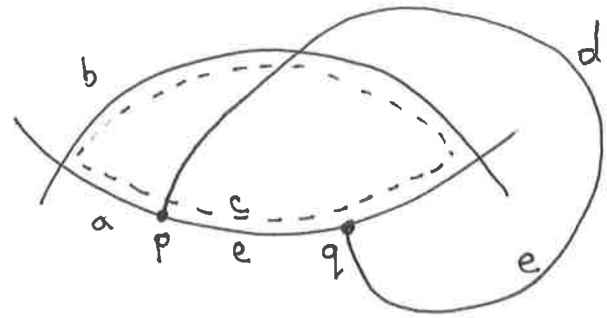


To find e: Consider 3 consec. intersections of d with c

(if fewer than 3, d is already close to c, so $e = d$).

Say 2 of these intersections are on the a-arc.

call them p, q:



Form e from the arc of d shown and the arc of $c \subseteq a$ as shown.

Note $i(c,e) \leq 2 \Rightarrow d(c,e) \leq 2$. ▨

GUESSING GEODESICS

see Bowditch "Uniform hyp" Prop 3.1 for a proof of the stronger one.

We'll prove something a little weaker than the lemma used above.

<u>Lemma</u>. (Hamenstädt) X = metric space. Suppose $\exists D$ s.t. $\forall x, y \in X$ there is a path $p(x,y)$ connecting them and so:

① $\operatorname{diam} p(x,y) \leq D$ if $d(x,y) \leq 1$

② $\forall x, y$ and $x', y' \in p(x,y)$, $d_{Haus}(p(x',y'),$ subpath of $p(x,y)$ from x' to $y') \leq D$

③ $p(x,y) \subseteq N_D(p(x,z) \cup p(z,y)) \quad \forall x, y, z$.

Then X is δ-hyp.

So to prove the theorem, need to either prove the stronger lemma (i.e. eliminate ② above) or check ② for $C'(S_g)$.

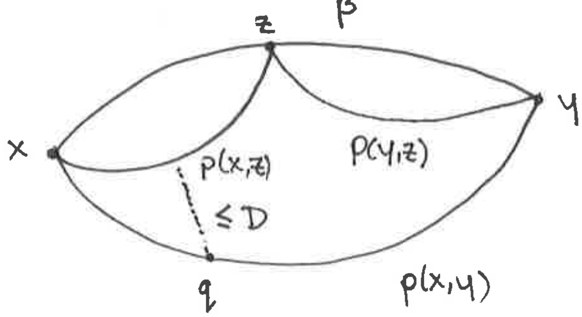
Idea: show the $p(x,y)$ are (close to) geodesics

Pf. Two steps.

Step 1. If β is any path $x \longrightarrow y$ then $p(x,y) \subseteq N_R(\beta)$
where $R \sim \log(\text{length } \beta)$.

recall: in $\mathbb{H}^n$ if a path leaves the R nbd of a geodesic its length is $\sim e^R$.

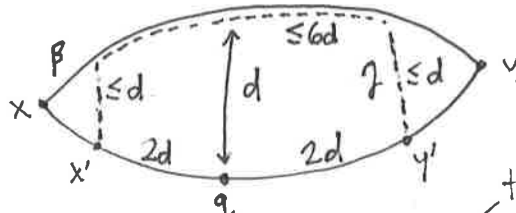
To prove this, let $q \in p(x,y)$ and split β in half, draw the p paths. Note q is close to one, using condition ③.



Induct. Base case given by condition ①.

Step 2. Improve this when β is geodesic: $p(x,y)$ is close to β.

Let q = furthest pt on $p(x,y)$ from β.
say $d(q,\beta) = d$.
Pick $x', y' \in p(x,y)$ before/after q at distance $2d$ — actual dist, not dist along $p(x,y)$
Have:



$\ell(\gamma) \leq 8d$

this fn only depends on the constants.

$\rightsquigarrow \quad d \leq d(q,\gamma) \leq O(\log d) \implies d$ bounded above.

look at pic. — by Step 1 and ② applied to x', y'.

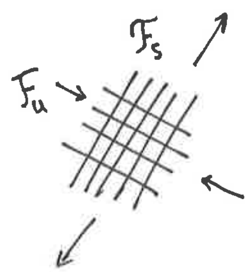
Step 3. β close to $p(x,y)$ (similar)

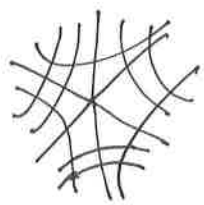
$\blacksquare$

Pseudo-Anosov Mapping Classes and Train Tracks

Nielsen-Thurston Classification. Each $f \in MCG(S)$ has a rep. φ of one of these types

1. finite order $\quad \varphi^n = 1$
2. reducible $\quad \varphi(C) = C \quad C = 1\text{-subman.}$
3. pseudo-Anosov : $\exists$ transverse meas. foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ s.t.
$\varphi \cdot (\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda \mu_u)$
$\varphi \cdot (\mathcal{F}_s, \mu_s) = (\mathcal{F}_s, 1/\lambda\, \mu_s)$





Analogous classification for $SL_2\mathbb{Z}$:

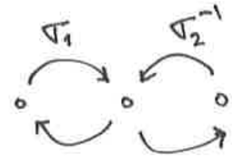
1. $|\text{trace}| = 0,1 \iff$ finite order $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
2. $|\text{trace}| = 2 \iff$ nilpotent $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
3. $|\text{trace}| \geq 3 \iff$ Anosov $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
$\rightsquigarrow$ 2 real eigenvalues, measured foliations*

For T^2, the classifications are the same.

Some questions.
1. How to construct pAs?
2. How to algorithmically determine the NT type?
3. How do pAs act on $C(S)$?

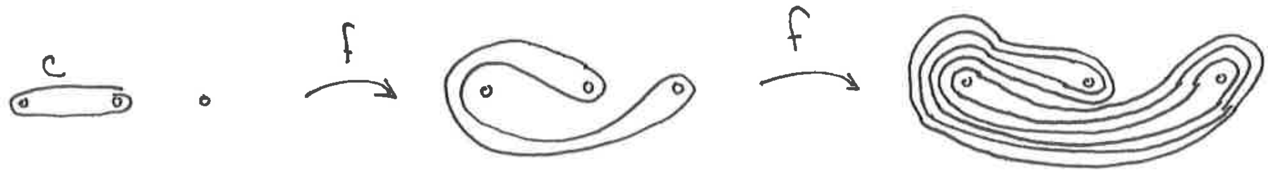
A goal: For f, h pA $\exists$ n s.t. $\langle f^n, h^n \rangle$ is either abelian or free.

Thurston's Train Tracks

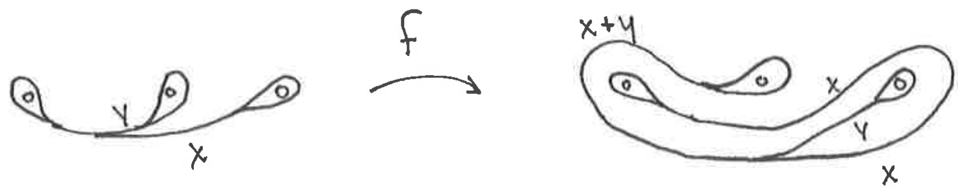
example. 

$f = \sigma_1 \sigma_2^{-1}$

Iterate f on a curve:



Replace with train track:



Transition matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \underbrace{\lambda}_{\text{PF eigenvalue}} = \frac{3+\sqrt{5}}{2}$$

Eigenvector gives foliation: replace each edge with a foliated rectangle.

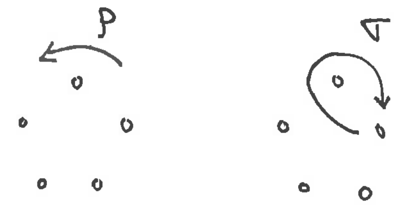
Summary:

mapping class $\overset{\text{guess}}{\rightsquigarrow}$ train track $\rightsquigarrow$ transition matrix $\rightsquigarrow$ eigenvalue/eigenvector $\rightsquigarrow$ foliations/stretch factor

Next: algorithm for finding train tracks.

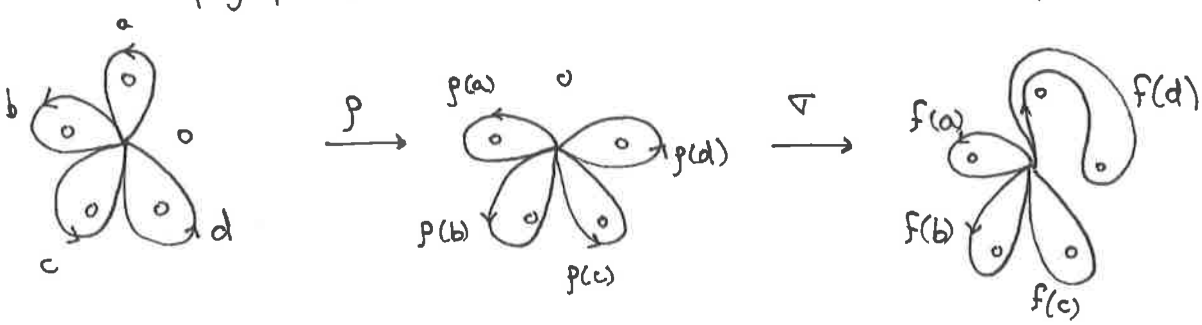
Bestvina-Handel Algorithm

Example on $S_{0,5} = S$



$f = \sigma\rho$.

Start with any graph (not smooth at vertices) that is a spine for S:



Collapse onto original graph:

$$\begin{aligned} a &\longrightarrow b \\ b &\longrightarrow c \\ c &\longrightarrow d \\ d &\longrightarrow \bar a \bar d \bar c \bar b \end{aligned}$$

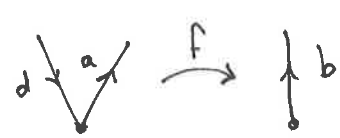
Main concern: Is there an edge that backtracks under an iterate of f?

