MA4J2: Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Matthew Pressland
Assisted by Anna Lena Winstel
and David Kitson

Lecture 1

One goal of topology is to classify manifolds up to homeomorphism. In dimension $n \geq 4$, this problem is undecidable; no algorithm, given two manifolds as an input, can decide whether or not they are homeomorphic.* We will classify manifolds in dimensions 0, 1 and 2 in the next few pages. The general topic is to classify 3–manifolds.

Definition 1.1. An *n–manifold* M^n is a Hausdorff topological space with a countable basis and such that every point $p \in M$ has an open neighbourhood U which is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$.

Remark. $\mathbb{R}^n_+$ is called the upper half space, and $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n \leq 0\}$ is called the lower half space.

Definition 1.2. ∂M is the set of points p in M such that no neighbourhood of p is homeomorphic to $\mathbb{R}^n$.

Proposition 1.1. *∂M is an $(n-1)$–manifold, and $\partial\partial M = \varnothing$.*

Definition 1.3. $\operatorname{int}(M) = M - \partial M$.

Definition 1.4. We use $I = [0,1] \subseteq \mathbb{R}$, $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and $\mathbb{D}^2 = \mathbb{B}^2$.

Definition 1.5. We give several equivalent definitions of the *sphere*:

(i) A submanifold definition: $S^n = \partial\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

(ii) A *one-point compactification* definition: S^n is the one-point compactification of $\mathbb{R}^n$, that is $S^n = \mathbb{R}^n \cup \{\infty\}$ topologized such that for any compact $K \subseteq \mathbb{R}^n$, the set $(\mathbb{R}^n - K) \cup \{\infty\}$ is a neighbourhood of ∞. Note here that $\mathbb{B}^n$ is the one-point compactification of $\mathbb{R}^n_+$.

(iii) A gluing definition: $S^n = \mathbb{B}^n_0 \sqcup \mathbb{B}^n_1 / \sim$ where $(x,0) \sim (x,1)$ if and only if $x \in \partial\mathbb{B}^n$. For example, S^1 can be obtained by joining two copies of $\mathbb{B}^1$ by their boundaries, and similarly for S^2 and $\mathbb{B}^2$.

*This result is due to A.A. Markov (1958).

Definition 1.6. We now give several equivalent definitions of *projective spaces*:

(i) A covering space definition: $\mathbb{P}^n = S^n / \sim$ where $x \sim -x$, taking S^n as in definition (i) above.

(ii) A gluing definiton: $\mathbb{P}^n = \mathbb{B}^n / \sim$ where $x \sim -x$ if and only if $x \in \partial\mathbb{B}^n$.

(iii) A moduli space definition:

$$\mathbb{P}^n = \{L \subseteq \mathbb{R}^{n+1} : L \text{ is a line through the origin}\} = (\mathbb{R}^{n+1} - \{0\}) / \sim$$

where $x \sim \lambda x$ for $\lambda \in \mathbb{R} - \{0\}$.[†]

Definition 1.7. We have three equivalent definitions of *tori*:

(i) A *Cartesian product* definition: $\mathbb{T}^n = (S^1)^n$, taking the Cartesian product.

(ii) A covering space definition: $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{R}^n / \sim$ with $x \sim y$ if and only if $x - y \in \mathbb{Z}^n$.

(iii) A gluing definition: $\mathbb{T}^n = I^n / \sim$ where $(x, 0, y) \sim (x, 1, y)$ if and only if $x \in I^k$ and $y \in I^{n-k-1}$ for any $k \in \{0, ..., n-1\}$.

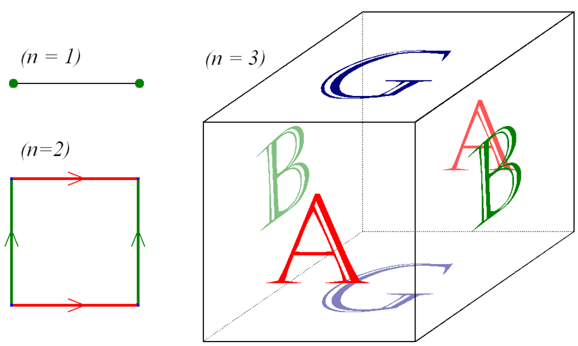

Figure 1: Construction of the first three n–tori $\mathbb{T}^n$. Identify opposite faces of I^n without twisting.

Note. $\mathbb{T}^1 \cong S^1$.

Exercise 1.1. For each set of three definitions above, prove that all three are equivalent.

In dimension zero, any compact manifold is a finite collection of points, so the classification is given by the number of points. All compact connected one-dimensional manifolds are homeomorphic to either S^1 or I.

[†] We will sometimes use $\mathbb{R}^*$ for $\mathbb{R} - \{0\}$.

Definition 1.8. Suppose M_i (for $i = 0, 1$) are orientable n–manifolds. Choose $\mathbb{B}_i^n \subseteq M_i$ and suppose $\varphi : \partial\mathbb{B}_0^n \to \partial\mathbb{B}_1^n$ is an orientation reversing homeomorphism. Define:

$$M_0 \# M_1 := ((M_0 - \mathrm{int}(B_0^n)) \sqcup (M_1 - \mathrm{int}(B_1^n)))/\sim$$

where $x \sim \varphi(x)$ whenever $x \in \partial\mathbb{B}_0^n$.

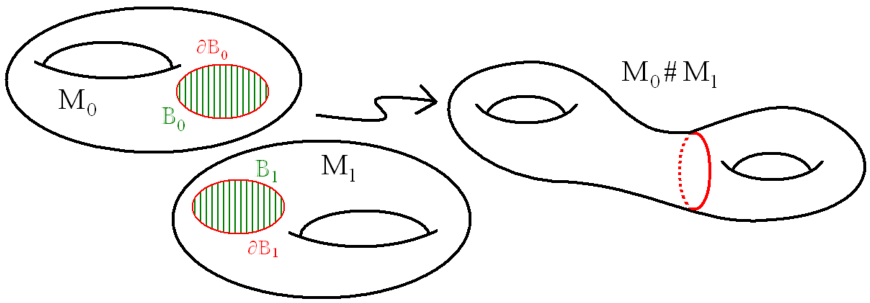

Figure 2: The connect sum. Remove the interiors of the disks $\mathbb{B}_i$ and glue along their boundaries.

Exercise 1.2. Show that $\#_3\mathbb{P}^2 \cong \mathbb{T} \# \mathbb{P}^2$.

Theorem 1.2. *Every compact connected two-dimensional manifold is homeomorphic to some $S_{g,n,c}$, where:*

$$S_{g,n,c} := (\#_g\mathbb{T}^2) \# (\#_n\mathbb{D}^2) \# (\#_c\mathbb{P}^2)$$

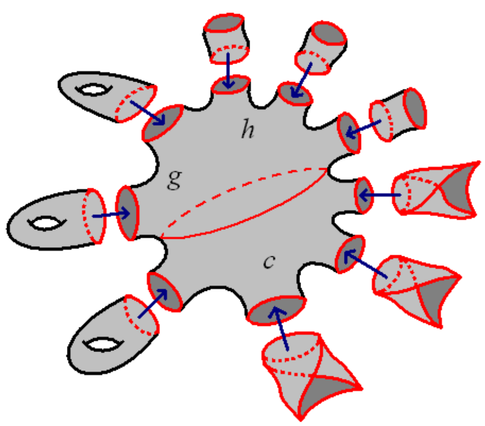

Figure 3: $S_{3,3,3}$ is the connect sum of the sphere with three tori, three Möbius strips and three 2–disks, glued along the boundary components (in red)

Example 1.1. Some spaces $S_{g,n,c}$ are homeomorphic, for example $S_{3,3,3} \cong S_{4,3,1}$.

Lecture 2

Example 2.2. We give some connect sums of three manifolds:

$$S^3 \# S^3 \cong S^3$$
$$\mathbb{T}^3 \# S^3 \cong \mathbb{T}^3$$

In general, S^n is a unit for the connect sum. $\mathbb{P}^3 \# \mathbb{P}^3$ is more interesting, as we will discuss later. On the other hand, $\mathbb{T}^3 \# \mathbb{T}^3$ invites splitting into two copies of $\mathbb{T}^3$ for a more interesting and fundamental geometry. In general, we shall find a decomposition theorem for 3–manifolds with respect to $\#$.

Definition 2.9. M^3 is *prime* if whenever $M = N \# L$ then either N or L is homeomorphic to S^3.

Remark. If $M = N \# L$ and $N \cong S^3$ then $L \cong M$, and vice versa.

Definition 2.10. M is *irreducible* if every smoothly embedded S^2 in M bounds a 3–ball.

Note. We have no examples yet of prime or irreducible 3–manifolds.

Definition 2.11. Suppose $X, Y \subseteq Z$. We say X is *ambient isotopic* (*diffeotopic*) to Y if there exists a continuous (smooth) map $F : Z \times I \to Z$ such that, defining $F_t(z) := F(t, z)$:

(i) For all $t \in I$, F_t is a homeomorphism (diffeomorphism).

(ii) $F_0 = \mathrm{Id}_Z$.

(iii) $F_1|X : X \to Y$ is a homeomorphism (diffeomorphism).

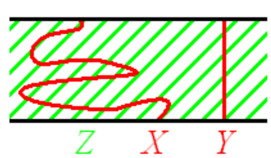

Figure 4: Here, X is ambient isotopic to Y in Z.

Theorem 2.3 (Alexander)**.** *Every smoothly embedded $S^2 \subset S^3$ is ambient isotopic to the equator.*

Compare this to:

Theorem 2.4 (Jordan-Schoenflies)**.** *Every smoothly embedded $S^1 \subset S^2$ is ambient isotopic to the equator.*

We will prove Alexander's theorem later, but for now give the following corollary.

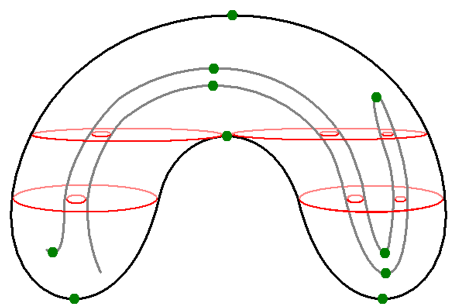

Figure 5: It is not always obvious which ball a sphere bounds

Corollary 2.5. *S^3 is prime.*

Proof. Suppose $S^3 = M \#_S N$. By Alexander's theorem, S is ambient isotopic to a round embedding of S^2 in S^3 (say the equator). Thus $M - \mathrm{int}(\mathbb{B}^3) \cong N - \mathrm{int}(\mathbb{B}^3) \cong \mathbb{B}^3$, and hence $M \cong N \cong S^3 \cong \mathbb{B}^3 \cup_\partial \mathbb{B}^3$. $\square$

It is important that the embedding is smooth, as the following result shows.

Theorem 2.6. *There exists a topological $S^2 \subset S^3$ which does not bound $\mathbb{B}^3$ on either side.*

Note. This is a generalization of the Alexander horned sphere.

Remark. The statement of Alexander's theorem with $S^2 \subset S^3$ replaced by $S^3 \subset S^4$ is an open problem, although it has been proved that a smoothly embedded $S^3 \subset S^4$ bounds a topological ball. Brown has proved the more general statement that a smoothly embedded $S^{n-1} \subset S^n$ bounds a topological ball.

Remark. It is worth making explicit the various categories involved:

(i) Topological (TOP).

(ii) Piecewise linear (PL).

(iii) Smooth (DIFF).

These categories are all equivalent in dimension at most 3, so we move between them freely.

Exercise 2.3.

(i) Prove that any irreducible manifold is prime.

(ii) Prove that M is orientable and $S \subset M$ is a non-separating 2–sphere, then $M = N \# (S^2 \times S^1)$.

(iii) Suppose M is orientable. Then M is prime and reducible if and only if $M \cong S^2 \times S^1$. Prove the forward direction.

(iv) State and prove analogous statements to (ii) and (iii) for non-orientable manifolds.

We give one more corollary to Alexander's theorem:

Corollary 2.7. *If $M \subseteq S^3$ is compact and has $|\partial M| \leq 1$ (at most one boundary component) then M is irreducible.*

Example 2.3. We give further examples of irreducible manifolds. Suppose $K \subset S^3$ is a knot, that is a smooth embedding of S^1. Let $N(K) \subseteq S^3$ be a closed regular neighbourhood (i.e. a tubular neighbourhood) of the knot. Let $n(K) = \mathrm{int}(N(K))$. Then the knot exterior $X_K := S^3 - n(K)$ is irreducible, by the previous corollary.

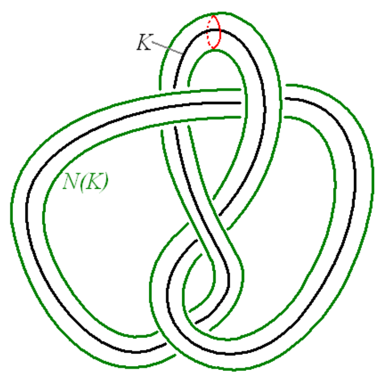

Figure 6: A tubular neighbourhood of the figure 8 knot

Lecture 3

We now prove Alexander's theorem. More precisely, we will prove that any (smoothly) embedded $S^2 \subset \mathbb{R}^3$ bounds a 3-ball, from which the theorem can be deduced as a corollary.

Exercise 3.4. Show how Alexander's theorem follows from this statement.

We need the following lemma:

Lemma 3.8. *Suppose that a manifold M^n and $\mathbb{B}_1^{n-1} \subseteq \partial M^n$ are given, as is a diffeomorphism $\varphi : \mathbb{B}_0^{n-1} \to \mathbb{B}_1^{n-1}$, where $\mathbb{B}_0^{n-1} \subseteq \partial \mathbb{B}^n$. Then $M^n \cup_\varphi \mathbb{B}^n \cong M^n$, as per Figure 7.*

As a consequence, if B and B' are n–balls, then $B \cup_\partial B'$ is a ball (Figure 8(a)), as is $\overline{B - B'}$ if $B' \subset B$ and $\partial B' \cap \partial B \cong \mathbb{D}^{n-1}$ (Figure 8(b)).

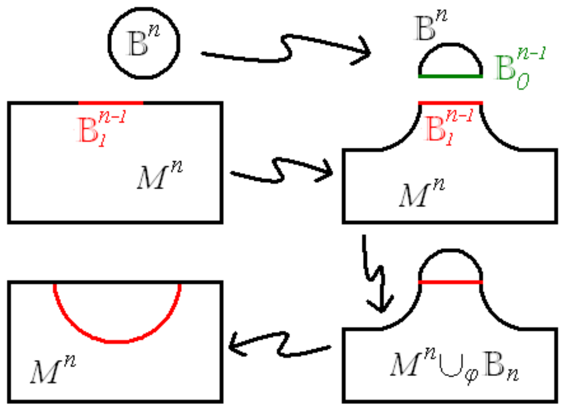

Figure 7: Glueing M^n to $\mathbb{B}^n$ along submanifolds of their boundaries is homeomorphic to M^n.

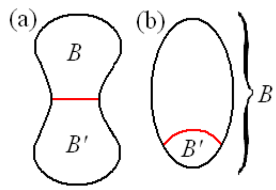

Figure 8: (a) B, B' balls $\Rightarrow B \cup_\partial B'$ a ball, and (b) $B' \subset B$ and $\partial B' \cap \partial B \cong \mathbb{D}^{n-1} \Rightarrow \overline{B - B'}$ a ball.

Theorem 3.9. *Any smoothly embedded $S^2 \subset \mathbb{R}^3$ bounds a 3–ball.*

Proof. Suppose $S^2 \cong S \subset \mathbb{R}^3$ is smooth. We can isotope S so that $z : S \to \mathbb{R}$ (the height function, giving the z co-ordinate) is a Morse function. Thus all critical points are of the standard three types; cups (minima), caps (maxima), and saddles, and all critical points occur at distinct heights (as illustrated in Figure 9). Choose $a_i \in \mathbb{R}$ such that $(-\infty, a_1), (a_1, a_2), ..., (a_{n-1}, \infty)$ each contain

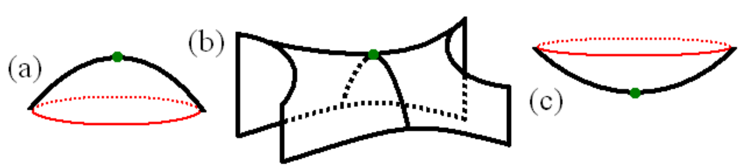

Figure 9: (a) A cap. (b) A saddle. (c) A cup.

exactly one critical value, as in Figure 10. Let:

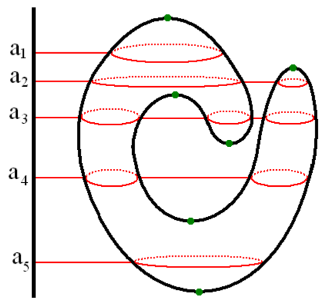

Figure 10: The red circles are regular values separating the critical points (green). Here we have $(n, w) = (6, 9)$.

$$\begin{aligned} L[a,b] &:= \{(x,y,z) : z \in [a,b]\} \\ L(a) &:= \{(x,y,z) : z = a\} \\ L_i &:= L(a_i) \end{aligned}$$

Define $n(S)$ to be the number of critical points. Define the *width* by:

$$w(S) = \sum_{i=1}^{n-1} |S \cap L_i|$$

This is the number of red circles in Figure 10. We will induct on $(n(S), w(S))$ lexicographically. Note that the components of $L_i \cap S$ are all simple closed curves, because each a_i is a regular value. So by the Jordan-Schoenflies theorem, they all bound disks. Say that β, a component of $L_i \cap S$, is *innermost* if D_β, the disk bounded by β, has the property that $D_\beta \cap S = \beta$. Notice that β also bounds a pair of disks in S. Label a_i with an A (resp. B) if there is some

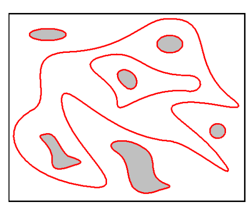

Figure 11: The intersection of the plane L_i with the sphere. Shaded components are innermost.

innermost curve $\beta \subseteq L_i \cap S$ such that one disk of $S - \beta$ contains exactly one critical point, a maximum (resp. minimum). Note that a_i could receive both labels. Note also that a_1 is labelled by B and a_{n-1} is labelled by A. We have cases:

Case 1: Some a_i is labelled both A and B.

Case 2: Some a_i is unlabelled.

Case 3: There exists i such that a_i is labelled B and a_{i+1} is labelled A.

Exercise 3.5. Check that we must always be in at least one of these cases.

We prove these in turn:

Case 1a: Some innermost $\beta \in L_i \cap S$ bounds a disk in S above and bounds a disk in S below, each with one critical point; this forms the base case of the induction, where $n(S) = 2$ and $w(S) = 1$. We claim that in this case S bounds a ball. To see this, cut off the two critical points with planes

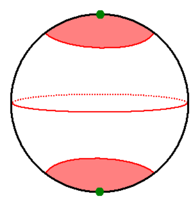

Figure 12: The base case.

slightly above the minimum and below the maximum, removing two 3–balls from S, and giving a compact cylinder. We claim that for every $a \in \mathbb{R}$ such that the set $L(a)$ intersects this compact cylinder, there exists $\varepsilon > 0$ such that $S \cap L[a, a+\varepsilon]$ bounds a 3–ball in $L[a, a+\varepsilon]$. This can be proved by the implicit function theorem and the isotopy extension theorem. See Hatcher's *Notes on basic 3–manifold topology* for more details. Note that

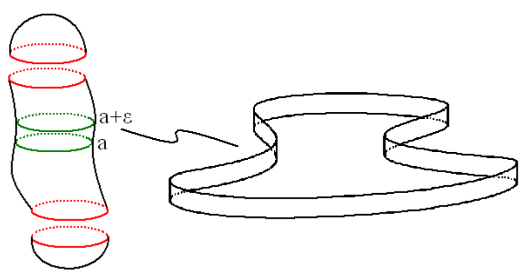

Figure 13: The slab bounded by $L_[a, a+\varepsilon]$.

the intersection $L(a) \cap S$ is a curve, so bounds a disk. Note that finitely many of the $L[a, a+\varepsilon]$ cover the compact cylinder. Glue these slabs together, and re-attach the cap and cup. By Lemma 3.8, this gives a 3–ball.

This proof continues in the lectures from week two.

Please let me know if any of the problems are unclear or have typos.

Exercise 1.1. Suppose that M^n is a manifold. Prove directly from the definition that ∂M is either empty or is an $n-1$–manifold. Prove that $\partial\partial M$ is empty.

Exercise 1.2. Show that the definitions of S^n given in class (as a submanifold of $\mathbb{R}^{n+1}$, as the one-point compactification of $\mathbb{R}^n$, and as the double of an n–ball) are equivalent.

Exercise 1.3. Give a map $T^n \to \mathbb{R}^{n+1}$ that is an *embedding*: a diffeomorphism onto its image. Show that any compact n–manifold embedded in $\mathbb{R}^n$ has non-empty boundary; deduce that T^n does not embed in in $\mathbb{R}^n$.

Exercise 1.4. [Hard] Verify the classification, up to homeomorphism, of compact connected 1–manifolds. For a detailed outline of the argument, see David Gale's article "The classification of 1-manifolds: a take-home exam", in the American Mathematical Monthly.

Exercise 1.5. Show that, up to homeomorphism, connect sum is commutative, associative, and that the n–sphere is an identity element: $M^n \# S^n \cong M^n$.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Anna Lena Winstel
Assisted by Matthew Pressland
and David Kitson

Lecture 4

(Proof continued from last lecture).

Case 1b There are innermost $\alpha, \beta \subset L_i \cap S$ so that α bounds D above, β bounds E below. Let D', E' be the disks bounded by α, β, inside of L_i. So, by the base case $D \cup D'$ $(E \cup E')$ bounds a 3–ball. Use this 3–ball to define an ambient isotopy that flattens D (E), pushing the critical point just below (above) the plane L_i.

Exercise 4.1. Show that this reduces $w(S)$.

Case 2 The regular value a_i is not labelled. For this case, we first have to introduce

Definition 4.1. Suppose $F^2 \subset M^2$ is properly embedded (i.e. a submanifold, i.e. embedded and $F \cap \partial M = \partial F$). We say $(D^2, \partial D) \subset (M, F)$ is a *surgery disk* for F if $D \cap F = \partial D$.

Let $n(\partial D)$ be an open annular neighbourhood of ∂D, in F. Let D_+, D_- be parallel copies of D in M. Define F *surgered along* D by $F_D := (F - n(\partial D)) \cup D_+ \cup D_-$, as in Figure 1.

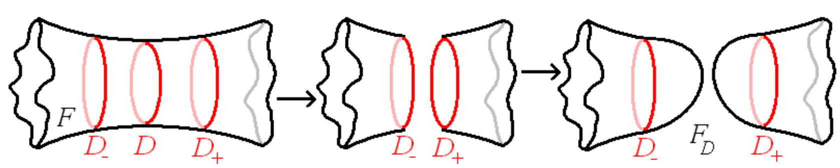

Figure 1: Surgery. $F_D := (F - n(\partial D)) \cup D_+ \cup D_-$.

We now return to case 2. Suppose $\beta \subset S \cap L_i$ is innermost. So, β bounds D above, E below, $D \cup_\beta E = S$ and D, E each contain at least 3 critical points. Say β bounds a disk $B \subset L_i$. So:

$$S_B = S_+ \cup S_-, \quad S_+ \cong D \cup B_+, \quad S_- \cong E \cup B_-.$$

Thus $n(S_+), n(S_-) < n(S)$ since $n(S_+) + n(S_-) = n(S) + 2$. By induction, S_+, S_- each bound a 3–ball X_+, X_- thus so did S, applying Lemma 1.3 in Hatcher's notes. In the first case $X_+ \cap X_- = B$ and so we take the union. In the second case $X_+ \subset X_-$, we take the difference. See Figure 2.

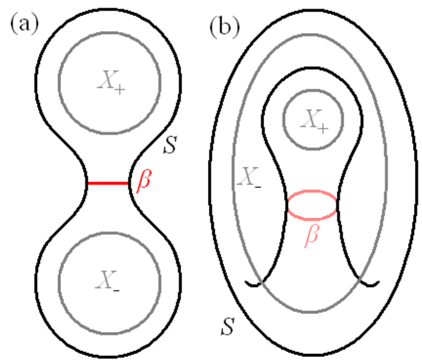

Figure 2: The case when (a) $x_+ \cap X_- = \varnothing$ or (b) $X_+ \subseteq X_-$.

Case 3 The regular value a_i is labelled only B and the regular value a_{i+1} is labelled only A. Between L_i and L_{i+1} we have $S \cap L[a_i, a_{i+1}]$ is a union of cylinders, caps, cups, pairs of pants, upside down pairs of pants and pants with inverted legs, as illustrated in Figure 3.

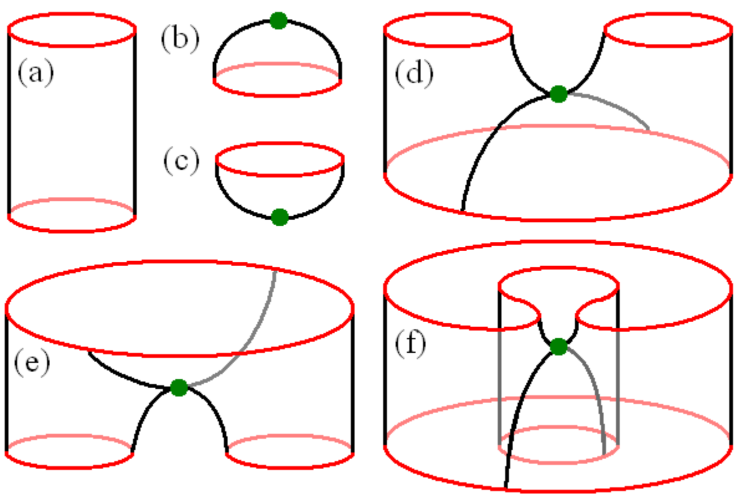

Figure 3: $S \cap L[a_i, a_{i+1}]$ is a union of (a) cylinders, (b) caps, (c) cups, (d) pairs of pants, (e) upside down pairs of pants, (f) pants with inverted legs and (g) an upside down version of (f) (not shown).

Note that there is at most one critical point in $S \cap L[a_i, a_{i+1}]$, so it is a saddle (check this using the labelling). Using the labelling deduce that either α or β is a cuff of the pants.

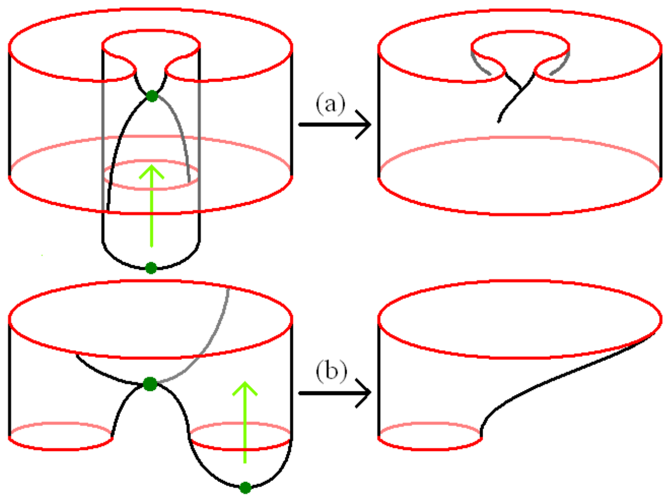

Figure 4: Two examples of how may isotope E to be in L_i and then upwards, canceling two critical points.

We have that β is innermost in L_i and β bounds (in S) a disk below, E, with a single critical point (minimum). Hence, by the base case, we may isotope E to be in L_i and then upwards to cancel two critical points, as in Figure 4. Thus, we have isotoped S to a sphere S' such that $n(S') = n(S) - 2$. This completes the induction step and so, the proof. $\square$

Lecture 5

Definition 5.2. Say a 2–sphere $S \subset M^3$ is *essential* if no component of $M - n(S)$ is a 3–ball.

Incompressible surfaces

Definition 5.3. Suppose $F^2 \subset M^3$ is properly embedded. Suppose $(D, \partial D^2) \subset (M, F)$ is a surgery disk. Say that D is a *trivial surgery* disk if $\partial D \subset F$ is equal to $\mathbf{1} \in \pi_1(F)$ where $\pi_1(F)$ is the fundamental group of F. We say that D is a *compressing disk* if $\partial D \subset F$ is not equal to $\mathbf{1} \in \pi_1(F)$.

An alternative definition is: D is a trivial surgery disk if ∂D bounds a disk in F. See Figure 5.

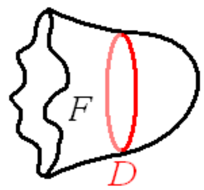

Figure 5: Here, D is a trivial surgery disc for F.

Exercise 5.2. Check that a simple closed curve $\alpha \subset F$ bounds a disk $E \subset F$ if and only if $[\alpha] = \mathbf{1} \in \pi_1(F)$.

Definition 5.4. Suppose $F \subset M$ is either proper embedded or $F \subset \partial M$ is a subsurface. Then we say that F is *compressible* if and only if there exists a compressing disk for F. Otherwise we call F *incompressible*.

Example 5.1. Let $T \subset S^3$ be the standard embedding, i.e. $\partial N(U)$ where U is the unknot. Then T is compressible since there are two compressing disks. We call them the *meridian disk* and the *longitude disk* respectively, as illustrated in Figure 6.

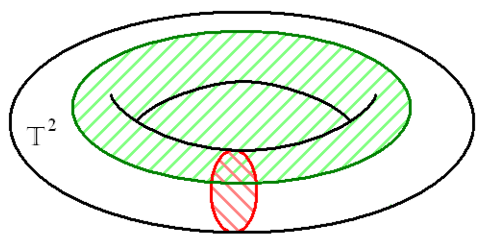

Figure 6: The meridian disk is in green while the longitude disk is red. The boundary of the meridian disk is a circle in T but its interior is in S^3.

Example 5.2. If $M = D \times S^1$ is a solid torus then $\partial M \subset M$ is compressible.

Exercise 5.3. Show that $T = \mathbb{T}^2 \times \{\frac{1}{2}\} \subset \mathbb{T}^2 \times I = M$ is incompressible.

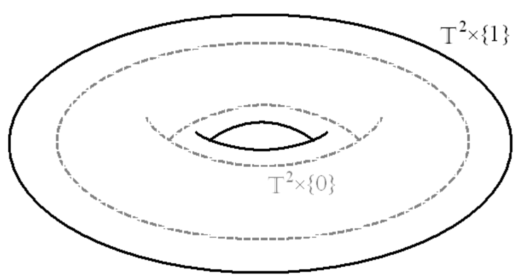

Figure 7: $T = \mathbb{T}^2 \times \{\frac{1}{2}\} \subset \mathbb{T}^2 \times I = M$ is incompressible.

Exercise 5.4. Suppose that M is an irreducible three-manifold and $F, G \subset \partial M$ are disjoint, incompressible subsurfaces. Suppose that $\varphi\colon F \longrightarrow G$ is a homeomorphism. Show that M/φ is irreducible.

Note. One can check $M = \mathbb{D}^2 \times S^1$ is irreducible but $D(M)$, the double of M, is not. Here $D(M) = M_0 \sqcup M_1/\sim$, where $(x, 0) \sim (x, 1)$ if and only if $x \in \partial M$ where $M_i = M \times \{i\}$.