Can see $f^2(d)$ backtracks $\quad d \xrightarrow{f} \bar a \bar d \bar c \bar b \xrightarrow{f} \underline{\underline{\bar b\,(b}}cda)\bar d \bar c$

More systematically, regard half-edges as "tangent vectors"
$\rightsquigarrow$ differential Df:

$$\begin{tikzcd} a \arrow[r] & b \arrow[r] & c \arrow[r] & d \arrow[d] \\ \bar d \arrow[ur] & \bar c \arrow[l] & \bar b \arrow[l] & \bar a \arrow[l] \end{tikzcd}$$

$\rightsquigarrow$ illegal turn da (or $\bar a \bar d$):

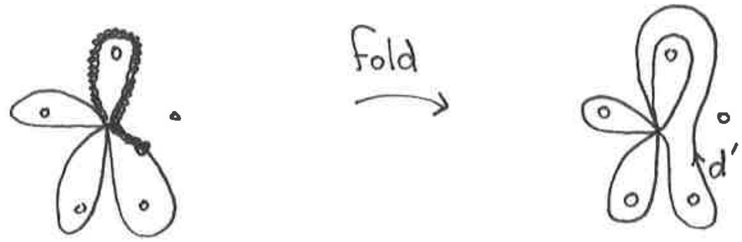


Then check if this illegal turn arises in image of f. As we said, it occurs in $f(d)$.

More generally, illegal turns are pairs of tangent vectors identified by some power of f. Suffices to look at Df.

In our example, last $1/4$ of d, all of a both map to b under f^2.

Folding. We can eliminate the problem by folding, i.e. identify the offending (partial) edges right from the start (à la Stallings).



$d = d'\bar{a}$

Get a new map of graphs using $d = d'\bar{a}$ and the fact that d' is the first $3/4$ of d:

$$\begin{aligned} a &\to b \\ b &\to c \\ c &\to d'a \\ d' &\to \bar{a}a\bar{d}'\bar{c} \xrightarrow{\text{tighten}} \bar{d}'\bar{c} \end{aligned}$$

Does the new map have any illegal turns?

$$Df: \begin{tikzcd} a \arrow[r] & b \arrow[r] & c \arrow[r] & d' \arrow[r] & \bar{d}' \arrow[ll, bend left] \\ & \bar{c} \arrow[ul] & \bar{b} \arrow[l] & \bar{a} \arrow[l] \end{tikzcd}$$

for instance

Yes: $\bar{b}\bar{d}'$ (and $d'b$).

But: this does not appear in the image of f

exercise: show this really ensures no folding under any iterate.

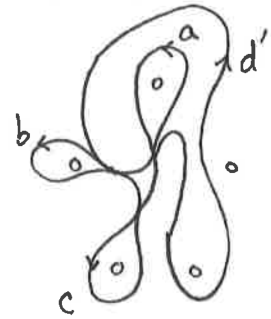
Finding the train track. Identify two tangent vectors if the are identified under some iterate of f (this is an equiv rel).

⇝ 3 equiv classes: $\{a, \bar{a}, d'\}$, $\{b, \bar{b}, \bar{d}'\}$, $\{c, \bar{c}\}$ "gates"

An illegal turn is exactly a pair from one equiv class. (in our convention reverse one of the two vectors)

But no such turn appears in f(edge).

⇝ Make a train track by squeezing together equivalence classes:



Finding the stretch factor. Transition matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Perron-Frobenius.

⇝ char poly $x^4 - x^3 - x^2 - x + 1$

⇝ PF eigenvalue ≈ 1.722

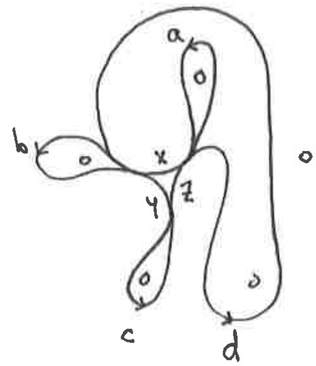
Finding the foliation. PF eigenvector $(0.316, .184, .545, .755)$

⇝ foliated rectangles instead of edges

⇝ foliation (collapse complementary region

Infinitesimal edges

In the above example we secretly added 3 "infinitesimal edges" x, y, and z:



What Bestvina-Handel tells you to do is to blow up each vertex and add these infinitesimal edges, connecting two gates whenever some f^n(edge) needs to travel between those gates.

$\rightsquigarrow$ augmented graph map:

$a \to b \qquad d' \to \bar{d'}\, z\, \bar{c}$

$b \to c \qquad x \to y \to z \to x$

$c \to d' x a$

$\rightsquigarrow$ augmented matrix:

$$\left(\begin{array}{ccc|cccc}
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)$$

5^{th} power:

$$\left(\begin{array}{ccc|cccc}
0 & 1 & 0 & 2 & 4 & 4 & 9 \\
0 & 0 & 1 & 0 & 2 & 4 & 4 \\
1 & 0 & 0 & 2 & 2 & 6 & 7 \\
\hline
0 & 0 & 0 & 1 & 2 & 2 & 4 \\
0 & 0 & 0 & 0 & 1 & 2 & 2 \\
0 & 0 & 0 & 2 & 2 & 5 & 6 \\
0 & 0 & 0 & 2 & 4 & 6 & 9
\end{array}\right)$$

So each real branch eventually traverses each branch, including infinitesimals. This happens in general.

Hyperbolic Isometries and Free Groups

Goal. f_1, f_2 pA.
If $[f_1, f_2] \neq 1$ then $\exists\ n$ s.t. $\langle f_1^n, f_2^n \rangle \cong F_2$

Idea. Use $MCG(S_g) \circlearrowright C(S_g) \leftarrow \delta$-hyp

Classification of isometries of δ-hyp spaces:

1. elliptic : $\exists$ bounded orbit
2. parabolic : $\exists!$ fixed pt in ∂X
3. hyperbolic : $\exists$ two f.p. in ∂X

$\rightsquigarrow$ invariant quasigeodesic : take one orbit and connect dots equivariantly.

Prove similarly to $\mathbb{H}^n$.

Prop. $f_1, f_2 \in \mathrm{Isom}(X)$ hyp. isoms w/ distinct fixed pts
$\exists\ n$ s.t. $\langle f_1^n, f_2^n \rangle \cong F_2$

Pf idea. A_i = quasigeodesic axis for f_i
for convenience, say $x_0 \in A_1 \cap A_2$
Take: $X_i = \{ x \in X : d(\pi_{A_i}(x), x_0) \geq M \}$ $\quad$ M large compared to δ.

(This is compatible with our pic for $\mathbb{H}^n$.)

Need to check $X_1 \cap X_2 = \emptyset$.
$f_i^n(X_j) \subseteq X_i$

Easy to see for trees. Then generalize. $\blacksquare$

Conclusion: Need to show $pA \circlearrowright C(S_g)$ is hyperbolic.

Nesting Lemma

Train track terminology. τ is recurrent if it has a positive measure

τ is large if all compl. regions are polygons or one-punctured polygons.

A diagonal extension of τ is a track obtained by adding edges with endpts in cusps of τ

$E(\tau)$ = set of diag. ext. of τ.

$P(\tau)$ = polyhedron of non-neg measures

$PE(\tau) = \bigcup_{\sigma \in E(\tau)} P(\sigma)$

$\text{int } P(\tau) \subseteq P(\tau)$ all measures strictly pos.

Nesting Lemma. τ = large, recurrent train track.

$$N_1(\text{int}(PE(\tau))) \subseteq PE(\tau)$$

N_1 = 1-nbd in $\mathcal{C}(S_g)$.

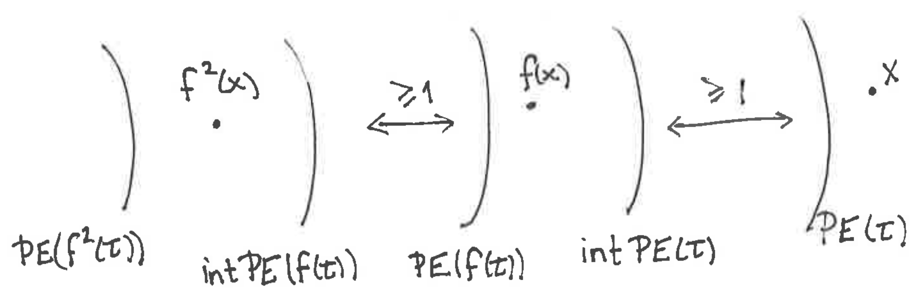
i.e. α carried by diag. ext. of τ,
α passes through each branch of τ
β disj. from α
$\Rightarrow \beta$ carried by some diag ext. of τ.

(on first pass, can pretend τ is maximal, i.e. $E(\tau) = \tau$; our example has this).

Here is how we apply this: τ = train track for f.

① $f^n(PE(\tau)) \subset \text{int } PE(\tau)$ $\quad n=5$ in above example.

② $N_1(\text{int } PE$



$\Rightarrow f$ acts hyperbolically!

Proof of Nesting Lemma

Let $\alpha \in \operatorname{int} PE(\tau)$

σ = smallest diag ext. of τ carrying α

$\rightsquigarrow$ $\alpha \in \operatorname{int} P(\sigma)$

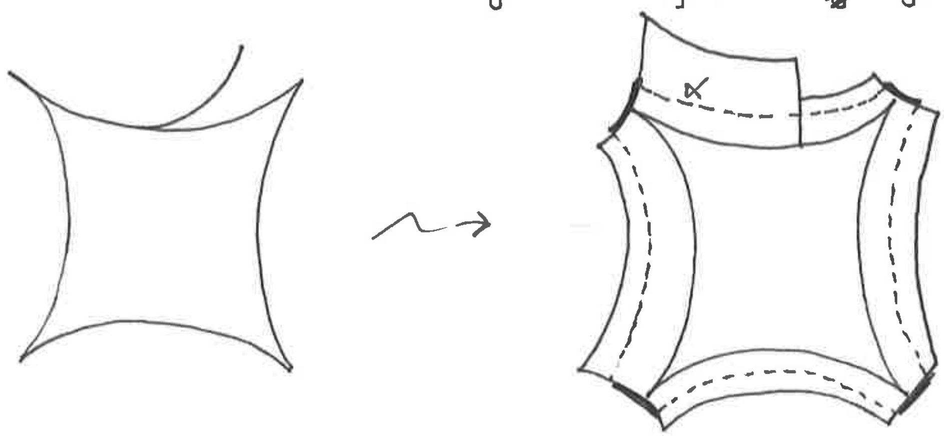
Suffices to show that if $\alpha \cap \beta = \emptyset$ then $\beta \in PE(\sigma)$.