Exercise 5.5.

1. If $F \subset S^3$ is closed, $F \neq S^2$, then F is compressible.
2. (Alexander) Any $\mathbb{T}^2 \subset S^3$ bounds a solid torus ($\mathbb{D}^2 \times S^1$) on at least one side.

Definition 5.5. Let V_g be the *handlebody* of genus g, i.e.

$$V_g = \underbrace{\mathbb{D}^2 \times S^1 \ \cup_{\mathbb{D}^2} \ \mathbb{D}^2 \times S^1 \ \cup_{\mathbb{D}^2} \ \dots \ \cup_{\mathbb{D}^2} \ \mathbb{D}^2 \times S^1}_{g \text{ times}}$$

By convention, $V_0 = \mathbb{B}^3$. See Figure8.

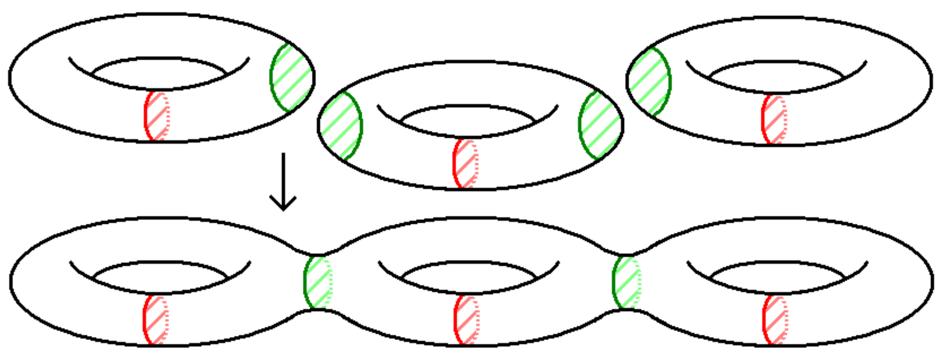

Figure 8: The handlebody V_3. Note that V_g is "solid", and not a surface.

Example 5.3. Find $S_2 \hookrightarrow S^3$ which does not bound a handlebody on either side. Here S_2 denotes a surface of genus 2.

Remark. $\partial V_g = \#_g \mathbb{T}^2 = S_g$ because $\partial(\mathbb{D}^2 \times S^1) = S^1 \times S^1 = \mathbb{T}^2$.

Products and Bundles

A map $\rho\colon Z \longrightarrow X$ is a Y*–bundle* (or a *fibre bundle*) if for all $x \in X$ there exists a neighbourhood $x \in U \subset X$ and a homeomorphism $h_U\colon Y \times U \longrightarrow \rho^{-1}(U)$ such that the composition $\rho \circ h_U$ is the projection onto the second coordinate. Here, Z is called the *total space*, X the *base space*, Y the *fibre* and h_U is called a *local trivialization*.

Example 5.4. Let $Z = \mathbb{D}^2 \times S^1$ and denote ρ_i be the projection onto the i–th coordinate. Then $\rho_1 \colon Z \longrightarrow \mathbb{D}^2$ is a S^1–bundle map and $\rho_2 \colon Z \longrightarrow S^1$ is a $\mathbb{D}^2$–bundle map.

Lecture 6

Bundles and Neighbourhoods

See Lackenby §6.

Definition 6.6. We say $Z \xrightarrow{\rho} X,\ Z' \xrightarrow{\rho'} X$ are *equivalent* Y–bundles if there is a homeomorphism $h \colon Z' \longrightarrow Z$ making the following diagram commute

$$\begin{tikzcd} Z' \arrow[r, "h"] \arrow[d, "\rho'"'] & Z \arrow[d, "\rho"'] \\ X \arrow[r, "\mathrm{Id}_X"] & X \end{tikzcd}$$

Corollary 6.1 (See Corollary 6.3 in Lackenby's notes)**.** *If X is contractible then any Y–bundle $Z \xrightarrow{\rho} X$ is equivalent to the product bundle $Y \times X \xrightarrow{\rho_2} X$.*

Exercise 6.6. Prove this directly for $X = \mathbb{B}^1, \mathbb{B}^2$.

Exercise 6.7. Find a S^1–bundle over S^2 that is not equivalent to the product bundle. It follows that the fundamental group $\pi_1(X, x) = \{\mathbf{1}\}$ is not sufficient hypothesis for Corollary 6.1.

Lemma 6.2 (See Lemma 6.4 in Lackenby's notes)**.** *For all $n \in \mathbb{N}$ there are exactly two $\mathbb{B}^n$–bundles over S^1 up to equivalence. These are*

- *the trivial bundle $\mathbb{B}^n \times S^1$*
- *the twisted bundle $\mathbb{B}^n \widetilde{\times} S^1 = \mathbb{B}^n \times I/(x, 0) \sim (r(x), 1)$, where $r(x_1, \ldots, x_n) = (x_1, \ldots, -x_n)$ is a reflection.*

Version of the Tubular Neighbourhood Theorem

Definition 6.7. Suppose $\rho \colon Z \longrightarrow X$ is a bundle. Then a map $s \colon X \longrightarrow Z$ is a *section* of ρ if $\rho \circ s = \mathrm{Id}_X$.

Theorem 6.1. *Suppose $F^{n-k} \subset M^n$ is properly embedded. Then there is a closed neighbourhood $N = N(F) \subset M$ of F and a $\mathbb{B}^k$–bundle map such that*

1. *the inclusion $i \colon F \longrightarrow N(F)$ is the zero section, i.e. $i(x) = 0 \in \mathbb{B}^k = \rho^{-1}(x)$,*
2. *N is a codimension 0 submanifold of M (with corners) and*

3. *any $N'(F)$ satisfying the properties (1) and (2) is ambient isotopic to $N(F)$ fixing F pointwise.*

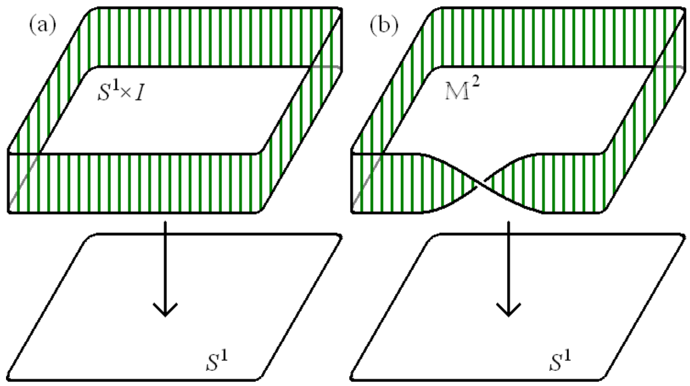

Figure 9: Two inequivalent bundles over S^1: (a) $\mathbb{B}^1 \times S^1$ and (b) $\mathbb{B}^1 \widetilde{\times} S^1$.

Notation: We denote by $n(F)$ the interior of $N(F)$. Furthermore, M *cut along* F, is the manifold (perhaps with corners) $M - n(F)$. When F is codimension 1 manifold there is a regluing map $M - n(F) \overset{\text{reglue}}{\longrightarrow} M$.

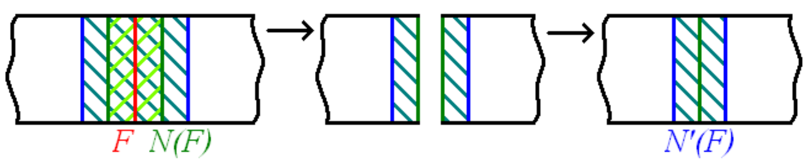

Figure 10: Cut open along $N(F)$ and glue back along $N'(F)$.

Exercise 6.8. All I–bundles over S^2 are trivial.

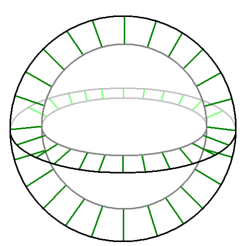

Figure 11: The trivial I–bundle over S^2.

Please let me know if any of the problems are unclear or have typos.

Exercise 2.1. Extend the definition of connect sum to non-orientable surfaces and prove that $\#_3\mathbb{P} \cong \mathbb{T}\#\mathbb{P}$.

Exercise 2.2. [Alexander Trick] Suppose that $\phi\colon S^2 \to S^2$ is a homeomorphism and $\mathrm{Id}\colon S^2 \to S^2$ is the identity. Find an explicit homeomorphism between $M = \mathbb{B}^3 \cup_\phi \mathbb{B}^3$ and $S^3 = \mathbb{B}^3 \cup_{\mathrm{Id}} \mathbb{B}^3$. (It follows, in dimension three, that M is diffeomorphic to the three-sphere. The Alexander trick works in all dimensions but the promotion to smoothness does not. See Milnor's paper "On manifolds homeomorphic to the 7–sphere".)

Exercise 2.3.

- Show that if M^3 is irreducible then M is prime.
- Show that if M is orientable and $S \subset M$ is a non-separating two-sphere embedded in M then $M = S^2 \times S^1 \# N$.
- Suppose that M^3 is orientable. Then: M^3 is prime and reducible iff $M \cong S^2 \times S^1$. Prove the forward direction.

Exercise 2.4. [Medium] Prove the backwards direction of part (iii) of Exercise 2.3. [Idea: rewrite the proof of Alexander's theorem.]

Exercise 2.5. [Medium] Prove the Jordan-Schoenflies theorem: every smoothly embedded S^1 in $\mathbb{R}^2$ bounds a disk. [Idea: rewrite the proof of Alexander's theorem for dimension two.]

Exercise 2.6. Suppose that $K^1 \subset S^3$ is a knot. Let $X_K = S^3 - n(K)$ be the *knot complement*. Show that ∂X_K is compressible iff K is the unknot (isotopic to a round circle).

Exercise 2.7. Suppose that M is an irreducible three-manifold and $F, G \subset \partial M$ are disjoint, incompressible subsurfaces. Suppose that $\phi\colon F \to G$ is a homeomorphism. Show that M/ϕ is irreducible.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Matthew Pressland
Assisted by Anna Lena Winstel,
Victoria Scott and David Kitson

Lecture 7

Suppose that $\rho\colon G^2 \to F^2$ is a double cover. Roughly, this corresponds to an index two subgroup of $\pi_1(F)$, and hence to a homomorphism $\pi_1(F) \to \mathbb{Z}/2\mathbb{Z}$. Then for all $x \in F$, $|\rho^{-1}(x)| = 2$, so there is a canonical involution $\tau\colon G \to G$, where $\tau(y)$ is defined to be the unique element of $\rho^{-1}(\rho(y)) - \{y\}$. For an example, see Figure 1.

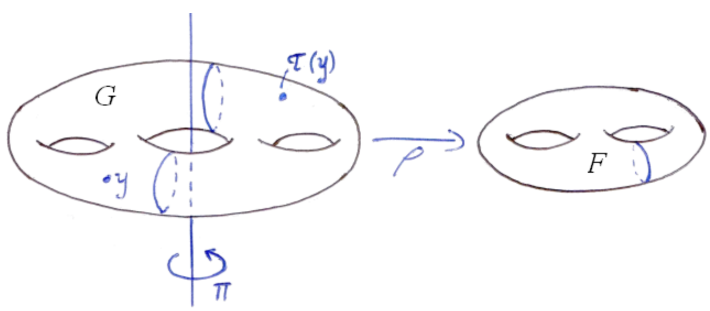

Figure 1: Here the involution τ is rotation by π about an axis.

Define $T = (G \times I)/\sim$, where $(y, 0) \sim (\tau(y), 0)$. Then $P\colon T \to F$ given by $(y, t) \mapsto \rho(y)$ is an I–bundle over F. Now suppose that $\rho\colon G \to F$ is the orientation double cover; so $G = F \times \{0, 1\}$ if F is orientable, and G is orientable if F is not; for example $\mathbb{T}^2 \xrightarrow{\times 2} \mathbb{K}^2$ (Figure 2).

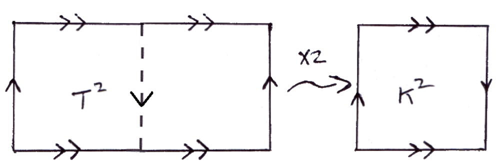

Figure 2: The torus is a double cover for the Klein bottle.

Then $P\colon T \to F$ as above is called the *orientation I–bundle* (Figure 3).

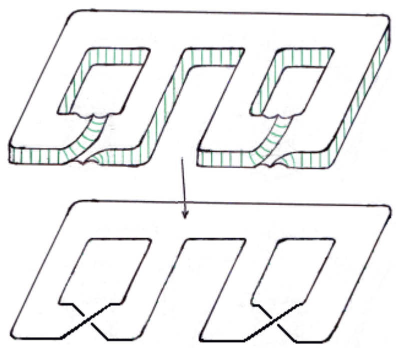

Figure 3: The orientation I–bundle over $\mathbb{K}^2 - \text{int}\left(\mathbb{D}^2\right)$.

We have the following:

Theorem 7.1. *Suppose that $(F^2, \partial M) \subset (M^3, \partial M)$ is properly embedded. Then $N(F)$ is bundle equivalent to an I–bundle over F. If additionally M is orientable, then $N(F)$ is bundle equivalent to the orientation I–bundle over F.*

Example 7.1. Figure 4 shows the I–bundle for $\mathbb{T}^2$.

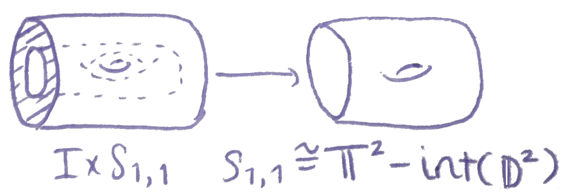

Figure 4: The orientation I–bundles are the only I–bundles one can draw in three-space.

Definition 7.1. We say that $F \subset M$ is *two-sided* if F separates $N(F)$. Otherwise F is *one-sided*.

Example 7.2. The core curve α in the Möbius band $\mathbb{M}^2$ is one-sided. $\mathbb{D}^2 \times \{p\} \subset \mathbb{D}^2 \times S^1$ is two-sided for any $p \in S^1$. We can also find a Möbius band in $\mathbb{D}^2 \times S^1$ that is one-sided. $\mathbb{M}^2 \times \left\{\frac{1}{2}\right\}$ is two-sided in $\mathbb{M}^2 \times I$; see Figure 5.

Exercise 7.1. If $F \subset M$ is properly embedded, give a relationship between the orientability of M and F, and the number of sides of F.

Definition 7.2. If $\rho\colon T \to F$ is an I–bundle, then $X \subset T$ is *vertical* if X is a union of fibres.

Definition 7.3. The *vertical* boundary of an I–bundle $\rho\colon T \to F$ is $\partial_v T := \rho^{-1}(\partial F)$.

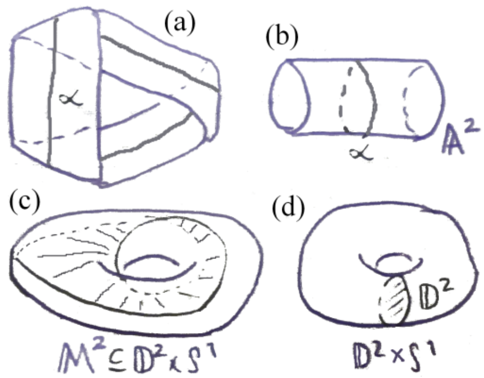

Figure 5: (a) α is one-sided in $\mathbb{M}^2$. (b) α is two-sided in $\mathbb{A}^2$ (c) $\mathbb{M}^2$ is one-sided in $\mathbb{D}^2 \times S^1$ (d) $\mathbb{D}^2$ is two-sided in $\mathbb{D}^2 \times S^1$.

Definition 7.4. The *horizontal* boundary of an I–bundle $\rho\colon T \to V$ is $\partial_h T = \partial T - \mathrm{int}(\partial_v T)$.

Exercise 7.2. $\partial_v T$, $\partial_h T$ and the zero section are all incompressible in T, except for $\partial_v T$ when $T = I \times \mathbb{D}^2$.

Exercise 7.3. If $\partial F \neq \varnothing$, F is compact and connected, and $\rho\colon T \to F$ is the orientation I–bundle, then T is a handlebody.

Before moving on, we summarize examples of 3–manifolds discussed so far.

Example 7.3. We have seen:

(i) S^3, $\mathbb{P}^3$ and $\mathbb{T}^3$, which are closed.

(ii) V_g, the handlebodies.

(iii) I–bundles and S^1–bundles over surfaces.

8 Lecture 8: Triangulations

Definition 9.1. Define the *k–simplex* by:

$$\Delta^k = \left\{(x_0, \ldots, x_k) \in \mathbb{R}^{k+1} : \sum x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i\right\}$$

Definition 9.2. The *facet* $\delta_I \subset \Delta^k$ is the subsimplex of the form:

$$\delta_I = \{(x_0, ..., x_k) \in \Delta^k : x_i = 0 \text{ for all } i \in I\}$$

Definition 9.3. If $\delta \subset \Delta$ and $\delta' \subset \Delta'$ are *faces* (codimension 1 facets), then a *face pairing* is an isometry $\varphi\colon \delta \to \delta'$.

Definition 9.4. We call a collection T of simplices and face pairings a *triangulation*.

Remark. We require that for every face pairing $\varphi \in T$ that if $\varphi\colon \delta \to \delta'$ then $\delta \neq \delta'$.

Definition 9.5. The number of simplices is written $|T|$. The *underlying space* is written $||T||$, and is defined by:

$$||T|| := \left(\bigsqcup \Delta_i\right) / \{\varphi_j\}$$

Definition 9.6. The quotient map is given by $\pi\colon \bigsqcup \Delta_i \to ||T||$ and we define $\pi_i\colon \Delta_i \to ||T||$ by restriction: $\pi_i = \pi|\Delta_i$.

Example 9.1. If T is the pair of simplices in Figure 6 with face pairings given by the arrows, then $||T|| \cong \mathbb{T}^2$.

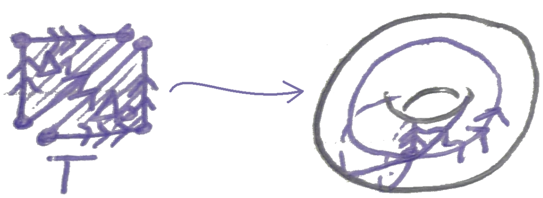

Figure 6: $||T|| \cong \mathbb{T}^2$.

Similarly, if we draw T as in Figure 7 then $||T|| = \mathbb{M}^2$.

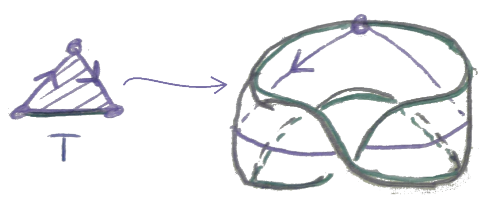

Figure 7: $||T|| \cong \mathbb{M}^2$.

Exercise 9.1. Find necessary and sufficient combinatorial conditions on T so that $||T||$ is a (PL) manifold of dimension 1, 2 or 3.

Hauptvermutung (Moise). *Every topological* 3*–manifold admits a triangulation, unique up to subdivision. In particular, for any* M^3*, there exists a triangulation* T *such that* $||T|| \cong M$*.*

Remark. This is one important step in showing, in dimension three, that the categories TOP, PL and DIFF are all equivalent.

Definition 9.7. Suppose (M^3, T) is a triangulated manifold. An *orientation* of M is a choice of orientation for all $\Delta \in T$, such that all face pairings reverse the induced orientation on faces.

Example 9.2. The annulus is orientable, but the Möbius band is not. See Figure 8.

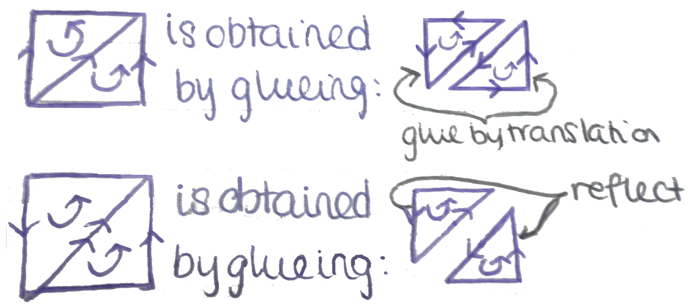

Figure 8: The annulus is orientable as all face pairings reverse the induced orientation on faces.

Proposition 9.1 (Proposition 6.5 in Lackenby)**.** *An n–manifold (M^n, T) is orientable if and only if for every simple closed curve $\alpha \in M$ we have $N(\alpha) \cong \mathbb{B}^{n-1} \times S^1$.*

Remark. We can also determine orientability in DIFF using sign(det(Dh)) where h ranges over the overlap maps, as in Figure 9. We can also define orientation in TOP using homology.

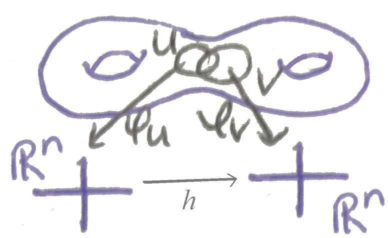

Figure 9: Orientation in DIFF arises from overlap maps of charts.

Definition 9.8. Define $\Delta^{(k)}$ to be the union of k–dimensional facets of Δ. If (M, T) is a triangulated 3–manifold, define $M^{(k)}$, the *k–skeleton* of M to be the manifold with triangulation $T = \bigcup_{i=1}^{|T|} \pi_i(\Delta^{(k)})$. Figure 10 shows the k-skeleta of Δ.

Example 9.3. Figure 11 shows two examples of identifications.

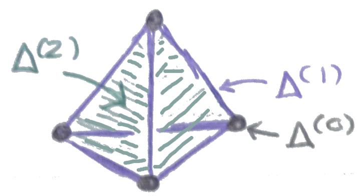

Figure 10: The k–skeleta of Δ.

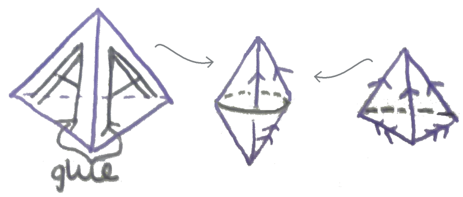

Figure 11: Two different views of the same triangulation for $\mathbb{B}^3$.

Exercise 9.2. Verify that the triangulation in Figure 12 is a three-manifold, and recognise it.

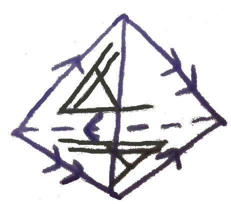

Figure 12: Which three-manifold is this?

Definition 9.9. An isotopy $F\colon M \times I \to M$ is *normal* with respect to a triangulation T of M if for all $t \in I$, the homeomorphism F_t preserves $M^{(k)}$ for all k, and $F_0 = \mathrm{Id}_M$. See Figure 13 for an example.

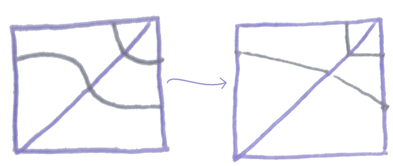

Figure 13: A normal isotopy.

Remark. Thus $M^{(0)}$ is fixed pointwise, and all other facets are fixed setwise.

Definition 9.10. Say an arc $(\alpha, \partial\alpha) \subset (\Delta^2, \partial\Delta)$ is *normal* if the points of $\partial\alpha$

are in distinct edges of Δ, and $\alpha \cap \Delta^{(0)} = \varnothing$. See Figure 14 for some examples and a non-example.

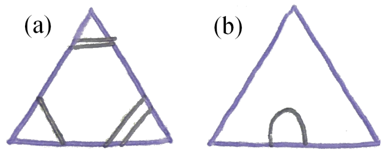

Figure 14: (a) Normal arcs. (b) This is not a normal arc.

Definition 9.11. A disk $(D, \partial D) \subset (\Delta^3, \partial\Delta)$ is a *normal disk* if ∂D is transverse to $\Delta^{(1)}$, ∂D meets each edge of $\Delta^{(1)}$ at most once, and $D \cap \Delta^{(0)} = \varnothing$. See Figures 15(a) and (b) for examples and 15(c) and (d) for non-examples.

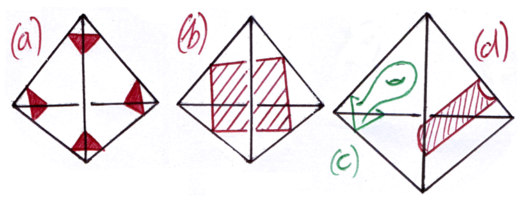

Figure 15: (a) There are four normal triangles. (b) There are three normal quadrilaterals. (c) This is not even a disk, let alone normal. (d) This is also not a normal disc.

Exercise 9.3. Prove that:

(i) There are only three normal arcs up to normal isotopy.

(ii) There are only seven normal disks up to normal isotopy.

Recall that $\pi_i \colon \Delta_i \to M$ is defined by $\pi_i = \pi|\Delta_i$, where π is the quotient map.

Definition 9.12. Suppose $S \subset M$ is a surface. Say S is normal if $\pi_i^{-1}(S)$ is a disjoint collection of *normal* disks for all i.

Example 9.4. The three normal disks in the tetrahedron shown in Figure 16 give a normal surface under the identification indicated by the arrows.

Exercise 9.4. Show that, with triangulations as in Figure 17, (a) and (b) are three manifolds, and recognise them.

Theorem 9.2 (Haken-Kneser Finiteness)**.** *Suppose (M, T) is a connected, compact triangulated 3–manifold. Suppose $S \subset (M, T)$ is an embedded normal surface. Then if $|S| \geq 20|T| + 1$ there are components $R, R' \subset S$ so that R, R' cobound a product component of $M - S$.*

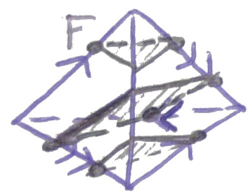

Figure 16: Recognise the normal surface F by computing $|\partial F|$, $\chi(F)$ and the orientability.

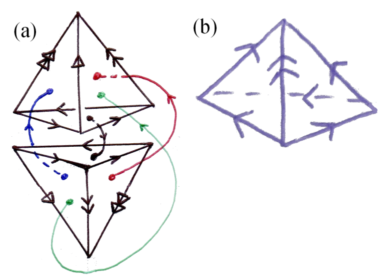

Figure 17: Show that (a) and (b) are three manifolds and recognise them.

Remark. Figures 18 and 19 show examples of parallel surfaces.

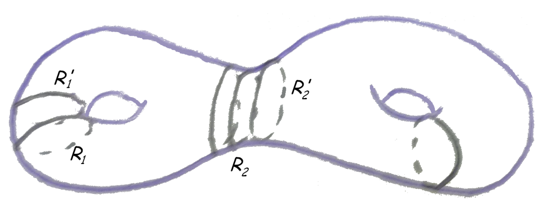

Figure 18: Here both R_1 & R_1' and R_2 & R_2' bound copies of $\mathrm{D}^2 \times I$.

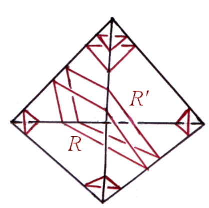

Figure 19: R and R' bound a product.

Proof of Theorem 9.2. Recall that $S \cap \Delta$ for $\Delta \in T$ is a finite collection of normal disks. Consider the subcollection of disks of a fixed *type*, that is a normal isotopy class. Call the outermost disks *ugly*, the second outermost disks *bad*, and all other disks *good*, as illustrated in Figure 20.

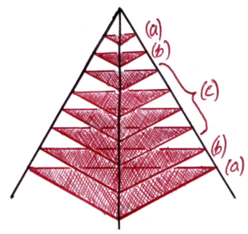

Figure 20: (a) Ugly disks. (b) Bad disks. (c) Good disks.

Thus there is a component $F \subset S$, such that F is a union of good disks. To see this, note that there are at most $20|T|$ ugly and bad disks in total. There are at most five types of disk in each $S \cap \Delta$, and at most four of each can be ugly or bad; see Figures 21(a) and (b).

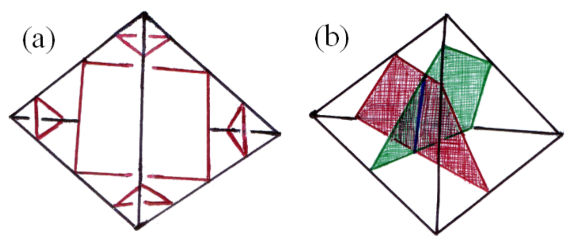

Figure 21: (a) There are at most five types of disk in each $S \cap \Delta$ because (b) two normal quadrilaterals of different types must intersect.

Now let N be the closure of the union, over all Δ_i, of all components of $\Delta_i - S$ that are adjacent to F, as in Figure 22.

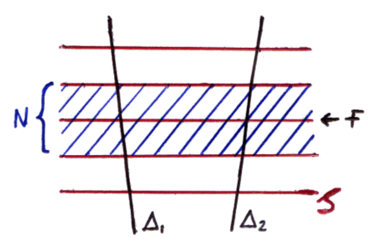

Figure 22: N is the closure of the union over all Δ_i of all components of $\Delta_i - S$ that are adjacent to F.

Exercise 9.5. Prove that N is an I–bundle and either N is ambient isotopic to $N(F)$ *or* F is two-sided and parallel to $\partial_h N$.

Please let me know if any of the problems are unclear or have typos.

Exercise 3.1. Suppose that $\alpha \subset F^2$ is a simple closed curve. Show that $\alpha = \mathbb{1} \in \pi_1(F)$ if and only if α bounds a disk in F.

Exercise 3.2. [Medium] Suppose that $F^2 \subset S^3$ is smoothly embedded, connected and without boundary. Prove that if F is not a sphere then F compresses. [Idea: rewrite the proof of Alexander's theorem.]

Exercise 3.3. [Jordan-Brouwer separation] Suppose that $F^2 \subset S^3$ is smoothly embedded, connected and without boundary. Show that F separates S^3. Deduce that F is two-sided. Deduce that F is orientable.

Exercise 3.4. Prove that every smoothly embedded two-torus $T^2 \subset S^3$ bounds a solid torus ($D^2 \times S^1$) on at least one side. Find an embedded S_2 in S^3 that does not bound a handlebody on either side.

Exercise 3.5. Suppose that $\rho\colon M' \to M^3$ is a covering map. Show that if M' is irreducible then M is as well.

Exercise 3.6. Suppose that $\rho\colon M' \to M^3$ is a covering map. Suppose that $F \subset M$ is an properly embedded surface. Show that if $F' = \rho^{-1}(F)$ is incompressible then F is as well.

Exercise 3.7. [Hard] Show that all I–bundles over S^2 are equivalent. Find infinitely many inequivalent S^1–bundles over S^2.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Anna Lena Winstel
Assisted by Matthew Pressland
and David Kitson

Lecture 10

We recall properties of π_1:

Definition 10.1. Suppose A and B are groups. Then if $A = \langle a_i \mid r_k \rangle$ and $B = \langle b_j \mid s_l \rangle$, their free product $A * B$ is given by

$$A * B = \langle a_i, b_j \mid r_k, s_l \rangle .$$

Theorem 10.1 (van Kampen)**.** *If $W = X \cup_Z Y$ and Z is path connected (as in Figure 1), then, choosing a base point $p \in Z$, $\pi_1(W,p) \cong \pi_1(X,p) * \pi_1(Y,p)/N$, where N is the normal subgroup generated by:*

$$\{i_*(z)(j_*(z))^{-1} : z \in \pi_1(Z,p)\}$$

where $i : Z \hookrightarrow X$ and $j : Z \hookrightarrow Y$ are the inclusions.