Fatten branches of σ to rectangles; widths given by α.

Cut S_g along α and vertical sides of rectangles.

$\rightsquigarrow$ two kinds of pieces: ① rectangles inside the above rectangles

② $2k$-gons coming from k-gons in $S_g \setminus \sigma$



8-gon

If $\beta \cap \alpha = \emptyset$ β has no choice but to follow along rectangles as in ① and/or cut across the $2k$-gons. $\blacksquare$

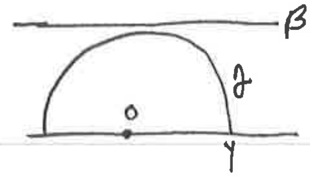
Subsurface Projections

Projections in hyp space

Fact 1. $\exists\, M$ s.t. $\forall$ horocycles β, geod γ with $\beta \cap \gamma = \emptyset$ we have $\pi_\beta(\gamma) \le M$

diam

exercise: $M = 2$ for $\mathbb{H}^2$.

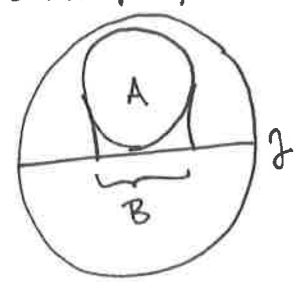


Fact 2. $\exists\, B$ s.t. $\forall$ geod γ, compact ball A with $A \cap \gamma = \emptyset$

$\operatorname{diam} \pi_\gamma(A) \le B$

"contraction property"

exercise: find B for $\mathbb{H}^2$, trees.



Masur-Minsky: If a metric space X has a coarsely transitive path family Γ with the contraction property then X is δ-hyp and elts of Γ are quasi-geodesics.

$\pi_\gamma(\alpha \cup \beta)$ large

Fact 3*. $\exists\, C$ s.t. $\forall$ geod α, β, γ disjoint, at most one of

$\pi_\alpha(\beta \cup \gamma)$, $\pi_\beta(\alpha \cup \gamma)$, $\pi_\gamma(\alpha \cup \beta)$

has diam $> C$.

exercise: prove $C = 0$ for trees (see Bestvina-Bromberg-Fujiwara)

* For this fact, need to assume a discrete family of geodesics, e.g. lifts of geodesics in a hyp. surf.

Facts 3,4 work for horocycles as well.

Fact 4. Same discreteness assumption as Fact 3, same C.

For fixed α, the set of geods β with $\operatorname{diam} \pi_\alpha(\beta) > C$ is finite.

BOUNDED GEODESIC IMAGE THM

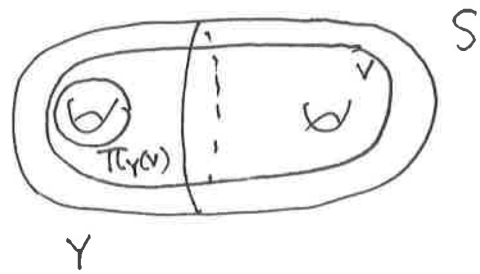
Want analogues of all of these facts. Need analogues of horocycles and projections.

Subsurface projections

S = surface
Y = subsurface
$\rightsquigarrow$ coarsely defined map

$$\pi_Y \, C(S) \longrightarrow C(Y)$$

e.g.



When Y is an annulus, need special definition.
There is a cover $S_Y \to S$ corresponding to Y
(induces $\pi_1(S_Y) \xrightarrow{\cong} \pi_1(Y)$).
Can compactify to closed annulus $\overline{S_Y}$
$C(Y)$ has vertices for proper arcs in $\overline{S_Y}$, edges for disjointness.
not discrete!
Given $v \in C(S)$ can look at preimage in S_Y hence arc in $\overline{S_Y}$.
(all such are disjoint, so lie in one simplex).
This is $\pi_Y(v)$.

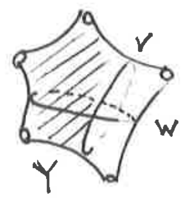
Bounded Geodesic Image Thm

this part relies on uniform hyp'ity.

Thm (Masur–Minsky) $\exists M$ (indep. of S) s.t. if $Y \subsetneq S$ and g is a geodesic in $C(S)$ all of whose vertices intersect Y then $\operatorname{diam} \pi_Y(g) \le M$.

Webb: $M = 100$.

Applications ① Consider



Let $f \in MCG(Y) \subseteq MCG(S)$ pA
Can choose n s.t.
$d_{C(Y)}(w, f^n(w)) > M$.

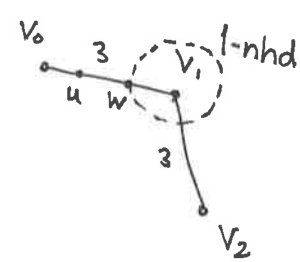
BGI $\Rightarrow$ every geodesic in $C(S)$ from w to $f^n(w)$ must pass through v.
(similar for v a nonsep curve in S_g).

② A construction of Aougab–Taylor.

Say $d(v_0, v_1) = 3$.
Let $v_2 = T_{v_1}^{M+1}(v_0)$.

Claim: $d(v_0, v_2) = 4$.

Pf: To see ≥ 4 use BGI: any geod $v_2 \longrightarrow v_0$ must pass through 1 nbd of v_1.
To see ≤ 4 find a path:
$v_0,\ u,\ w,\ T_{v_1}^{M+1}(u),\ v_2$



Can keep going: $v_3 = T_{v_2}^{M+1}(v_0)$.
Get distances $6, 10, 18, 34, \ldots$

LEASURE'S QUASIGEODESICS

Problem: compute distance in $C(S)$.

If $C(S)$ were locally finite could do a brute force search for geodesics.

Assume $d(v,w) \geq 3$. Will find a nice (2,2) quasigeodesic $v \rightsquigarrow w$.

Note $v \cup w$ cuts S_g into a union of disks.
A vw-cycle is a loop that intersects each disk in at most one arc
Take a geodesic $v = v_0, \ldots, v_n = w$
Truncate each v_i to a vw-cycle v_i' : follow v_i (starting anywhere) and when you return to the same disk twice, do a surgery.

Observation: $i(v_i', v_{i+1}') = 2$

Pf: only intersections are in disks where we did surgery and only one arc of each curve in such a disk.

$\Rightarrow d(v_i', v_j') \leq 2|i-j|$

If $d(v_i', v_j') < |i-j|$, choose a geodesic $v_i' \longrightarrow v_j'$ and convert to vw-cycles again.

At end: (2,2)-quasigeodesic. ← can get scrunching of more than 1/2 if you don't do this.

Moral: can approximate distance with uncomplicated curves.
Will do this with BGI.

Proof of Bounded Geodesic Image Theorem (Webb)

$AC(Y)$ = arc and curve complex of Y

qi to $C(Y)$.

$\pi_Y : C^\circ(S) \to P(AC^\circ(Y))$ subsurface proj.

<u>Thm</u> $\exists M$ s.t. if $Y \subseteq S$

$g = (u_i)$ = geod in $C(S)$

with $\pi_Y(u_i) \neq \emptyset \ \forall i$

then $\operatorname{diam} \pi_Y(g) \leq M$

Proof idea: simplify g wrt Y à la Leasure.

<u>vw-loops</u>

$u, v, w \in C(S)$.

Say u is a vw-loop if for each arc $\alpha \subseteq w \setminus v$ either have

① $|u \cap \alpha| \leq 1$

② $|u \cap \alpha| = 2$ and signs of intersection are opposite.

Will apply to $v = \partial Y$, $w = u_i$

To show: Given any $g = (u_i)$, v, w

can replace u_i with u_i' to get

quasigeod $g' = (u_i')$. (like Leasure).

Recipe for vw-loop conversion $u \rightsquigarrow u'$

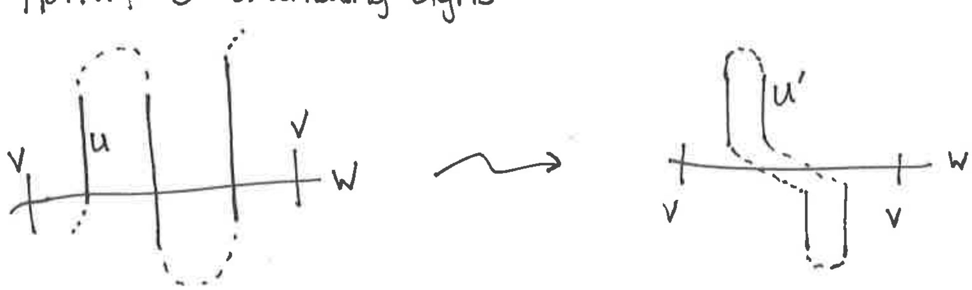
If u already a vw-loop, $u' = u$.
Otherwise, let β = a minimal arc of u failing the defn
note $\partial\beta \subseteq \alpha$ where $\alpha \subset w \setminus v$ is the arc where the failure happens.

Case ① $|\beta \cap \alpha| = 2$, signs of int are same.

Case ② $|\beta \cap \alpha| = 3$, nonalternating signs.

Similar to Case ①

Case ③ $|\beta \cap \alpha| = 3$ alternating signs



Can show u' is ① essential
② in min pos with v, w
③ a vw-loop.

Claim: If we apply this recipe to a geod $g = (u_i)$ we get a path $g' = (u'_i)$ that is a $(4,0)$-quasi-geod.

Pf: Same as Leasure. Use $i(u'_i, u'_{i+1}) \leq 4$. ▨

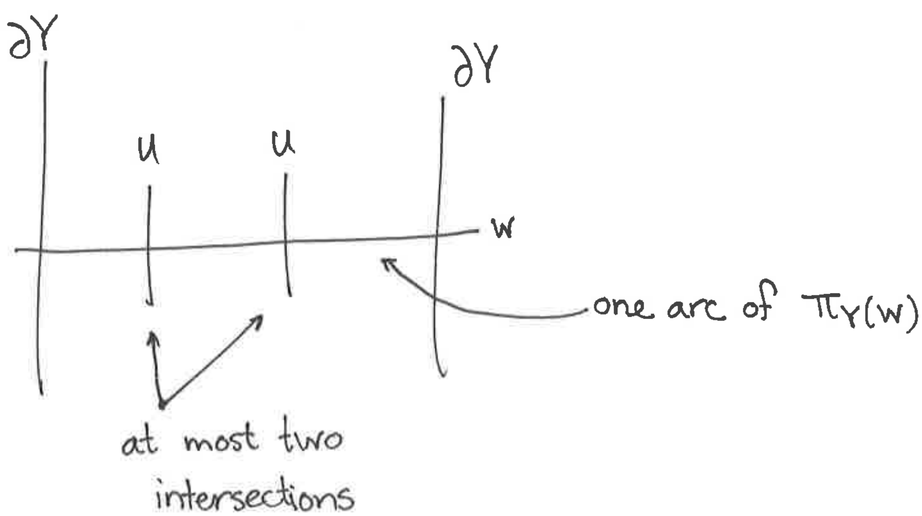
Now for the magic:

Lemma. $Y \subseteq S$. Say $v \in \partial Y$, w fill* S i.e. $d(v,w) \geq 3$.
$u = vw$-loop, $i(u,v) \neq 0$ i.e. $d(u,v) \geq 2$

Then: ① $d_Y(u,w) \leq 2$ Y nonannular

② $d_Y(u,v) \leq 5$ Y annular.

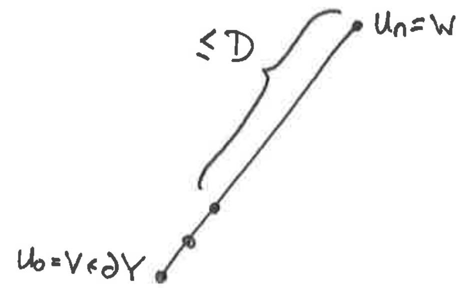
Pf of ①.



Arcs/curves with at most two intersections cannot fill i.e. cannot have distance 3. ▨

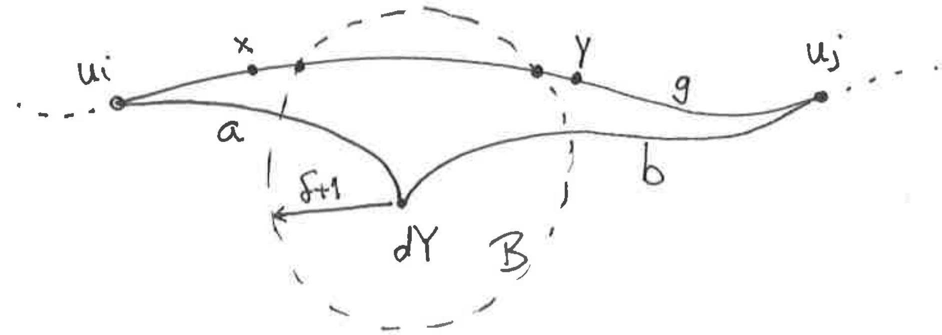
* Webb requires $d \geq 3$ in the claim and the Lemma.

Lemma. $\exists D$ s.t. $\forall\, Y \subseteq S$ $\forall\, v \in \partial Y$
$\forall$ geod $v = u_0, \ldots, u_n = w$ $n \geq 3$
have: $d_Y(u_i, u_n) \leq D$ $i \geq 2$.



Pf. Replace $g = (u_i)$ with $g' = (u_i')$ a $(4,0)$-quasigeod.
Each u_i is D'-close to g' $D' = $ fn of $4, \delta$.
So: u_i' close to u_n' in Y by prev. lemma
u_i close to some u_j' (quasigeods are unif close to geods) ▨

Proof of Thm. Let $B = (\delta+1)$-ball around ∂Y:



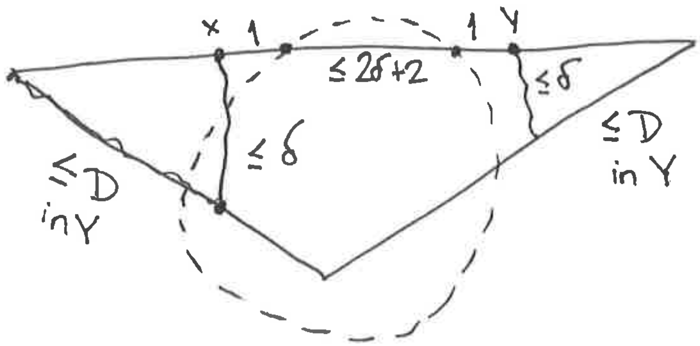
i, j arbitrary

a, b = other two sides of $u_i, u_j, \partial Y$ triangle
x/y = vertices right before/after g passes thru B. (otherwise $x = u_i$, $y = u_j$)

Key: x, y have distance $\delta+2$ from ∂Y so any path of length δ has all vertices intersecting ∂Y.

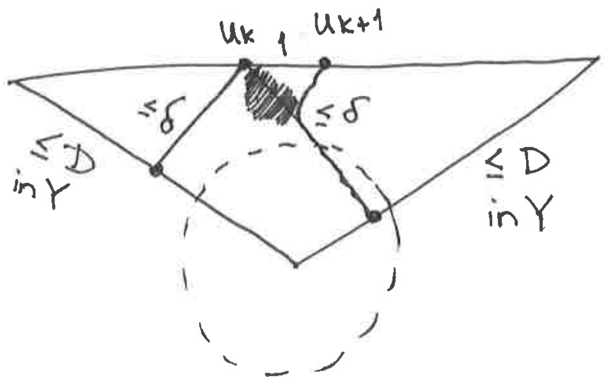
Now, the points of $(u_i, \dots, u_j)$ are within δ of $a \cup b$.
At some point they switch from close-to-a to close-to-b
That can happen in B or out of B.

<u>Case</u> ① x within δ of a
y within δ of b.



Get a path of length $\leq 2D + 4\delta + 4$ in Y.

<u>Case</u> ② $\exists\, u_k, u_{k+1}$ outside B with u_k δ-close to a
u_{k+1} δ-close to b



Get path of length $\leq 2D + 2\delta + 1$.

▨

BEHRSTOCK LEMMA

$\xi(S)$ = complexity = $3g-3+n$ = dim $C(S)+1$.

Lemma. $Y, Z \subseteq S$ overlapping
$\xi(Y), \xi(Z) \geq 4$.
x = curve with $\pi_Y(x), \pi_Z(x) \neq \emptyset$.
Then $d_Y(x, \partial Z) \geq 10 \Rightarrow d_Z(x, \partial Y) \leq 4$

i.e. can't both be large.
This is analogous to Fact 3 above. (think of x as ∂X).

Facts. Let $U \subseteq S$ $\xi(U), \xi(S) \geq 4$.
$u, v \in C(S)$
a_u, a_v projection arcs in U
$\pi_U(u), \pi_U(v)$ projection curves.

1. $i(a_u, a_v) = 0 \Rightarrow d_U(u, v) \leq 4$
2. $i(u,v) > 0 \Rightarrow i(u,v) \geq 2^{(d_U(u,v)-2)/2}$
3. $i(u,v) \leq 2 + 4 \cdot i(a_u, a_v)$.

Pf of Lemma (Leininger). $d_Y(x, \partial Z) \geq 10 > 2 \Rightarrow$ distance realized by curves $u \in \pi_Y(x)$, $v \in \pi_Y(\partial Z)$ s.t. $i(u,v) \geq 2^4 = 16$ (Fact 2). Now, u & v come from arcs a_u, a_v with $i(a_u, a_v) \geq (16-2)/4 > 3$ (Fact 3). Note $a_u \subseteq x$, $a_v \subseteq \partial Z$. One arc of a_u b/w pts of intersection with a_v lies in Z. This arc is disjoint from x-arcs in Z, so $d_Z(x, \partial Y) \leq 4$ (Fact 1). ▨

More Free Groups in MCG

We showed: $f_1, f_2 \in MCG$ pA $\rightsquigarrow \exists\, n$ s.t. $\langle f_1^n, f_2^n \rangle$ is abelian or free.
That proof generalizes to $f_1, \ldots, f_k$ pA.
Want to generalize in two more ways: ① f_i are partial pA
② $k = \infty$.

First...

<u>More free groups in Isom($\mathbb{H}^2$)</u>

Say $a, b \in \text{Isom}(\mathbb{H}^2)$ parabolic.
WTS $\exists\, n$ s.t. $\langle a^n, b^n \rangle \cong F_2$.

Key is "BGI": If A, B, C are horoballs with $d(\pi_C(A), \pi_C(B)) > M$ then the geodesic from A to B passes thru C.

Choose horoballs A, B preserved by a, b and distance 1 apart.
Replace a, b with powers s.t. $d_A(B, aB) \geq 2M$
$d_B(A, bA) \geq 2M$
Create an "electrified space" by coning off each horoball in the $\langle a,b \rangle$-orbit of A, B.

Let $\omega = a_b^{p_1} b^{p_2} \cdots a/b^{p_L} \in \langle a, b \rangle$
$= s_1 \cdot \cdots \cdot s_L$

To show: $d(\omega(B), B) \geq L$ in electrified space

$\Rightarrow \omega \neq id \Rightarrow \langle a, b \rangle \cong F_2$.

Let $B_i = s_1 \cdots s_i (B)$ $\quad i$ odd

$\qquad = s_1 \cdots s_i (A)$ $\quad i$ even

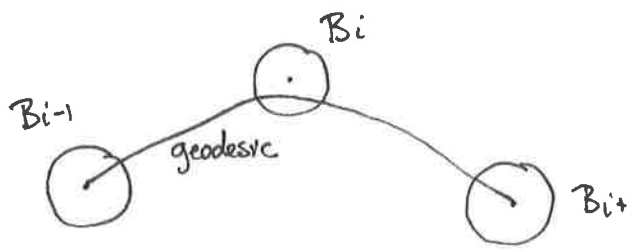
and $B_{-1} = B$.