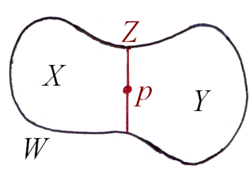

Figure 1: If $W = X \cup_Z Y$, then $\pi_1(W) = (\pi_1(X) * \pi_1(Y))/N$.

Corollary 10.1. *If $\pi_1(Y,p) = \{\mathbf{1}\}$ then $\pi_1(W,p) = \pi_1(X,p)/N$ where N is the normal subgroup generated by:*

$$\{i_*(z) : z \in \pi_1(Z,p)\}.$$

Corollary 10.2. *If $\pi_1(Z,p) = \{\mathbf{1}\}$ then $\pi_1(W,p) = \pi_1(X,p) * \pi_1(Y,p)$.*

Proposition 10.3. *If (M,T) is triangulated then $\pi_1(M) = \pi_1\left(T^{(2)}\right)$.*

Exercise 10.1. Prove Proposition 10.3. See Figure 2 for a hint.

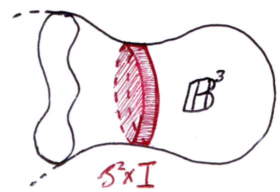

Figure 2: Hint: Attach 3-balls one by one.

Proposition 10.4. *$\pi_1(T^{(2)}) = \pi_1(T^{(1)})/N$ where N is the normal subgroup generated by boundaries of two-simplices in T. Note that $\pi_1(T^{(1)})$ is a free group, as $T^{(1)}$ is a connected graph.*

We now give several example computations.

Example 10.1. Consider Figure 3, where the faces are glued according to the arrows.

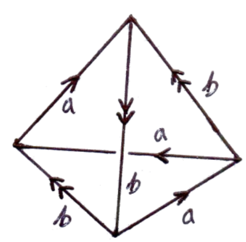

Figure 3: What is the fundamental group of this manifold?

Exercise 10.2. Check that this is a 3–manifold.

Step 1: Find a spanning tree for $T^{(1)}$. Here $T^{(1)}$ is the graph shown in Figure 4 and so the spanning tree is just the vertex.

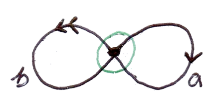

Figure 4: $T^{(1)}$ in this case. The spanning tree is the single vertex circled in green.

Step 2: Give labels to the non-tree edges of $T^{(1)}$, as in Figure 4.

Step 3: Read off relations from faces of $T^{(2)}$. There is one relation per face in the quotient. Here we have $\langle a, b \mid a^2 = b,\ b^2 a = \mathbf{1} \rangle$.

Step 4: (optional) Use Tietze transformations to simplify:

$$\langle a, b \mid a^2 = b,\ b^2a = \mathbf{1}\rangle \cong \langle a \mid (a^2)^2a = \mathbf{1}\rangle \cong \mathbb{Z}/5\mathbb{Z}.$$

Example 10.2. (A non-Abelian example.) The *one-quarter turn space* Q is the quotient of the unit cube as shown in Figure 5:

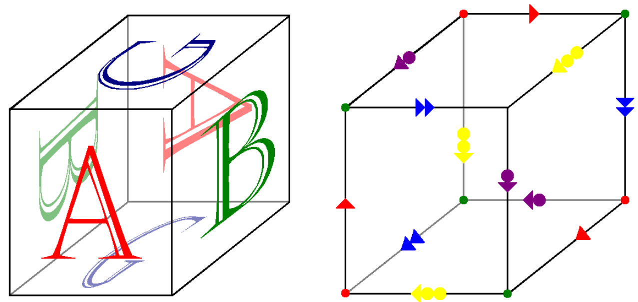

Figure 5: Two visualisations of how to glue faces to get Q.

Step 1: The 1–skeleton is the graph in Figure 6(a) with four edges and two vertices. We take the circled edge as the spanning tree.

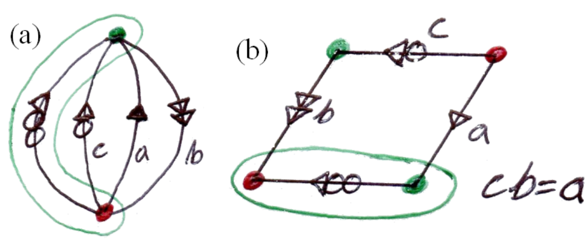

Figure 6: (a) The 1–skeleton and spanning tree. (b) After labelling the non-tree edges, read off relators from the faces. Edges of the spanning tree do not contribute to the relators.

Step 2: Label the non-tree edges with a, b, c.

Step 3: The three squares give relations and we have the following presentation

$$\pi_1(Q) = \langle a, b, c \mid a = cb,\ ba = c,\ abc = \mathbf{1}\rangle\,.$$

Exercise 10.3. Recognize $\pi_1(Q)$. In particular, it is not Abelian.

Abelian groups

Definition 10.2. Suppose that Z is an Abelian group. Define $N := \{z \in Z : z$ is finite order$\}$. Then $N < Z$ is called the *torsion subgroup* of Z.

Recall that $A \oplus B$ is the *direct product* of A and B.

Proposition 10.5. *Suppose Z is a finitely generated Abelian group. Then there exist unique $k \in \mathbb{N}$ and N a finite group so that $Z \cong \mathbb{Z}^k \oplus N$.*

Proof. This follows from the classification of finitely generated Abelian groups. □

Definition 10.3. We call k the *rank* of Z, and use the notation $\mathrm{rk}(Z) = k$.

Definition 10.4. Let G be any (finitely generated) group. The *commutator subgroup* of G is $[G, G]$, the subgroup of G generated by all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$.

$$[G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle \triangleleft G.$$

Definition 10.5. We define the *Abelianization* of G to be $G^{\mathrm{Ab}} = G/[G, G]$.

Definition 10.6. We define the *first homology group* of M^3 to be $H_1(M, \mathbb{Z}) := [\pi_1(M)]^{\mathrm{Ab}}$.

Example 10.3. Let $M^3 = N^3 \# P^3$. Then it follows by van Kampen's theorem that $\pi_1(M) \cong \pi_1(N) * \pi_1(P)$. Therefore $H_1(M) = H_1(N) \oplus H_1(P)$.

Exercise 10.4. Show that $(A * B)^{\mathrm{Ab}} = A^{ab} \oplus B^{\mathrm{Ab}}$.

Example 10.4. As in the last example we have that

$$\pi_1(\#_g S^2 \times S^1) = F_g \cong *_g \mathbb{Z},$$

so $H_1(\#_g S^2 \times S^1) = \mathbb{Z}^g$ has rank g. We denote $\#_g S^2 \times S^1$ by M_g.

Proposition 10.6. *If M is connected, orientable, compact and $M \cong N \# M_g$, then $g \leq \mathrm{rk}(H_1(M))$.*

Note here that π_1 is finitely generated since M is compact.

Proof. We know that $H_1(M) = H_1(N) \oplus H_1(M_g)$, so:

$$\mathrm{rk}(H_1(M)) = \mathrm{rk}(H_1(N)) + g.$$ □

This is the first step in the existence proof for connect sum decompositions. For the next step, we need the following proposition:

Proposition 10.7. *Suppose M is connected, orientable and compact. Then there exists a decomposition*

$$M \cong \#_{i=1}^k N_i \# (\#_g S^2 \times S^1) \# (\#_n \mathbb{B}^3)$$

where each N_i is irreducible and not S^3, $\mathbb{B}^3$ or $S^2 \times S^1$.

Proof.

Step 1: Let n be the number of components of ∂M that are 2–spheres. Let F be the frontier of a "tree-like" union of arcs and two-sphere boundary components, as shown in Figure 7. Form $M - n(F)$ and cap off $F^{\pm}$ by 3–balls. From now on we assume that $n = 0$.

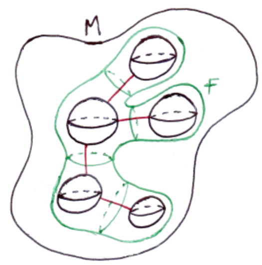

Figure 7: F is the frontier of a "tree-like" union of arcs and two-sphere boundary components.

Step 2: Proposition 10.6 gives us an upper bound on the number of summands of M homeomorphic to $S^2 \times S^1$. Thus from now on we may assume that $g = 0$. It follows that any 2–sphere embedded in M separates.

For Step 3, we require the following definitions.

Definition 10.7. We define $S^3_k := \#_{i=1}^k \mathbb{B}^3$ and we call this a *ball with holes* or a *punctured sphere*. See Figure 8.

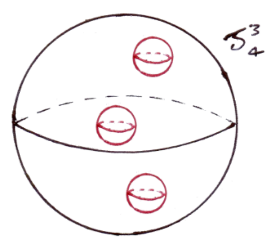

Figure 8: Here, $n = 4$.

Exercise 10.5. Show that $(\#_n \mathbb{B}^3) \cup_{S^2} (\#_m \mathbb{B}^3) \cong \#_{n+m-2} \mathbb{B}^3$.

Definition 10.8. We call $S \hookrightarrow M$ a *sphere system* if S is an embedding of a disjoint collection of 2–spheres; see Figure 9.

Definition 10.9. A system $S \hookrightarrow M$ is *reduced* if no component of $M - n(S)$ is homeomorphic to a punctured sphere. The sphere system in Figure 9 is reduced.

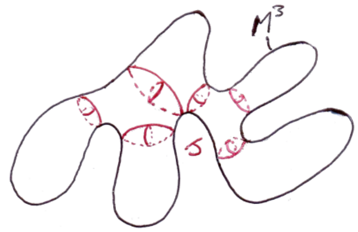

Figure 9: A reduced sphere system S in M.

Lectures 11 and 12

Step 3: If M is irreducible we are done. If $M \cong S^3$ we are done by Alexander's theorem. So suppose that M contains an essential 2–sphere. For the reminder of the proof, we fix a finite triangulation T of M. So our assumptions give us a reduced sphere system $S \subset M$.

Normalization Lemma. For any reduced sphere system $S \subset M$ there is a normal, reduced sphere system S' such that $|S'| \geq |S|$.

If we assume this lemma, we get the following proposition:

Proposition 11.8. *(Existence) Let M be defined as above. Then $M \cong \#_{i=1}^{n} N_i$ such that all N_i are irreducible and $N_i \ncong S^3, \mathbb{B}^3$.*

Proof. Let S_1 denote an essential 2–sphere, so $M = N_1 \#_{S_1} N_2$. If N_1 is homeomorphic to $\#_k \mathbb{B}^3$ for $k \geq 1$, then we have a contradiction. So, $S = \{S_1\}$ is a reduced sphere system. Let $\overline{S}$ be a maximal sphere system (i.e. of maximal size). This exists because any normal reduced system has at most $20|T|$ components; this follows from the Haken-Kneser finiteness and the normalization lemma. Since $\overline{S}$ is maximal , if we cut M along $\overline{S}$ and cap off with 3–balls the resulting manifolds $\{N_i\}$ are all irreducible. $\square$

To prove the normalization lemma, we must *normalize* the given system S.

Proof of Normalization Lemma. Isotope S to be transverse to $T^{(k)}$ for $k = 0, 1, 2$, i.e. $S \cap T^{(0)} = \varnothing$, $|S \cap T^{(1)}| =: w(S)$ (the *weight* of S) is finite, $S \cap T^{(1)}$ is transverse and $S \cap \partial\Delta_i$ is a finite collection of simple closed curves; see Figure 10. We alternatingly apply *surgery* and the *baseball move.*

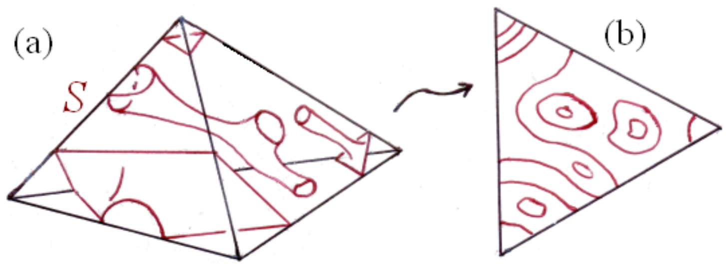

Figure 10: (a) The sphere system can look unpleasant in the triangulation. (b) A possible picture of $S \cap T^{(2)}$.

Surgery: Suppose $(D, \partial D) \subset (M, S)$ is a surgery disk, i.e. $D \cap S = \partial D$. Suppose $D \cap S \subset F$ is a component of S. As before, define $F_D = F - n(D) \cup D^+ \cup D^-$. Define $S_D = (S - F) \cup F_D$. Notice that ∂D separates F, so $F_D = F^+ \cup F^-$. See Figure 11.

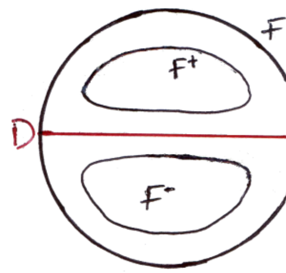

Figure 11: Notice that ∂D separates F, so $F_D = F^+ \cup F^-$.

Let $X, Y \subset M - n(S)$ be the components adjacent to F and suppose $D \cap X \neq \emptyset$. So let $X^+ \cup X_0 \cup X^- = X - n(F_D)$ where X_0 meets D and $X^\pm$ are adjacent to $F^\pm$, respectively. See Figure 12.

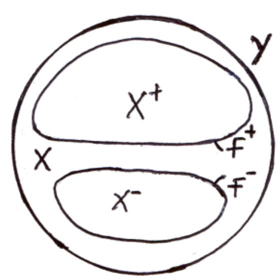

Figure 12: $X^+ \cup X_0 \cup X^- = X - n(F_D)$, where X_0 meets D, and $X^\pm$ are adjacent to $F^\pm$.

Note that $X_0 \cong \#_3 \mathbb{B}^3$. Since we assumed S is a reduced sphere system, we find Y is not a punctured sphere.

Exercise 11.6. $Y \cup_F X_0$ is not a punctured sphere.

Claim. At most one of X^+, X^- is a punctured sphere.

Proof. If both are punctured spheres then so is $X = X^+ \cup_{F^+} X_0 \cup_{F^-} X^-$, a contradiction. This proves the claim. $\square$

Let $S' = S - F$ thus either $S^+ = S' \cup F^+$ or $S^- = S' \cup F^-$ or $S_D = S' \cup F_D$ is a reduced system.

Using surgery: For every tetrahedron $\Delta \in T^{(3)}$, the surface S meets $\partial\Delta$ is a collection of simple closed curves. See Figure 13 for a possible intersection pattern.

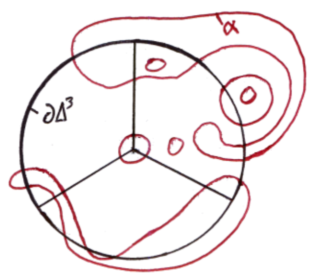

Figure 13: A possible intersection of S with the boundary of a tetrahedron.

For every simple closed curve $\alpha \subset \partial\Delta \cap S$ we do the following. Pick a disk $D \subset \partial\Delta$ bounded by α. Isotope D into Δ (∂D stays in S), as in Figure 14.

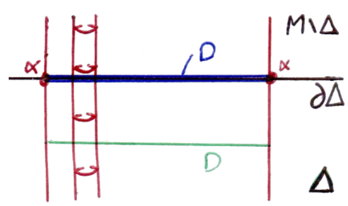

Figure 14: Isotope D into Δ.

Use D (in Δ) to surger all curves of $S \cap D$, innermost first. When this is done, $S \cap \Delta$ is a collection of disks (for all Δ).

Claim. After surgery, for all Δ and for all simple closed curves $\alpha \subset \partial\Delta \cap S$, α meets $\Delta^{(1)}$.