Claim. $d_{B_i}(B_{i-1}, B_{i+1}) \geq 2M$ $\qquad$ (dist of proj's)

Pf. Say i odd.

$$\begin{aligned} d_{B_i}(B_{i-1}, B_{i+1}) &= d_{s_1 \cdots s_i (B)}\left(s_1 \cdots s_{i-1}(A),\ s_1 \cdots s_{i+1}(A)\right) \\ &= d_B\left(s_i^{-1}(A),\ s_{i+1}(A)\right) \\ &= d_B(A, s_{i+1}(A)) = d_B(A, b^k A) \\ &\geq 2M \end{aligned}$$

By BGI have this picture:



Want to string these together: if the geodesic from B_0 to B_L passes through all B_i, the distance is at least L.

Assume by induction that any geodesic from B_0 to B_{k-1} passes through $B_0, \ldots, B_{k-1}$.

Claim. $\exists$ geodesic from B_0 to B_{k-2} avoiding B_{k-1}

Pf. Say γ from B_0 to B_{k-2} passes in B_{k-1}.

By induction the intial segment from B_0 to B_{k-1}

passes thru $B_{k-2} \rightsquigarrow \gamma$ can be shortened.

(use the coning off!)

By Claim and BG1, $d_{B_{k-1}}(B_0, B_{k-2}) \leq M$

Now: $$\begin{aligned} d_{B_{k-1}}(B_0, B_k) &\geq d_{B_{k-1}}(B_{k-2}, B_k) - d_{B_{k-1}}(B_0, B_{k-2}) \\ &\geq 2M - M \\ &= M \end{aligned}$$

By BG1 any geod from B_0 to B_k passes thru B_{k-1}

And by induction such a geod passes thru $B_0, \ldots, B_k$

To conclude $d(B_0, B_L) \geq L$ remains to show the B_i are pairwise disjoint. Suppose $z \in B_i \cap B_{i+k}$. By the above, the constant geodisic z passes thru $B_i, \ldots, B_{i+k} \Rightarrow z \in B_i \cap B_{i+1}$, a contradiction. $\blacksquare$

To Do: ① Redo the argument without coning. Instead use Behrstock inequality. (see email from Mangahas 11/12/14)

② Show all elements of $\langle a,b \rangle$ not conj to power of generator are hyperbolic isometries. Key: parabolics/elliptics move pts sublinearly.

FREE GROUPS FROM PARTIAL PSEUDO-ANOSOVS (MANGAHAS)

Simple case. $A, B \subseteq S$

$\alpha = \partial A$, $\beta = \partial B$ ← $\partial A, \partial B$ conn.

$d_{C(S)}(\alpha, \beta) \geq 3$.

a, b partial pAs supp. on A, B.

Basically the same argument. Need to say what horoballs are:

$$C_A = \{ v \in C(S) : \pi_A(v) = \emptyset \} \subseteq N_1(\alpha)$$

similar $\quad C_B \subseteq N_1(\beta)$

Note: $d(\alpha, \beta) \geq 3 \Rightarrow C_A \cap C_B = \emptyset$.

Replace a, b with high powers s.t.

$d_A(C_B, a(C_B)) \geq 2M+4$ ← d_A means diam of union of two proj's.

$d_B(C_A, b(C_A)) \geq 2M+4$

First one implies: $d_A(v, a^k(v')) \geq 2M \quad \forall \; v, v' \in C_B$.

since $\operatorname{diam} C_B = 2$.

etc. Just run through the same argument.

Since pA's are only elts with unbounded orbits, immediately get that all elements of $\langle a, b \rangle$ not conj to a power of a or b is pA.

Behrstock Lemma

$\xi(S)$ = complexity = $3g-3+n$ = dim $C(S)+1$.

<u>Lemma.</u> $Y, Z \subseteq S$ overlapping
$\xi(Y), \xi(Z) \geq 4$.
x = curve with $\pi_Y(x), \pi_Z(x) \neq \emptyset$.
Then $d_Y(x, \partial Z) \geq 10 \Rightarrow d_Z(x, \partial Y) \leq 4$

i.e. can't both be large.
This is analogous to Fact 3 above. (think of x as ∂X).

<u>Facts.</u> Let $U \subseteq S$ $\xi(U), \xi(S) \geq 4$.
$u, v \in C(S)$
a_u, a_v projection arcs in U
$\pi_U(u)$, $\pi_U(v)$ projection curves.

① $i(a_u, a_v) = 0 \Rightarrow d_U(u, v) \leq 4$
② $i(u,v) > 0 \Rightarrow i(u,v) \geq 2^{(d_U(u,v)-2)/2}$
③ $i(u,v) \leq 2 + 4 \cdot i(a_u, a_v)$.

Pf of <u>Lemma</u> (Leininger). $d_Y(x, \partial Z) \geq 10 > 2 \Rightarrow$ distance realized by curves $u \in \pi_Y(x)$, $v \in \pi_Y(\partial Z)$ s.t. $i(u,v) \geq 2^4 = 16$ (Fact ②). Now, u & v come from arcs a_u, a_v with $i(a_u, a_v) \geq (16-2)/4 > 3$ (Fact ③). Note $a_u \subseteq x$, $a_v \subseteq \partial Z$. One arc of a_u b/w pts of intersection with a_v lies in Z. This arc is disjoint from x-arcs in Z, so $d_Z(x, \partial Y) \leq 4$ (Fact 1). ∎

Free Groups via Ping Pong (Manganas à la Ishida & Hamidi-Tehrani)

a, b pA with supports A, B

$\xi(A), \xi(B) \geq 4$

$A \cap B \neq \emptyset$.

Choose n s.t. translation distance of a on $\mathcal{C}_A(S)$ is ≥ 14
and same for b.

<u>Prop.</u> $\langle a^n, b^n \rangle \cong F_2$

<u>Pf.</u> Ping pong

needed?

$$X_a = \{ v : \pi_A(v), \pi_B(v) \neq 0 \ , \ d_A(v, \partial B) \geq 10 \}$$

X_b similar. Note $X_a \cap X_b = \emptyset$ by Behrstock.

Take $v \in X_a$.

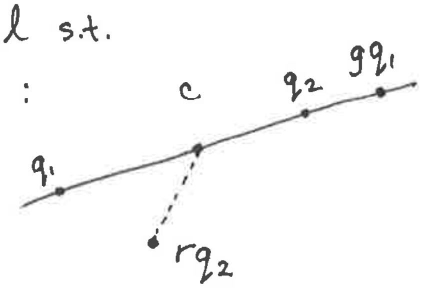
$$\begin{aligned} \text{Behrstock} &\Rightarrow d_B(v, \partial A) \leq 4 \\ &\Rightarrow d_B(b^n(v), \partial A) \geq 10 \\ &\Rightarrow b^n(v) \in X_b \end{aligned}$$

$\blacksquare$

Broad outline of proof. First we cone off the $Q_i \subseteq X$
and show result is δ-hyp
(use: fellow traveller condition)
The R_i now rotate about cone points
moving family $\rightsquigarrow$ rotating family
large inj rad $\rightsquigarrow$ very rotating : if we take a pt x sufficiently far from a cone pt c, then rotate about c by g then the geodesic from x to gx passes thru c (like BGI).
In this sense, the proof is reminiscent of last lecture.

Windmills. A windmill is a subset $W \subseteq X$ with
① W almost convex
② $N_{40\delta}(W) \cap C = W \cap C \neq \emptyset$ $\qquad$ C = set of cone pts
③ $G_W = \langle G_c : c \in W \cap C \rangle$ preserves W $\qquad$ G_c = rotating elt
④ $\exists\ S_W \subseteq W \cap C$ s.t. $G_W \cong \ast_{c \in S_W} G_c$
⑤ (Greendlinger condition) Every elliptic in G_W lies in some G_c, $c \in S_W$. Other elts have invar. geod. axis ℓ s.t. $\ell \cap C$ contains at least 2 g-orbits of pts at which there is a shortening elt

Shortening elt $\quad$ ℓ = axis for g, contains $c \in C$
shortening elt is $r \in G_c \setminus id$ s.t. $\exists\ q_1, q_2 \in \ell$ s.t.
$d(q_1, q_2) \in [24\delta, 50\delta]$ but $d(q_1, rq_2) \leq 20\delta$:
Triangle $\leq$ $\Rightarrow$ rg has shorter transl. length than g.



Infinitely Generated Free Groups

Thm (Dahmani-Guirardel-Osin) $f \in MCG(S)$ pA.
$\exists\, n$ s.t. $\langle\langle f^n \rangle\rangle \cong F_\infty$
and all nontrivial elements pA.

Inspired by:

Thm (Gromov) $\exists\, m = m(k,\delta)$ s.t. if $\gamma_1, \ldots, \gamma_k$ are hyp. elements of a δ-hyp gp the normal closure of the $\gamma_i^{m_i}$ is free when $m_i \geq m\ \forall i$.

Aside: Whittlesey's groups

$F_i : MCG(S_{0,n}) \to MCG(S_{0,n-1})$ forget i^{th} marked pt
$\mathrm{Brun}(S_{0,n}) = \bigcap \ker F_i$ "Brunnian"

Thm. For $n \geq 5$ $\mathrm{Brun}(S_{0,n})$ is all pA (it is obviously normal).

Pf. By NT Classification, suffices to rule out periodic, reducible.

A Brunnian braid

Easy to rule out periodic, either by Birman exact seq or classification of torsion in $MCG(S_{0,n})$.

Say an elt f of $\mathrm{Brun}(S_{0,n})$ has a reducing curve c.
On one side of c, f is doing something nontrivial.
Forget a marked pt on the other side $\rightsquigarrow F_i(f) \neq \mathrm{id}$. ▨

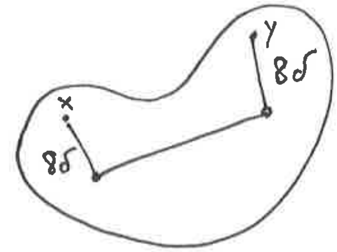
Small Cancellation Theory.

$X = \delta$-hyp space

$G \circlearrowright X$ by isoms.