Proof. Suppose α has weight 0 and $\alpha \subset f \subset \Delta^{(2)}$ a face. We surgered along both $D^{\pm}$, so the component sphere containing α bounds a ball as in Figure 15.

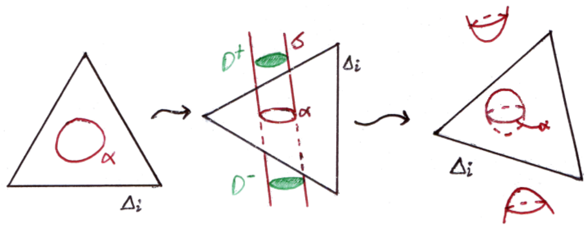

Figure 15: We surgered along both $D^{\pm}$, so the component sphere containing α bounds a ball.

But surgery deletes trivial spheres. This proves the claim. $\square$

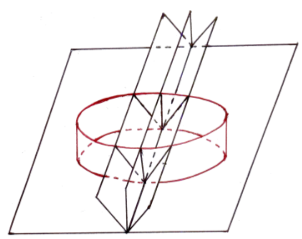

Figure 16: The intersection of S with the two-skeleton; outside of Δ it can be complicated.

Please let me know if any of the problems are unclear or have typos.

Exercise 4.1. Show that there are exactly two inequivalent $\mathbb{B}^k$–bundles over S^1. The same holds for S^1– and S^2–bundles over S^1.

Exercise 4.2. Prove that M^n is orientable if and only if for every simple closed curve $\alpha \subset M$ the regular neighborhood $N(\alpha)$ is a trivial bundle.

Exercise 4.3. Classify, up to bundle equivalence, two-fold covers of $\mathbb{T}^2$. Do the same for S_2, the closed orientable surface of genus two. Which of these can you give pictures for?

Exercise 4.4. Suppose that $\rho\colon T \to F^2$ is an $\mathbb{B}^1$–bundle. Show that the vertical boundary $\partial_v T$, the horizontal boundary $\partial_h T$, and the zero-section are all incompressible in T. (Here we exclude the case of the product I–bundle over D^2.) On the other hand: show that if F has boundary and T is orientable, then T is a handlebody. Thus ∂T is often compressible.

Exercise 4.5. Suppose that M is an irreducible three-manifold and $F, G \subset \partial M$ are disjoint, incompressible subsurfaces. Suppose that $\phi\colon F \to G$ is a homeomorphism. Suppose that $H \subset M$ is a properly embedded incompressible surface and disjoint from F and G. Show that the image of H is M/ϕ is incompressible. Deduce that if that $\rho\colon T \to S^1$ is an F^2–bundle then all fibers (point preimages) are incompressible.

Exercise 4.6. [Easy] Suppose that $V = V_2$ is a handlebody of genus two. Prove or find a counterexample: any surface properly embedded in V is compressible or is ambient isotopic into a regular neighborhood of ∂V.

Exercise 4.7. [Medium] Suppose that T is a finite triangulation. Give necessary and sufficient combinatorial conditions so that $||T||$ is homeomorphic to a topological manifold M^n, for $n \leq 3$.

Suppose that (F, T) is a triangulated surface. Call a simple closed curve $\alpha \subset F$ *normal* if α is transverse to the skeleta of T and $\alpha \cap \Delta$ is a finite collection of normal arcs, for every triangle $\Delta^2 \subset F$. The *weight* of α is $w(\alpha) = \left|\alpha \cap T^{(1)}\right|$.

Exercise 4.8. Suppose that (F, T) is the boundary of a three-simplex. Show that if α is a normal curve and if α meets every edge of $T^{(1)}$ at most once then α has weight three or four. Deduce that there are seven normal disks in a tetrahedron.

Exercise 4.9. In the proof of Haken-Kneser finiteness we defined N to be the closure of the union, over all $\Delta \in T^{(3)}$, of all components of $\Delta - S$ meeting F. Prove that N is an I–bundle and either N is ambient isotopic to $N(F)$ *or* F is two-sided and parallel to $\partial_h N$.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Matthew Pressland
Assisted by Anna Lena Winstel and David Kitson

Lecture 13

Proof. We complete the proof of the existence of connect sum decomposition.

Procedure 2: Baseball move. We perform this move after surgery along all curves of $S \cap \partial\Delta$ for all $\Delta^3 \in T$. Suppose α is a simple closed curve of $S \cap \partial\Delta$, where $\Delta^3 \in T$. So α bounds disks D_0 and D_1 in $\partial\Delta$. Suppose that there is an edge $e \in \Delta^{(1)}$ with $|\alpha \cap e| \geq 2$, as illustrated in Figure 1.

Exercise 13.1. Without loss of generality, there is a component $d \subset D_0 \cap e$ such that $d \cap \Delta^{(0)} = \varnothing$, as in Figure 1.

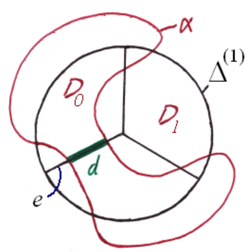

Figure 1: α bounds two disks D_0 and D_1, and there is an edge $e \in \Delta^{(1)}$ such that $|\alpha \cap e| = 2$.

Now let $D = D_0$. By an innermost arc argument we may assume that $d \cap S = \partial d$. Let $D' \subset S \cap \Delta$ be the disk bounded by α, as in Figure 2.

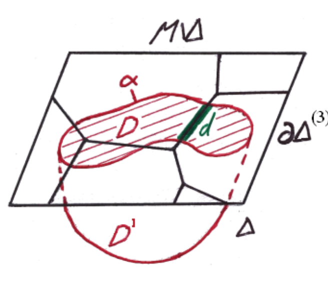

Figure 2: D' is the disk bounded by α.

Since $D \cup D' \cong S^2$, they cobound a three-ball, B, by Alexander's theorem, and so we may choose an embedded arc $d' \subset D'$ so that d and d' cobound a disk $E \subset B$, as in Figure 3.

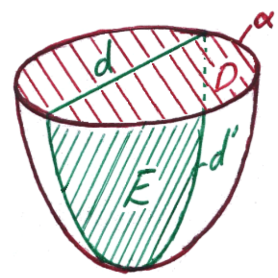

Figure 3: The arcs d and d' cobound a disk $E \subset B$.

Let C be the 3–ball obtained from $N(E)$ by cutting along S and retaining the component containing E; see figure 4.

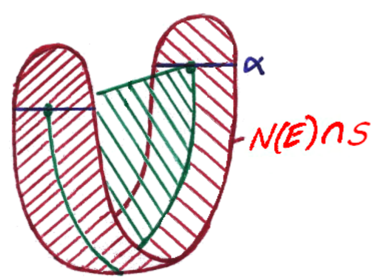

Figure 4: A picture of $N(E) \cap S$.

Write $\partial_- C = C \cap S$ and $\partial_+ C = \overline{\partial C - \partial_- C}$. The *baseball curve* is the common boundary $\partial\partial_+ C = \partial\partial_- C$, as in Figure 5.

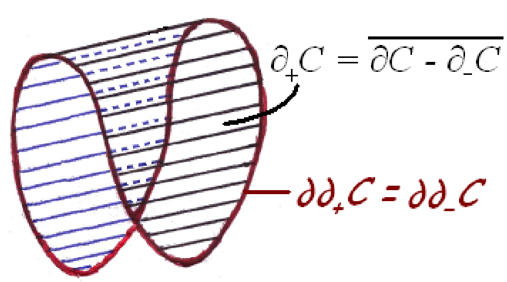

Figure 5: The baseball curve is the common boundary $\partial\partial_+ C = \partial\partial_- C$.

Since C is a 3–ball, there is an isotopy, called the *baseball move*, taking $\partial_- C$ to $\partial_+ C$; see Figures 6(a) or (b). This gives an isotopy of S to S'. Notice that $w(S') = w(S) - 2$.

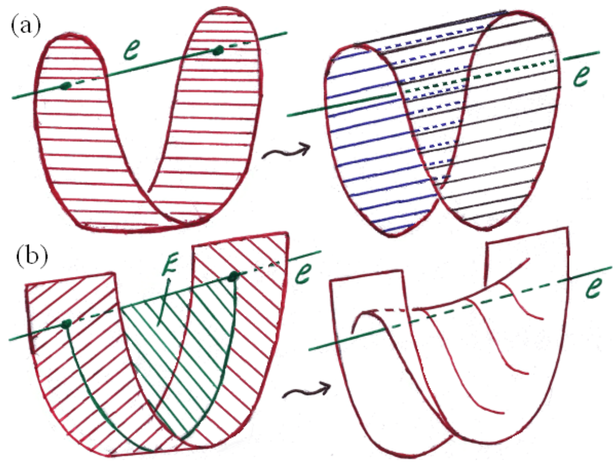

Figure 6: Two visualisations of the baseball move.

So alternate between surgery along all curves and single baseball moves. As $w(S)$ is decreasing, this process terminates with S in normal position. If $w(S) = 0$ then $S = \varnothing$ and this is a contradiction as surgery never decreases the initial number of essential spheres. So this completes the proof of existence. $\square$

Following Hatcher, for uniqueness we use lemma 13.1.

Definition 13.1. If M is a 3–manifold, define $\widehat{M}$ to be M with all $S^2 \subset \partial M$ capped off by 3–balls, and discarding 3–sphere components.

Lemma 13.1. *Suppose that $S \subset M$ is a sphere system (not necessarily reduced) so that:*

$$\widehat{M - n(S)} = \bigsqcup_{i=1}^{k} N_i$$

is a disjoint union of irreducible manifolds. Suppose that $(D, \partial D) \subset (M, S)$ is a surgery disk. Then:

$$\widehat{M - n(S_D)} = \bigsqcup_{i=1}^{k} N_i.$$

Exercise 13.2. Prove this lemma. For a hint, see Figure 7.

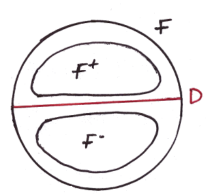

Figure 7: Hint for Exercise 13.2.

So we may now complete the proof of uniqueness of prime decomposition.

Proof of uniqueness. Suppose S and T are sphere systems so that:

$$M - n(S) = \bigsqcup_{i=1}^{k} P_i$$

and

$$N - n(T) = \bigsqcup_{j=1}^{l} Q_j$$

where the P_i and Q_j are irreducible. Now, if $S \cap T = \varnothing$ we have:

$$\begin{aligned}
\bigsqcup P_i &= \widehat{\bigsqcup P_i - n(T)} \\
&= \widehat{M - n(S \cup T)} \\
&= \widehat{\bigsqcup Q_j - n(S)} = \bigsqcup Q_j
\end{aligned}$$

On the other hand, if $S \cap T \neq \varnothing$ then surger S along an innermost disk of T and apply Lemma 13.1. Finally, if $M \cong N\#\left(\#_l S^2 \times S^1\right)$ and $M \cong N\#\left(\#_k S^2 \times S^1\right)$ then:

$$\operatorname{rank}(H_1(N)) + l = \operatorname{rank}(H_1(M)) = \operatorname{rank}(H_1(N)) + k$$

and so $l = k$. $\square$

Lecture 14

Exercise 14.3. Suppose that (M, T) is orientable, compact, connected, irreducible and triangulated. Suppose $F \subset M$ is embedded, closed ($\partial F = \varnothing$, compact) and orientable. Show that if G is incompressible, it is isotopic to a normal surface.

Definition 14.2. Say F properly embedded in M is *boundary parallel* if there is an isotopy (relative to ∂F) pusing F into ∂M. More precisely, there is an isotopy $H\colon F \times I \to M$ such that:

(i) H_t is an embedding of F into M for all $t < 1$.

(ii) H_1 is an embedding of F into ∂M.

(iii) $H_0 = \mathrm{Id}$.

(iv) $H_t|\partial F = \mathrm{Id}$.

Equivalently $M - n(F)$ has a component $X \cong F \times I$ with $F \times \{0\} = F^+ \subset N(F)$ and $F \times \{1\} \subseteq \partial M$. See Figure 8.

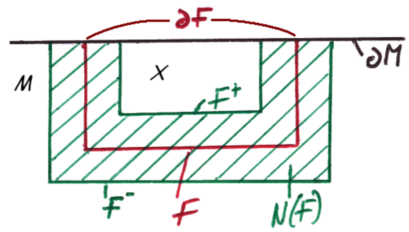

Figure 8: F is boundary parallel to M.

Example 14.1. (See Figure 9)

(i) The equatorial disk $\mathbb{B}^2 \subset \mathbb{B}^3$ is boundary parallel.

(ii) Take $K \subset T = \partial(\mathbb{D}^2 \times S^1)$. Let $N(K)$ be a closed neighbourhood in $\mathbb{D}^2 \times S^1$. Let $G = N(K) \cap T$. So $G \subset T = \partial(\mathbb{D}^2 \times S^1)$. Let $F = \overline{\partial N(K) - G}$, so F is boundary parallel; in fact parallel to G.

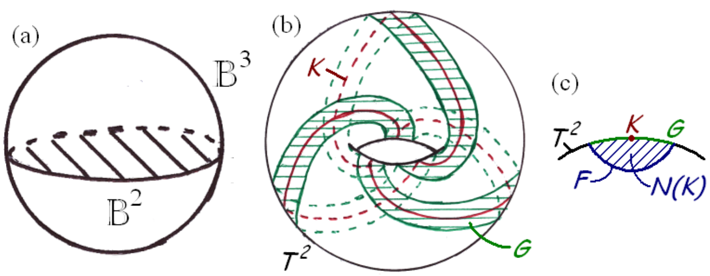

Figure 9: (a) Example (i). (b) Example (ii). (c) Cross section for Example (ii).

Note. F in example (ii) above is boundary parallel in essentially a unique way, unlike $\mathbb{B}^2 \subset \mathbb{B}^3$, or the following. Take $\mathbb{B}^1 \times S^1 \subseteq \mathbb{D}^2 \times S^1$. Then this is boundary parallel in two ways; see Figure 10.

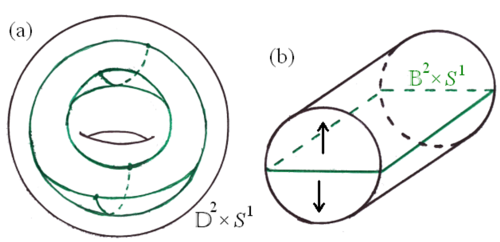

Figure 10: (b) is a cross section of (a), and $\mathbb{B}^1 \times S^1$ can be isotoped either up or down into $\mathbb{T}^2 = \partial\left(\mathbb{D}^2 \times S^1\right)$.

Example 14.2. $\mathbb{M}^2 \subseteq \mathbb{D}^2 \times S^1$ is not boundary parallel; see Figure 11.

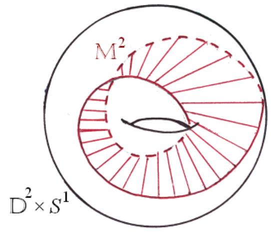

Figure 11: $\mathbb{M}^2$ is not boundary parallel in $\mathbb{D}^2 \times S^1$.

Definition 14.3. A torus $T \subset M$ is essential if it is incompressible and not boundary parallel.

Definition 14.4. Suppose M is irreducible, orientable, compact and connected. Then the manifold M is *toroidal* if there exists an essential torus $T \subset M$. M is *atoroidal* if there are no essential tori embedded in M.

Example 14.3. Suppose $K \subset S^3$ is a knot. Define the *knot exterior* $X_K := S^3 - n(K)$. If $K = L \# L'$ is a non-trivial connect sum of knots, then X_K is toroidal.

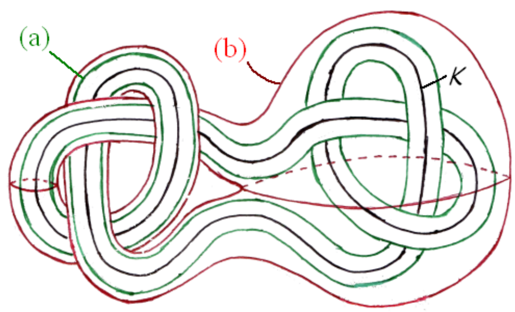

Figure 12: (a) $n(K)$. (b) An essential torus in X_K.

As shown in the previous lecture, when dealing with essential 2–spheres, we cut and cap off with 3–balls. However, there is no canonical way to cap off $\mathbb{T}^2 \subset \partial M$. So we must live with the possibility of incompressible tori, but at least we may eliminate essential tori.

Definition 14.5. Fix K, a knot in S^3, called the *companion knot*. Fix $L \subset \mathbb{D}^2 \times S^1$, the *pattern knot*. Fix a homeomorphism $\varphi\colon \mathbb{D}^2 \times S^1 \to N(K)$. Then $\varphi(L) \subset S^3$ is a *satellite knot* with pattern L and companion K. See Figure 13.

Example 14.4. All non-trivial connect sums are satellite knots.

Remark. If K is not the unknot and $L \subset \mathbb{D}^2 \times S^1$ is *disk busting* (for all compressing disks $D \subset \mathbb{D}^2 \times S^1$, $|L \cap D| \geq 1$, and L is not isotopic to $\{0\} \times S^1$), then $X_{\varphi(L)}$ is toroidal.

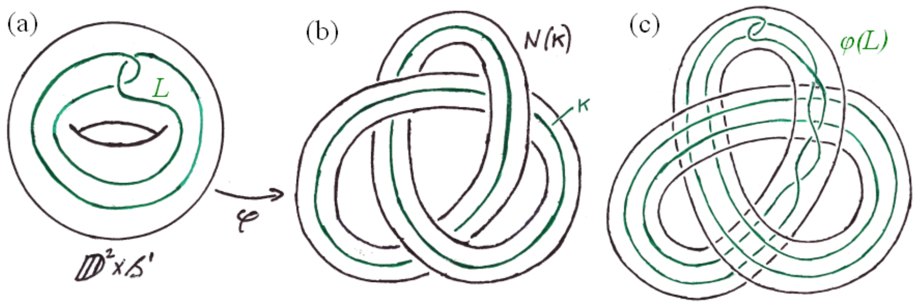

Figure 13: (a) L is the pattern knot, (b) K is the companion knot and (c) $\varphi(L)$ is the satelite knot.

Theorem 14.2 (Thurston)**.** *Every knot $K \subset S^3$ other than the unknot is either a satellite knot, a torus knot or a hyperbolic knot, as respectively X_K is toroidal, X_K is atoroidal but cylindrical, or X_K is atoroidal and acylindrical.*

Exercise 14.4. Show that X_K is irreducible.

Example 14.5. S^3 is atoroidal, but $\mathbb{T}^3$ is not; see Figure 14.

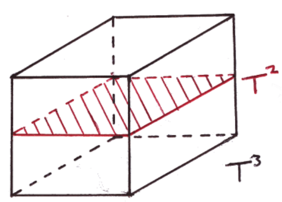

Figure 14: $\mathbb{T}^3$ contains $\mathbb{T}^2$ as an essential torus, and so is toroidal.

Lecture 15

Exercise 15.5. Suppose $F \subset M$ is properly embedded and suppose that $i_*\colon \pi_1(F) \to \pi_1(M)$ is injective. Show that F is incompressible (i.e., all surgery disks are trivial).

The final part of the course will be devoted to proving a partial converse to Exercise 15.5, via the loop theorem, the disk theorem and Dehn's lemma. An application of this converse will give us the following example:

Example 15.6. A knot $K \subset S^3$ is isotopic to a round circle (that is K is unknotted) if and only if $\pi_1(X_K) \cong \mathbb{Z}$.

Definition 15.6. A *torus system* is a finite union of disjoint, non-parallel, essential tori.

Proposition 15.3 (Corollary 1.8 in Hatcher)**.** *Suppose that M is compact, connected, orientable and irreducible. Then there is a torus system $S \subset M$ (where we allow $S = \varnothing$), so that all components of $M - n(S)$ are atoroidal.*

Proof. If M is atoroidal then take $S = \varnothing$. Otherwise, fix a triangulation T of M and suppose that $F \subset M$ is an essential torus. So $S = \{F\}$ is a torus system. We now induct on $|S|$. By Exercise 14.3 we may normalize S. By Haken-Kneser finiteness we find that $|S| \leq 20|T|$, so if there exists a component $N \subseteq M - n(S)$ which is toroidal then we find $F' \subset N$ an essential torus. So F' is not parallel to any component of S. Let $S' = S \cup \{F'\}$. Then S' is again a torus system. $\square$

Remark. The final step uses Exercise 4.5 in Exercise Sheet 4.

Example 15.7. Suppose $\varphi\colon F \to F$ is a homeomorphism of a surface F. Define $M_\varphi = F \times I/(x,1) \sim (\varphi(x),0)$. Then M_φ is a surface bundle over S^1 via $\rho\colon M_\varphi \to S^1$, where $\rho\colon (x,t) \mapsto t \in \mathbb{R}/\mathbb{Z}$; see Figure 15.

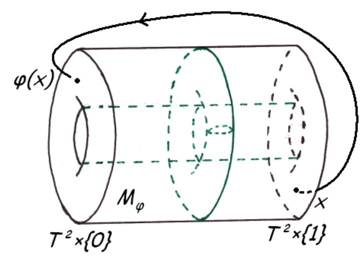

Figure 15: M_φ is a $\mathbb{T}^2$–bundle over S^1.

Exercise 15.6. Show that every fibre $T_t = \rho^{-1}(t)$ is incompressible (in fact π_1–injective) in M_φ.

Note. If $F = T \cong \mathbb{T}^2$, and $T \subset M_\varphi$ is a fibre, then $M_\varphi - n(T) \cong T \times I$. So we cannot avoid sometimes having a product component after cutting.

Remark. We have that $\mathbb{T}^3$ is the torus bundle M_{Id} in the above notation.

We now discuss lens spaces. Take $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 2\}$. Let y be the loop $\{|w| = 2\}$ and x be the loop $\{|z| = 2\}$, oriented as shown in Figure 16.

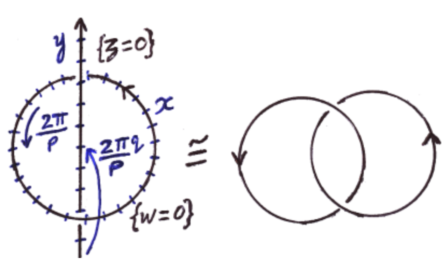

Figure 16: The great circles $\{z = 0\}$ and $\{w = 0\}$ in $S^3 \subset \mathbb{C}^2$ with this orientation are together homeomorphic to the right Hopf link.

Then define:

$$\begin{aligned} V &= \{(z,w) \in S^3 : |w| \leq 1\}, \\ W &= \{(z,w) \in S^3 : |z| \leq 1\}, \\ T &= V \cap W \\ &= \{(z,w) \in S^3 : |z| = |w| = 1\} \cong \mathbb{T}^2. \end{aligned}$$

Recall that $D \times S^1$ is a *solid torus.* We refer to any curve of the form $\partial D \times \{z\} \subset D \times S^1$ as a *meridian.* Now, as indicated in Figure 17 we take μ and λ to be generators of $\pi_1(T)$. Thus μ and λ are meridians of the solid tori V and W, respectively. We give μ and λ the orientations shown in Figure 17.

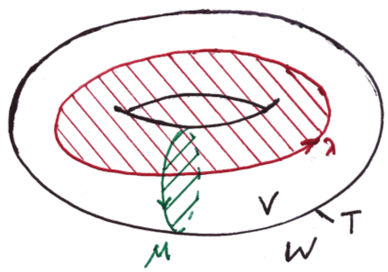

Figure 17: The curves μ and λ are oriented so that μ, λ and the outward normal for V form a right-handed frame.

Definition 15.7. Write $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{\alpha \in \mathbb{C} : \alpha^p = 1\}$ for $p \neq 0$, and fix $q \in \mathbb{Z}$ with $\gcd\{q,p\} = 1$. This acts on S^3 via:

$$\alpha \cdot (z,w) = (\alpha z, \alpha^p w).$$

Definition 15.8. Define $L(p,q) = \mathbb{Z}_p \backslash S^3$, the (p,q)–*lens space.*

Exercise 15.7. $L(p,q)$ is an orientable 3–manifold.

Example 15.8. We have $L(1,0) = S^3$.

Exercise 15.8. Show that $L(2,1) \cong P^3$.

Proposition 15.4. *Suppose $V, W \cong \mathbb{D}^2 \times S^1$ and $\varphi \colon \partial W \to \partial V$ is a homeomorphism. Show that $M = V \cup_\varphi W$ is either a lens space or is $S^1 \times S^2$.*

Note. We have $\pi_1(L(p,q)) \cong \mathbb{Z}_p$. Thus if $L(p',q') \cong L(p,q)$ then $p' = p$.

Exercise 15.9. Show that if $q' = \pm q^{\pm 1}$ modulo p, then $L(p,q') \cong L(p,q)$.

Remark. The converse holds, but is much harder to prove (see Brody 1960).

Remark. Whitehead (1941) showed that $L(p,q) \simeq L(p,q')$ (the spaces are homotopy equivalent) if and only if $qq' = \pm k^2$ modulo p for some k.

Example 15.9. We have $L(7,1) \simeq L(7,2)$, but these spaces are not homeomorphic.

Please let me know if any of the problems are unclear or have typos.

Exercise 5.1. For each of the two triangulations shown in Figure 1 prove that the underlying space is a three-manifold. Compute the fundamental groups and identify each manifold (by giving a homeomorphism).

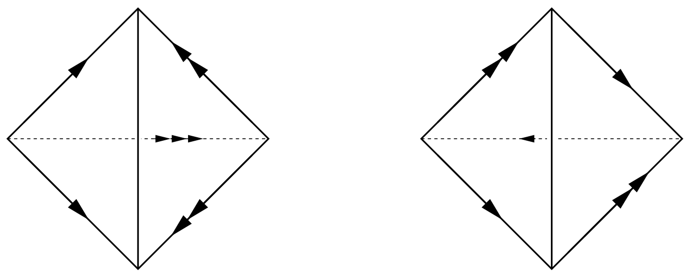

Figure 1: Each one-tetrahedron triangulation has exactly one face pairing between the back two faces.

Exercise 5.2. For each of the two triangulations shown in Figure 2 prove that the underlying space is a three-manifold. Compute the fundamental groups and identify each manifold (by giving a homeomorphism).

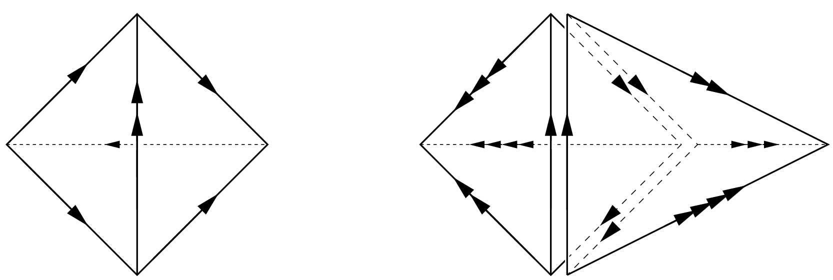

Figure 2: The left triangulation has two face pairings gluing the back two and the front two faces. The right triangulation has four face pairings.

Exercise 5.3. Classify, up to normal isotopy, all normal curves in (F^2, T) where:

- (F, T) is the usual triangulation of the torus, with two triangles. [Medium]
- (F, T) is the usual triangulation of the Klein bottle with two triangles. [Medium-hard]
- (F, T) is the two-sphere, triangulated as the two-skeleton of a tetrahedron. [Hard]

Exercise 5.4. For the cubing shown in Figure 3 prove that the underlying space Q is a three-manifold. Compute the fundamental group $\Gamma = \pi_1(Q)$ and show that Γ is finite and not Abelian. [Harder: Compute the universal cover $\widetilde{Q} \to Q$ and the associated action of Γ on $\widetilde{Q}$.]

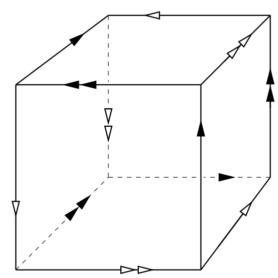

Figure 3: The *quarter-turn* space: opposite faces are idenified by a right handed quarter-turn. What do you get if you use a one-half turn instead? What manifolds arise from similarly nice face pairings of other Platonic solids?

Exercise 5.5. Suppose that $F \subset (M, T)$ is a closed incompressible embedded surface. Suppose that M is irreducible. Show that F is isotopic to a normal surface. (That is, there is a map $H\colon F \times I \to M$ so that $H_0 = \mathrm{Id}\,|F$, H_t is an embedding for all t, and $H_1(F)$ is normal.) Can you extend your proof to the case where F has boundary and is properly embedded?

Exercise 5.6. [Easy] Let $B_n = \#_n\mathbb{B}^3$ be the n–times *punctured three-sphere*. Here are two statements left over from the proof of existence of prime factorizations.

- Suppose that P, Q are three-manifolds and $\phi\colon S \to T$ is a homeomorphism of two-sphere boundary components $S \subset P$, $T \subset Q$. Prove that P, Q are both punctured three-spheres if and only if $P \cup_\phi Q$ is a punctured three-sphere.
- If M has n boundary components that are two-spheres then $M \cong N\#B_n$ where N has no two-sphere boundary components.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Anna Lena Winstel
Assisted by Matthew Pressland and David Kitson

today

Lecture 16

For lens spaces, we have the following definitions:

- The quotient space $\mathbb{Z}_p \backslash S^3$.
- The gluing $V \cup_\varphi W$, the union of solid tori, which is either a lens space or $S^2 \times S^1$.
- The following construction: let $B = \{(z, t) \in \mathbb{C} \times \mathbb{R} : \ |z|^2 + t^2 \leq 1\}$, a 3–ball. Let $D^\pm$ be the upper (respectively lower) hemisphere of ∂B, as in Figure 1.

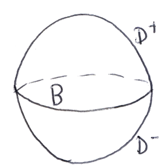

Figure 1: $D^\pm$ are the upper and lower hemispheres of ∂B.

Fix $\alpha = \exp(2\pi i/p)$ and glue D^- to D^+ by $\varphi \colon D^- \longrightarrow D^+$, where $\varphi(z, t) = (\alpha^q z, -t)$. See Figure 2.

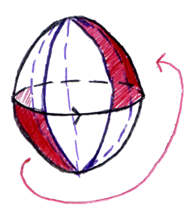

Figure 2: The lower hemisphere is glued to the upper by a $2\pi \cdot q/p$ twist.

Notice that, as Figure 2 indicates, there is a nice triangulation of B by a collection of p tetrahedra, all sharing the z–axis as an edge. Notice also that a neighborhood of the midpoint of any edge is a half-ball

$$B^3_+ \cong \{(x,y,z): \ z \geq 0, \ x^2+y^2+z^2 \leq 1\}$$

and p copies of these are glued, each to the next. So "geometrically", an edge has $p\pi$ dihedral angle which is $(p-2)\pi$ too much. So we consider a lens with dihedral angle $2\pi/p$ at the equator, as in Figure 3.

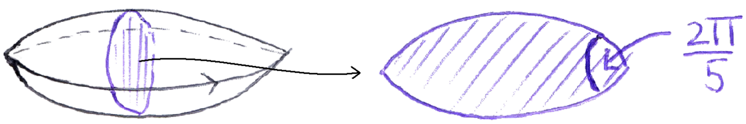

Figure 3: A lens with dihedral angle $2\pi/p$ at the equator (here, $p = 5$).