$(Q_i)_{i\in I}$ almost-convex subspaces : $\forall\, x, y$

(think: axes)



$(R_i)_{i\in I}$ $\quad R_i \triangleleft \mathrm{Stab}_G Q_i$

(think: hyp. elts)

$G \circlearrowright I$ with $Q_{gi} = gQ_i$

$R_{gi} = gR_ig^{-1}$

$\mathcal{F} = \{(Q_i), (R_i)\}$ "moving family"

<u>Injectivity radius</u>: $\mathrm{inj}(\mathcal{F}) = \inf\{d(x,gx) : i \in I,\ x \in Q_i,\ g \in R_i \setminus \mathrm{id}\}$

<u>Fellow traveling const</u>: $\Delta(Q_i, Q_j) = \mathrm{diam}\ N_{20\delta}(Q_i) \cap N_{20\delta}(Q_j)$

note: $Q_i \setminus$ this intersection is far from Q_j by δ-hyp.

$$\Delta(\mathcal{F}) = \sup_{i \neq j} \Delta(Q_i, Q_j)$$

$\mathcal{F}$ satisfies (A,ε)-small cancellation if

① $\mathrm{inj}(\mathcal{F}) \geq A\delta$

② $\Delta(\mathcal{F}) \leq \varepsilon\, \mathrm{inj}(\mathcal{F})$

<u>THM</u> (DGO) $\exists\, A_0, \varepsilon_0$ s.t. if $\mathcal{F}$ satisfies (A,ε)-small cancell. with $A \geq A_0$, $\varepsilon \geq \varepsilon_0$ then

$\langle\langle R_i \rangle\rangle$ is a free product on some of the R_i.

<u>THM</u>: MCG satisfies small canc. with $R_i = f_i^N$, f_i pA $\quad Q_i =$ axes.

Tight Geodesics

Problems with $C(S)$: ① not locally finite $\rightsquigarrow$ hard to do algorithms

② MCG action not prop disc $\rightsquigarrow$ hard to glean info about MCG.

Will remedy this somewhat.

Tight geodesics

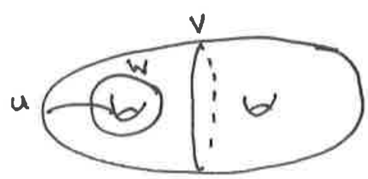
A tight geodesic from v to w is a seq. of simplices

$$v = \sigma_0, \ldots, \sigma_n = w$$

s.t. ① $\sigma_i = \partial F(\sigma_{i-1}, \sigma_{i+1})$ $\quad$ F = span of $\sigma_{i-1}, \sigma_{i+1}$ = smallest subsurface containing both

② $d(v_i, v_j) = |i-j|$ $\forall\, v_i \in \sigma_i, v_j \in \sigma_j$ $i \neq j$.

example.



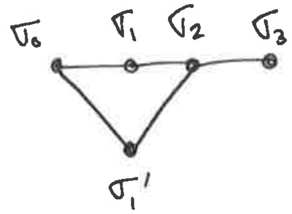
v is the canonical choice to get from u to w.

Tightening

Given a geodesic $v_0, \ldots, v_n$ can tighten at v_i: replace v_i by $\partial F(v_{i-1}, v_{i+1})$

Prop. If we tighten at v_i then tighten at v_{i-1}, result is still tight at v_i. In particular, tight geodesics exist.

Pf. Say $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ already tight at σ_2 and we tighten at σ_1:



New path is still geodesic (it has same length as a geodesic).

$\Longrightarrow$ all components of σ_1' & σ_3 intersect

$\Longrightarrow$ $F(\sigma_1', \sigma_3)$ connected.

$i(\sigma_1', \sigma_2) = 0 \Longrightarrow \sigma_1' \subseteq F(\sigma_1, \sigma_3)$ since $\sigma_2 = \partial F(\sigma_1, \sigma_3)$

$\Longrightarrow F(\sigma_1', \sigma_3) \subseteq F(\sigma_1, \sigma_3)$ (use connectedness).

Need: σ_1', σ_3 fill $F(\sigma_1, \sigma_3)$.

So let $\alpha \subseteq F(\sigma_1, \sigma_3)$ and say $i(\alpha, \sigma_3) = 0$.

$\rightsquigarrow$ need $i(\alpha, \sigma_1') \neq 0$.

$i(\alpha, \sigma_3) = 0 \longrightarrow$ $i(\alpha, \sigma_1) \neq 0$, $i(\alpha, \sigma_0) \neq 0$ since these pairs fill $F(\sigma_1, \sigma_3)$ and S resp.

But $\sigma_1 \notin F(\sigma_0, \sigma_2)$

$\rightsquigarrow$ α must cross $\underbrace{\partial F(\sigma_0, \sigma_2)}_{= \sigma_1'}$ to get from σ_1 to σ_0.

∎

Prop. There are finitely many tight geodesics between two vertices v, w.

Pf. Say $d(v,w) = n$.

Suffices to show $\exists$ finitely many choices for σ_1 on a tight

$$v = \sigma_0, \sigma_1, \ldots, \sigma_n = w$$

Cut S along v.

$\sigma_n = w \rightsquigarrow$ filling simplex of arc complex τ_n

σ_{n-1} also gives filling simplex τ_{n-1}

Note: $i(\tau_n, \tau_{n-1}) = 0$.

Fact: Given a filling simplex τ in arc complex $\exists$ only finitely many simplices τ' with $i(\tau, \tau') = 0$.

By induction, finitely many choices for τ_2.

By tightness, one choice of σ_1 for each choice of τ_2. ▨

In the above argument, we can algorithmically list all the τ_i & σ_i's.

Cor. $\exists$ algorithm to compute distance in $C(S)$.

Pf. Assume have algorithm to distinguish distances $1, \ldots, n-1$ and $> n-1$.

Want an alg to dist. distances $1, \ldots, n$ and $> n$.

Let $v, w \in C(S)$. By induction we can tell if $d(v,w)$ is $1, \ldots, n-1$ or $> n-1$.

If it is $1, \ldots, n-1$ we are done so assume $d(v,w) \geq n$.

Need to tell if $d(v,w)$ is n or $> n$.

List all candidate σ_1's on a tight path of length n as above.

If any such σ_1 has $d(\sigma_1, w) = n-1$ (using induction), $d(v,w) = n$.

Otherwise $d(v,w) > n$. ▨

Applications of tight geodesics

Thm. Any pA in $MCG(S)$ has a power with an honest geodesic axis. ← not. nec. tight!

Pf Sketch. Say f is pA with limit pts $a, b \in \partial C(S)$.

$\mathcal{L}_T$ = set of all tight geodesics from a to b.

$\mathcal{L}$ = set of all geodesics from a to b. ← locally finite!

G = subgraph of $C(S)$ given by union of elts of $\mathcal{L}_T$.

$\mathcal{L}_G$ = set of geodesics contained in G. Note $\mathcal{L}_T \subsetneq \mathcal{L}_G$!

$G/\langle f \rangle$ is finite

$\rightsquigarrow$ label the directed edges $1, \ldots, p$.

Say $g \in \mathcal{L}_G$ is lexicographically least if $\forall\ x, y \in g$ the sequence of labels along g is lex. least among all geodesics from x to y in G.

$\mathcal{L}_L$ = set of lex. least geods $\subseteq \mathcal{L}_G$.

$\rightsquigarrow$ this is f-invariant.

Claim 1. $\mathcal{L}_L \neq \emptyset$.

Pf. Take longer and longer lex. least geods

local finiteness $\Longrightarrow$ some sub seq. converges.

Claim 2. $|\mathcal{L}_L| < \infty$.

Now take any $g \in \mathcal{L}_L$. The finitely many elts are permuted by f so some power of f fixes a geodesic. ▨

Cor. Stable translation length for a pA on $C(S)$ is rational.

$$\tau(f) = \liminf_{n \to \infty} \frac{d(f^n(x), x)}{n}$$

Ingredients for Acylindricity

<u>Thm 1.</u> $d(a,b) \geq 3 \implies |\mathrm{Stab}_{MCG}(a) \cap \mathrm{Stab}_{MCG}(b)| \leq N_0 = N_0(S)$

<u>Pf idea.</u> $a \cup b \rightsquigarrow$ cell decomp of S

topological lemma: any $f \in \mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ has a rep that preserves the cell decomp.

The resulting auto. of the cell decomp is determined by where it sends one 2-cell.

But the number of nonrectangular 2-cells is at least one and is bounded by a fn of S. ☒

$G(a,b;r)$ = curves that lie on some tight geod. from a' to b' where $d(a,a') \leq r$, $d(b,b') \leq r$.

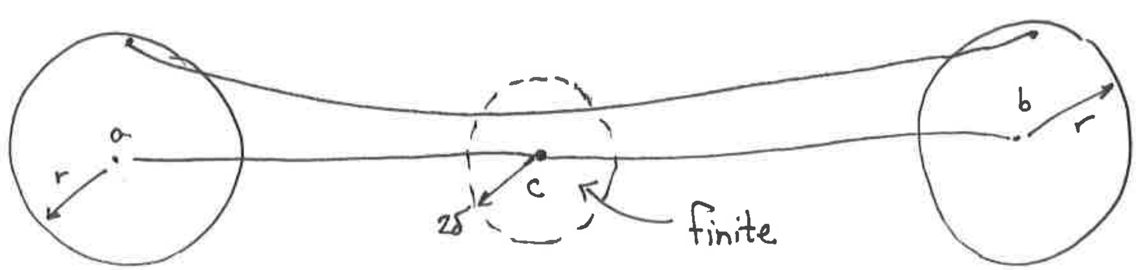
<u>Thm 2.</u> Fix $r \geq 0$.

$a, b \in \mathcal{C}(S)$ with $d(a,b) \geq 2r + 2(10\delta + 1) + 1$

$c \in \gamma$ = geod. from a to b.

$c \notin N_{r+10\delta+1}(a) \cup N_{r+10\delta+1}(b)$

$$\rightsquigarrow \quad |G(a,b;r) \cap N_{2\delta}(c)| \leq D = D(S)$$



Proof of Acylindricity

$R = 4r + 24\delta + 7$

$N = N_0\,(2r + 4\delta + 1)(8\delta + 7)D$

Say $d(a,b) \geq R$

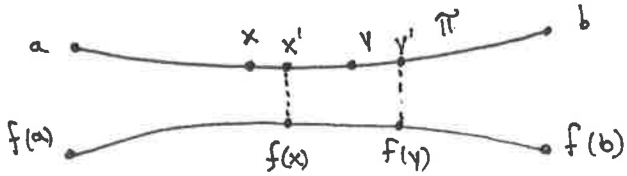
Pick $x, y \in \pi$ = tight geod from a to b.

s.t. ① $d(x,y) = 3$

② $d(\{x,y\}, \{a,b\}) \geq r + (10\delta + 1) + (2\delta + r) + 1$

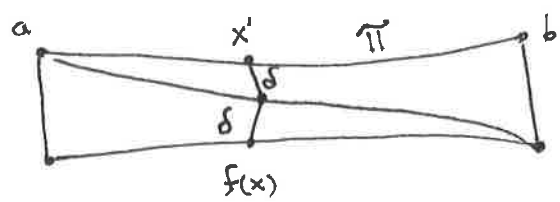
Say $f \in MCG(S)$ with $d(a, f(a)) \leq r$, $d(b, f(b)) \leq r$

Let x', y' proj's of $f(x), f(y)$ to π'.



Claim 1. $d(f(x), \pi') \leq 2\delta$, $d(f(y), \pi') \leq 2\delta$

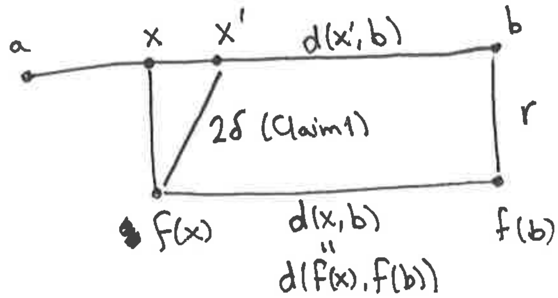
Pf.



Use δ-thinness plus fact that $f(x)$ is far from the vertical sides.

Claim 2. $d(x,x') \le r+2\delta \qquad d(y,y') \le r+2\delta$

Pf. Assume x' to right of x:

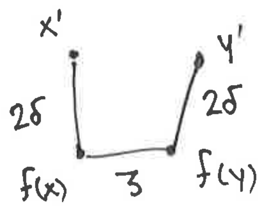


$$\begin{aligned} d(x,x') &= d(x,b) - d(x',b) \\ &\le (2\delta + d(x',b) + r) - d(x',b) \\ &= 2\delta + r \end{aligned}$$

If x' to left of x, replace b with a.

Claim 3. $d(x',y') \le 4\delta+3$

Pf.



Claim 4. $d(x',a)\,,\ d(y',b) \ge r + 10\delta + 2$

Pf. Immediate from Claim 2 & choice of x,y.

Claim 5. At most $2r+4\delta+1$ choices for x'.

Pf. Immediate from Claim 2.

Claim 6. Given x', at most:
- $(2r+4\delta+1)D$ choices for $f(x)$. Claim 4 + Thm 2
- $8\delta+7$ choices for y' (Claim 3)
- $(8\delta+7)D$ choices for $f(y)$ Claim 4 + Thm 2

Acylindricity now follows from Thm 1, with N as above. ▨

Bottlenecks

Remains to prove Thm 2. Here is a simpler version.

<u>Thm.</u> $a, b \in C(S)$

$c \in \pi$ = geod. from a to b.

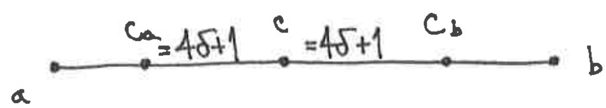
$\rightsquigarrow \quad |G(a,b) \cap N_\delta(c)| \leq D.$

$G(a,b) = G(a,b;0) =$ set of curves lying on some tight geod from a to b.

Pf. For simplicity, assume c is far from a, b:

$$d(c, \{a,b\}) \geq 4\delta + 1.$$

Choose c_a, c_b:



Enough to show that each elt of $G(a,b) \cap N_\delta(c)$ also lies on a tight filling multipath* from c_a to c_b of length at most $12\delta + 2$.

Indeed, when we gave the algorithm for distance we showed there is a constant $B = B(S, L)$ s.t. the number of curves that can lie on a tight filling multipath of length $\leq L$ is bdd above by B.

* A tight path (v_i) where $|i-j| \geq 3 \Rightarrow v_i, v_j$ fill.

THE DISTANCE FORMULA

σ = finite set of vertices of $\mathcal{C}(S)$ that fill S

$$[x]_M = \begin{cases} 0 & x \le M \\ x & x > M \end{cases}$$

<u>Thm</u> (Masur-Minsky) Let $f \in MCG(S)$

$$|f| \asymp \sum_{Y \subseteq S} \left[d_Y(\sigma, f(\sigma)) \right]_M$$

word length

up to bounded mult. & add. error

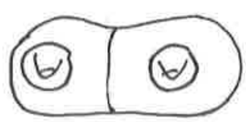
To prove this:

words in MCG $\longleftrightarrow$ moves on pants/markings $\longleftrightarrow$ hierarchies of geodesics in $\mathcal{C}(S)$

Idea of hierarchy: a geodesic in $\mathcal{C}(S)$ can be thickened to a path in pants complex or marking complex

<u>Pants complex</u>

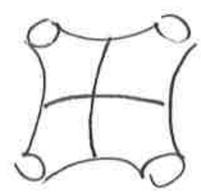
vertices



pants decomposition
= max simplex in $\mathcal{C}(S)$

edges

or

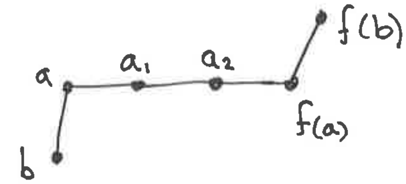


elementary move

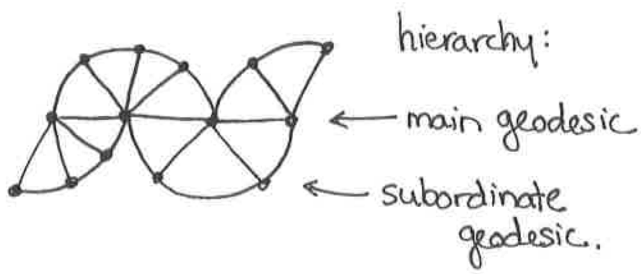
Marking complex: add twisting info

Example: $S_{0,5}$

pants dec. = edge in $\mathcal{C}(S_{0,5})$
Let $f \in MCG(S_{0,5})$
a — b = pants decomp
and geod from a to $f(a)$:



Key idea: can connect b to a_1 in $\mathcal{C}(S_{0,5} \setminus a)$ = Farey graph
$b = c_0, \ldots, c_m = a_1$
Each $(a, c_i) \longrightarrow (a, c_{i+1})$ is an edge in pants complex
Repeat for a_1, etc.
Get this picture:



subordinacy of geods $\approx$ nesting of domains.

A hierarchy can be <u>resolved</u> into a seq of pants decomps (or markings) each of which can be thought of as a <u>slice</u> of the hierarchy.

<u>Thm</u>. Any resolution of a hierarchy into a seq of complete markings is a quasigeod. in the marking complex

In general we construct hierarchies inductively as above.
Hyperbolicity $\Rightarrow$ choices of geodesics at each stage are essentially unique.
But more is true.

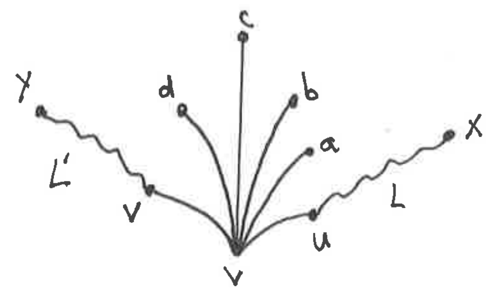
Common Links: If two hierarchies connect nearby pants dec/markings then they have (essentially) the same (long) geodesics (in the same domains).

Large Links: If two markings m_1, m_2 have $d_Y(m_1, m_2)$ large then any hierarchy connecting m_1 to m_2 has Y as a domain. The length of the corresp. geod is roughly $d_Y(m_1, m_2)$.

Both follow from Bounded Geodesic Image Thm.

<u>Example</u>: Genus 1 (Farey graph)

<u>Prop</u>. If a geodesic $x, \ldots, u, v, w, \ldots, y$ has $d_v(u,w) \geq 5$ then any geod from x to y must pass thru v.



<u>Pf</u>. Key: every edge of Farey graph separates.
Say h is a path x to y avoiding v.
Key $\Rightarrow$ h passes thru a, b, c, d
Also: $d(x,a) \geq L$ (otherwise original path not geod).
$\Rightarrow$ length$(h) \geq (L+2) + (L'+1) >$ length of original geod. ▨

Exercises: ① Still true if h connects x', y' adjacent to x, y
② Also h must enter $Lk(v)$ within 1 of u, w
(Large/Common Links)

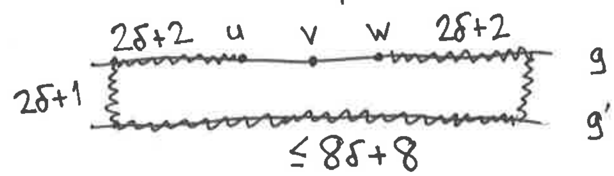
Example: Genus 2

$g = \ldots, u, v, w, \ldots$ geodesic in $C(S_2)$
g' = fellow traveler - say endpts are distance ≤ 1 from those of g.
Say distance from u, v, w to endpts of g is $\geq 2\delta+2$
Hyperbolicity $\Rightarrow$ g & g' are $2\delta+1$ fellow travelers

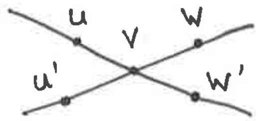
Suppose $d_Y(u,w) > 32\delta + 28$ $Y = S_2 \setminus v$.
Want to show: · g' must pass through/near v
· there is a geod in the g' hierarchy close to the geod in the g-hierarchy corresp. to Y.

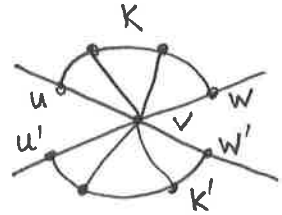
Case 1. v nonsep.

We claim g' must pass thru v.
Shortcut argument: If not, each vertex of g' intersects Y.
Consider this path:

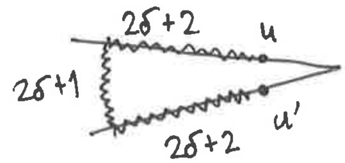


Each pt on the path intersects Y except u, w and length of path $\leq 16\delta + 14$
$\rightsquigarrow$ path in $C(Y)$ of length $\leq 32\delta+28$ (project), a contradiction.

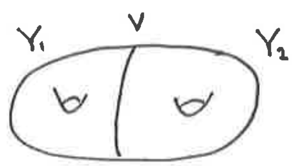
So we have: 

$\rightsquigarrow$ can continue the hierarchy: 

<u>Claim.</u> $d_Y(u,u') \leq 6\delta + 10$

<u>Pf.</u> similar shortcut argument: 

Since u,u' and v,v' close, the geodesics k,k' are close.

<u>Case 2</u> v separating. 

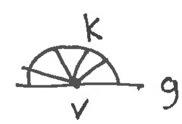
u,w must lie in same side, say Y_1.

Shortcut argument $\Rightarrow$ some curve v' of g' must miss Y_1

(still assuming $d_{Y_1}(u,w) > 32\delta + 28$).

$\Rightarrow$ $v' = v$ or v' essential in Y_2 (and is nonsep).

Suppose the latter.

Set $Y' = S_2 \setminus v'$ Goal: find geod in g'-hierarchy close to 

Shortcut argument $\Rightarrow$ $d_{Y'}(u',w') \leq 32\delta + 28$

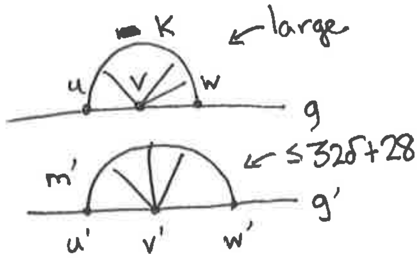
(otherwise, by Case 1 g must pass thru v';

this is a contradiction since $d(v,v') = 1$,

$v' \neq u,v,w$ and this would mean g

not geodesic).

Now have:

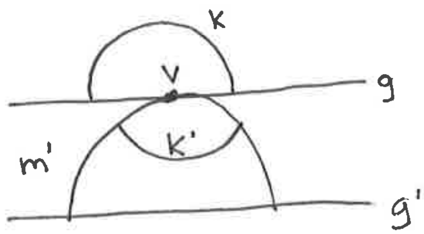


Claim that m' must have a vertex Z missing Y_1.

Suppose not. $\rightsquigarrow$ can find a path u to w missing v' and of (small) bounded length and so each vertex intersects Y_1, contradicting largeness of K.

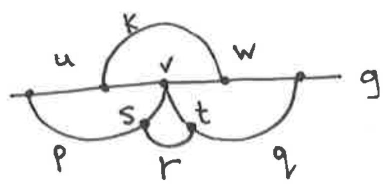
Z misses $v', Y_1 \Rightarrow Z = v$.

$\rightsquigarrow$ have:



$\rightsquigarrow$ construct K'. Similar arguments as before $\Rightarrow$ K close to K'. $\boxtimes$

None of K, K', m' have Y_2 as domain. But if we continue the g hierarchy, we will see Y_2:



The geodesic r lies in Y_2.

[say: r is forward subord. to q, backwards subord to p]

<u>Resolving the hierarchies</u>

g': v' (bottom level), v (next level), any $x \in K'$ form a pants decomp. = slice.

If x' is successor of x along K' then $(v'; v, x) \longrightarrow (v'; v, x')$ is elem. move.

g: v, $a \in K$, $b \in r$ $\rightsquigarrow$ (v, a, b) = pants decomp.

Again: to really understand MCG, need markings (pants + twisting data).

An MCG Action on Quasi-Trees.

Bestvina-Bromberg-Fujiwara: We have subsurface projections that behave like closest point projections in a δ-hyp space? So is there an ambient δ-hyp space lurking?

Setup: $\mathcal{Y}$ = collection of metric spaces

$\pi_X(Y)$ = projection of X to Y $\qquad \forall X, Y \in \mathcal{Y}$

$M \geq 0$

Axioms: 0. $\forall X, Y \in \mathcal{Y}$ diam $\pi_X(Y) \leq M$

1. $\forall X, Y, Z \in \mathcal{Y}$ at most one of

$$d_X(Y,Z) \quad d_Y(X,Z) \quad d_Z(X,Y)$$

is $> M$.

$d_A(B,C) =$ diam $\pi_A(B) \cup \pi_A(C)$

2. $\forall X, Y \in \mathcal{Y}$

$$\{Z \in \mathcal{Y} : d_Z(X,Y) > M\}$$

is finite.

Examples. ① $\mathcal{Y}$ = set of horizontal lines in $F_2 = \langle a, b \rangle$

= axes for conjugates of a

② $\mathcal{Y}$ = set of lifts to $\mathbb{H}^2$ of geodesic $\gamma \subseteq S_g$.

③ $\mathcal{Y}$ = set of $\mathcal{C}(Y)$ $\quad Y \subseteq S_g$

(really a subset where all Y pairwise intersect).

In example 3, what is the ambient space?

Thm (BBF) $\exists$ geodesic metric space $\mathcal{C}(\mathcal{Y})$
that contains isometric, totally geodesic, pairwise disjoint copies of the $Y \in \mathcal{Y}$.
and so $\forall\, X, Y \in \mathcal{Y}$ the nearest pt proj of Y to X in $\mathcal{C}(\mathcal{Y})$ is uniformly close to $\pi_X(Y)$.

There's more...

Quasi-trees

A quasi-tree is a geod. metric space quasi-isometric to a tree.

Asymptotic dimension

How to assign dim to a gp? Want $\dim(F_n) = 1$, $\dim \pi_1(S_g) = 2$, etc.

A metric space X has $\operatorname{asdim}(X) \leq n$ if $\forall\, R > 0$ $\exists$ covering of X by unif. bdd sets s.t. every metric R-ball intersects at most $n+1$ of the sets.
(large-scale analog of covering dim).

examples:
1. $\operatorname{asdim} \mathbb{Z}^n = n$
2. $\operatorname{asdim} F_n = 1$
3. $\operatorname{asdim} \pi_1 S_g = 2$
4. $\operatorname{asdim} F = \infty$ (Thompson's gp F contains $\mathbb{Z}^\infty$).

$\operatorname{asdim} G < \infty \Rightarrow G \hookrightarrow$ Hilbert space $\Rightarrow$ Novikov higher signature conj:
$\exists$ invariant of smooth type of $K(G,1)$
(defined in terms of pi)
which is really a homotopy invt.

Thm (BBF). $C(\mathcal{Y})$ also satisfies:

(i) the construction is equivariant wrt any group action on $\coprod Y$ that respects projections

(ii) if each Y is isometric to $\mathbb{R}$, $C(\mathcal{Y})$ is quasi-tree

(iii) if $\coprod Y$ is δ-hyp, $C(\mathcal{Y})$ is δ'-hyp.

(iv) if asdim $\coprod Y \leq n$ then asdim $C(\mathcal{Y}) \leq n+1$.

(ii) $\Rightarrow$ $C(\mathcal{Y})$ is a quasi-tree in example 2 above, not $\mathbb{H}^2$!

Projection Complex

$P(\mathcal{Y}) = C(\mathcal{Y})/\mathcal{Y}$ space obtained by collapsing each $Y \in \mathcal{Y}$ to pt.

Thm (BBF). $P(\mathcal{Y})$ is a quasi-tree.

Example

M^3 = closed hyp. 3-man

$\gamma \subseteq M$ geod.

$\mathcal{Y}$ = lifts of γ to $\mathbb{H}^3$.

$\rightsquigarrow$ action of $\pi_1(M)$ on quasitree where γ acts loxodromically.

Note: Any action of $\pi_1(M^3)$ on actual tree has a global fixed pt.

The Construction

Basic idea: Say Y is between X and Z if
$$d_Y(X,Z) \geq D$$
We connect each pt of X to each point of Z by a segment of length 1 if $\nexists\, Y$ between.

Mapping Class Groups

Goal: $MCG(S)$ equivariantly quasi-isometrically embeds in a finite product of quasitrees:
$$P(\mathcal{Y}_1) \times \cdots \times P(\mathcal{Y}_n)$$
For all $Y, Y' \in \mathcal{Y}_i$ $\pi_Y(Y')$ is defined, i.e. need to color the subsurfaces of S by finitely many colors s.t. disjoint subsurfs have diff colors.

Cor: $\operatorname{asdim} MCG(S) < \infty$.

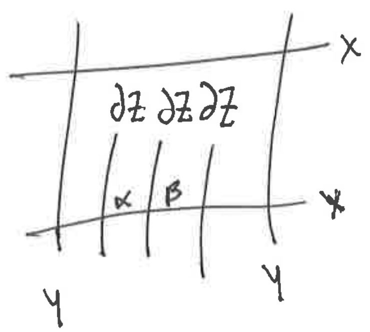
To get the qi embedding use the fact that each ∞-order elt of MCG acts loxodromically on ~~the~~ $C(Y)$ for some $Y \subseteq S$.

Axiom 2 for MCG

We'll prove something more general.

Lemma. $x,y \in C(S) \rightsquigarrow \exists$ finitely many $Z \subseteq S$ s.t.
$$d_Z(x,y) > 3.$$

Pf. Assume first x,y fill.
If $i(x,\partial Z) + i(y,\partial Z)$ large, see:



$\Rightarrow \exists$ arc of $x-y$ (or $y-x$) lying in Z and disjoint from y (namely α or β).
$\Rightarrow d_Z(x,y) \leq 3$
$\rightsquigarrow$ finite list of Z.

In general, let $R \subseteq S$ be subsurf filled by $x \cup y$.
If $Z \nsubseteq R$ $\exists$ curve in Z disjoint from $x \cap Z$, $y \cap Z$.
$$\Rightarrow d_Z(x,y) \leq 2.$$
If $Z \subseteq R$ we are in filling case with S replaced by R. $\square$