Now we can glue and get the right amount of dihedral angle. More precisely, the lens should live in S^3 and be cut out by great hemispheres, each meeting the next at angle $2\pi/p$. In Figure 4, you can see the lenses for $p = 10$. Glue pairs of these together to get lenses for $p = 5$.

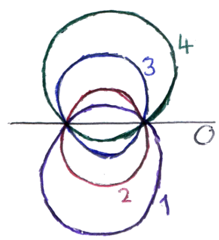

Figure 4: 10 copies of the lens tile S^3.

Exercise 16.1. Check that the three definitions agree.

Recall that we defined the meridian and longitude μ, λ for the torus $T = V \cap W \subset S^3$. See Figure 5.

Definition 16.1. If $K = s\mu + r\lambda$ then the *slope* of K is r/s.

Let $K = s\mu + r\lambda \in \pi_1(T)$, a simple closed curve. In Figure 6 for example, $K = 3\mu + 2\lambda$ has slope $2/3$ in T.

Notation. For $\alpha, \beta \in \pi_1(T)$ we define $\alpha \cdot \beta$ to be the signed intersection number. So

$$\begin{array}{cc} \mu \cdot \mu = 0 & \mu \cdot \lambda = +1 \\ \lambda \cdot \mu = -1 & \lambda \cdot \lambda = 0 \end{array}$$

and thus $\mu \cdot K = r$ and $K \cdot \lambda = s$.

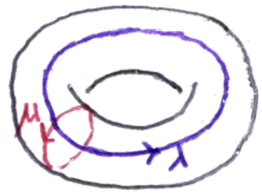

Figure 5: The torus T with meridian μ and longitude λ. Note that the orientation of μ, that of λ, and the outward normal to V, in that order, obey the right-hand rule.

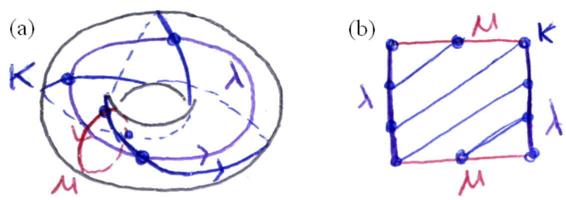

Figure 6: The right handed trefoil knot K has slope $2/3$. (a) K as seen in the torus T, and (b) K as seen in $\mathbb{R}^2/\mathbb{Z}^2 \cong T$.

Definition 16.2. Suppose $r, s \in \mathbb{Z}$ are coprime, with $|r|, |s| > 1$. We call $K = r\lambda + s\mu \subset T \subset S^3$ the (r,s)*–torus knot*. Then we define $X_K := S^3 - n(K)$, the *knot exterior*. Moreover, we define $V_K := V - n(K)$, $W_K := W - n(K)$ and $A = T_K = T - n(K)$.

In Figure 7, z is the core curve of $A = V_K \cap W_K$.

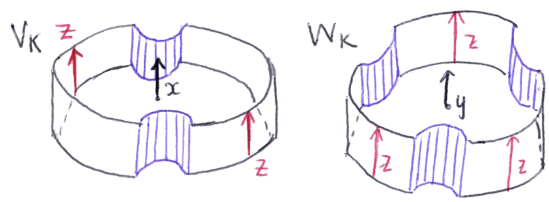

Figure 7: The cross-sections of V_K and W_K. The loop z is the core curve of $A = V_K \cap W_K$, and the loops x and y are the generators of $\pi_1(V_k)$ and $\pi_1(W_k)$ respectively.

Recall that the inclusions $i : A \hookrightarrow V_K$ and $j : A \hookrightarrow W_K$ induce maps i_* and j_* giving the following diagram:

$$\begin{tikzcd} & \pi_1(A) = \langle z \rangle \arrow[dl, hook, "i_*"] \arrow[dr, hook, "j_*"] & \\ \pi_1(V_k) = \langle x \rangle & & \pi_1(W_k) = \langle y \rangle \end{tikzcd}$$

Exercise 16.2. Show that $i_*(z) = x^r$ and $j_*(z) = y^s$ hence i_* and j_* are injective, where x and y are the loops shown in Figure 7.

By Seifert-van Kampen, assuming that $r, s \neq 0$, we get the following push-out where the lower maps are again inclusions:

$$\begin{tikzcd}
& \pi_1(A) = \langle z \rangle \arrow[dl, hook, "i_*"'] \arrow[dr, hook, "j_*"] & \\
\pi_1(V_k) = \langle x \rangle \arrow[dr, hook] & & \pi_1(W_k) = \langle y \rangle \arrow[dl, hook] \\
& \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} \cong \langle x, y \,|\, x^r = y^s \rangle =: \Gamma_{r,s} &
\end{tikzcd}$$

Via group theory, one can show that $\Gamma_{r,s} \cong \Gamma_{p,q}$ if and only if $\{|p|, |q|\} = \{|r|, |s|\}$.

Lecture 17

Aside. Note that

- $SO(2) \cong S^1$,
- $SO(3) \cong \mathbb{P}^3$ and
- $SL(2, \mathbb{R}) \cong \operatorname{int}(\mathbb{D} \times S^1) \cong \mathbb{R}^2 \times S^1$, the latter is not an isomorphism of groups.

Remark. We now have the following remarkable fact. Let $K \subset S^3$ be the trefoil knot and define $Y_K = S^3 - K$ be the *knot complement*, an open three-manifold. Then Y_K is homeomorphic to $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$.

For the following we assume that K is not the unknot, i.e. $|p|, |q| \geq 2$.

Theorem 17.1. *Suppose $K = K_{p,q}$ is the (p, q)–torus knot, then the annulus $A = T - n(K)$ is the unique* essential *annulus in X_K, up to isotopy.*

We will prove this later in the course.

Corollary 17.1. *Define $X_{p,q} = X_K$, where $K = K_{p,q}$. Then $X_{p,q} \cong X_{r,s}$ if and only if $\{|p|, |q|\} = \{|r|, |s|\}$.*

Non-uniqueness of torus decompositions

Now we closely follow Hatcher. Let $V_i \cong \mathbb{D} \times S^1$, $i = 1, 2, 3, 4$. Let $A_i \subset \partial V_i$ be an embedded annulus and suppose A_i winds q_i times about V_i with $q_i \geq 2$; for examples see Figure 8.

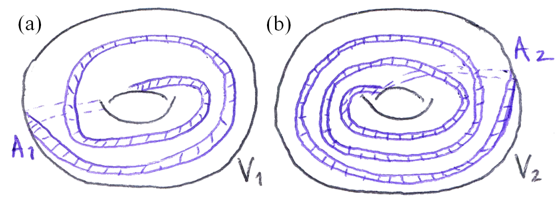

Figure 8: Two examples of a winding annulus; in (a) $q_1 = 2$ and in (b) $q_2 = 3$.

Another way to define q_i is the following: Let α_i be a core curve of A_i and define q_i via $q_i = |\alpha_i \cdot \partial D_i|$. Let $A'_i = \overline{\partial V_i - A_i}$ and pick $\varphi \colon A'_i \longrightarrow A_{i+1}$ where we take the indices modulo 4. Let $M = \sqcup V_i / \varphi_i$; see Figure 9(a). Let B_i denote the image of A_i in M. Now we define $M_i = V_i \cup_{\varphi_i} V_{i+1}$. Let $T_1 = B_1 \cup B_3$ and $T_2 = B_2 \cup B_4$. Thus $M = M_1 \cup_{T_1} M_3 = M_2 \cup_{T_2} M_4$.

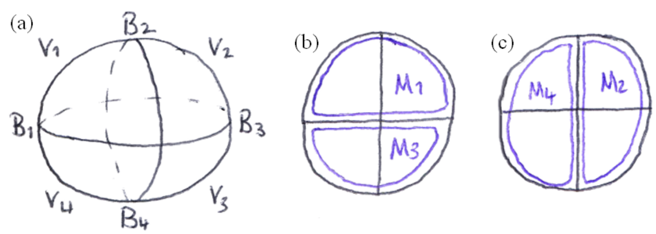

Figure 9: (a) A schematic of M. B_i is the image of A_i in M. (b) and (c) are schematics of two different torus decompositions.

Finally, we claim that $B_1 \cup B_3$ and $B_2 \cup B_4$ are incompressible tori in M. If we now choose the q_i to all be distinct and coprime then, for $i = 1, 2, 3, 4$, then manifold M_i is a torus knot exterior. So we have, for these choices of q_i, that M_1 is not homeomorphic to M_2 or M_4 and M_3 is not homeomorphic to M_2 or M_4. Thus the torus decompositions T_1 and T_2 are different; see Figures 9(b) and (c).

Remark (17.2)**.** This requires the following facts. If $X_{p,q} = S^3 - n(K_{p,q})$, then

- $\partial X_{p,q}$ is incompressible,
- $X_{p,q}$ is atoroidal and
- Theorem 17.1.

We will prove these facts later. To do so, and so to understand the non-uniqueness of torus decompositions, we must first understand *Seifert fibred spaces.*

Fibred solid tori

Fibre $D \times I$ by intervals of the form $\{x\} \times I$. We call $\{0\} \times I$ the *central fibre*. Let $\varphi\colon D \times \{1\} \longrightarrow D \times \{0\}$ be a $2\pi q/p$ rotation, $\varphi(z, 1) = (\alpha^q z, 0)$ where as usual p and q are coprime. Define $V_{p,q} = D \times I/\varphi$, the (p, q)–*fibred solid torus*. Notice that $\{0\} \times I$ now gives a circle as does the set of fibres $\{\alpha^k \cdot (z \times I) : \ \alpha^p = 1\}$. Note that $V_{p,q}$ is given a *fibring*, i.e. a decomposition into circles.

Definition 17.3. A *Seifert fibring* of a three-manifold M is a partition $\mathcal{F}$ of M into circles (the *fibres*) such that every fibre $\lambda \in \mathcal{F}$ has arbitrary small regular neighbourhoods $N(\lambda)$ all homeomorphic to $V_{p,q}$ for some fixed p, q. Here the homeomorphisms are all fibre-preserving.

Remark. The integers p, q only depend on λ.

Definition 17.4. We call p the *multiplicity* of λ.

Note that the space $V_{p,q}$ is Seifert fibred itself and the central fibre has multiplicity p while all other fibres have multiplicity equal to 1.

Definition 17.5. If λ has multiplicity greater than 1, then we call λ a *singular* fibre. All other fibres are called *generic*. See Figure 10.

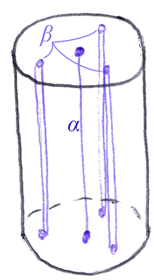

Figure 10: Inside of $V_{3,1}$ the central fibre α is singular (with multiplicity three) while all others, for example β, are generic.

Lecture 18

Exercise 18.3. If M is compact then there are only finitely many singular fibres, all contained in the interior of M.

Exercise 18.4. Show that $L_{p,q}$ is a Seifert fibred space with at most two singular fibres. Compute their multiplicities.

Exercise 18.5. Let $K = K_{p,q}$ be the (p, q)–torus knot. Show that X_K is a Seifert fibered space. Find the singular fibres and their multiplicities.

Example 18.1. Let $M = V_1 \cup V_2 \cup V_3 \cup V_4$ as in the last lecture. Then M is a Seifert fibred space with 4 singular fibres.

Definition 18.6. Suppose $(M, \mathcal{F})$ is a Seifert fibred space. Let $B = M/S^1$ be the *base orbifold*; that is, the quotient of M sending fibres to points.

Example 18.2. Suppose $M = V_{p,q}$. The quotient M/S^1 is a disk D with a cone point at the centre. The angle at the cone point is $2\pi/p$; see Figure 11.

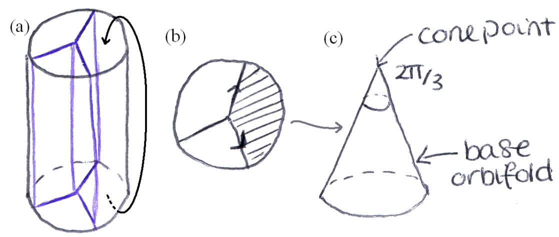

Figure 11: (a) The solid torus $V = V_{3,1}$. (b) A meridian disk for V. (c) The quotient V/S^1 is a cone with angle $2\pi/3$ at the cone point.

Exercise 18.6. In exercises 18.1 and 18.2, identify the base orbifolds.

Example 18.3. Notice that if $\rho \colon T \longrightarrow F$ is an S^1–bundle then $T/S^1 \cong F$.

Theorem 18.2 (1.9 in Hatcher)**.** *Let M be compact, irreducible and orientable. There exists a torus system $T \subset M$ such that all components of $M - n(T)$ are either atoroidal or Seifert fibred spaces. Furthermore any minimal such system is unique up to isotopy.*

Remark. The example from last lecture, M, contains infinitely many non-isotopic incompressible tori. Hence the uniqueness of Theorem 18.2 requires that we not cut along tori in Seifert fibred spaces.

Please let me know if any of the problems are unclear or have typos.

Exercise 6.1. [Easy] Suppose that $\alpha \subset \partial\Delta$ is a simple closed curve in the boundary of a tetrahedron Δ. Suppose that α is transverse to $\Delta^{(1)}$ and meets some edge $e \subset \Delta^{(1)}$ in at least two points. Show that there is a component $d \subset e - n(\alpha)$ disjoint from $\Delta^{(0)}$.

Exercise 6.2. [Old] Suppose that $F \subset (M, T)$ is a closed incompressible embedded surface. Suppose that M is irreducible. Show that F is isotopic to a normal surface. [Note that this is a duplicate of Exercise 5.6 from last week.]

Exercise 6.3. [Easy] Using the definition via triangulations or otherwise, show that lens spaces are orientable.

Exercise 6.4. [Easy] Show that $P^3 \cong L(2,1)$. Thus P^3 is orientable.

Exercise 6.5. [Medium] Show that there are exactly two I–bundles over P^2, up to equivalence. Show that the non-trivial bundle is homeomorphic to $P^3 - \text{interior}(B^3)$ (a once-punctured projective space).

Exercise 6.6. We gave three definitions of a lens space in class: as the quotient of the three-sphere, as the gluing of solid tori, and as the quotient of a lens (that is, of a three-ball). Show that the three definitions are equivalent by providing the necessary homeomorphisms. Mind your p's and q's!

Exercise 6.7. Suppose that $F \subset M$ is a properly embedded surface. Suppose that the induced map on fundamental groups $\iota_* \colon \pi_1(F) \to \pi_1(M)$ is injective. Show that F is incompressible.

Exercise 6.8. [Medium] Show that the the three-sphere S^3 and the three-ball B^3 are atoroidal. Show that the solid torus $D \times S^1$ is atoroidal. Show that T^3 is toroidal.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Matthew Pressland
Assisted by Anna Lena Winstel and David Kitson

Lecture 19

Suppose $F \subset M$ is properly embedded, and M is compact, irreducible and orientable. Recall that $(D, \partial D) \subset (M, F)$ is a *surgery disk* for F if $D \cap F = \partial D$. D is *trivial* if ∂D bounds a disk in F. If D is not trivial, then it is a *compressing disk* for F.

Definition 19.1. A disk D with $\partial D = \alpha \cup \beta$ such that α and β are connected and $\alpha \cap \beta = \partial\alpha = \partial\beta$ is a *bigon*; see Figure 1.

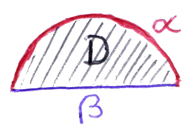

Figure 1: A bigon D.

Definition 19.2. Say $D \subset M$ is a *surgery bigon* for $F \subset M$ if D is a bigon, $D \cap F = \alpha$ and $D \cap \partial M = \beta$. Say that D is *trivial* if there is a bigon $D' \subset F$ so that $\partial D' = \alpha' \cup \beta'$, $\alpha = \alpha'$ and $D' \cap \partial M = \beta'$, as in Figure 2. If D is not trivial, call it a *boundary compressing bigon*, or simply a *boundary compression*.

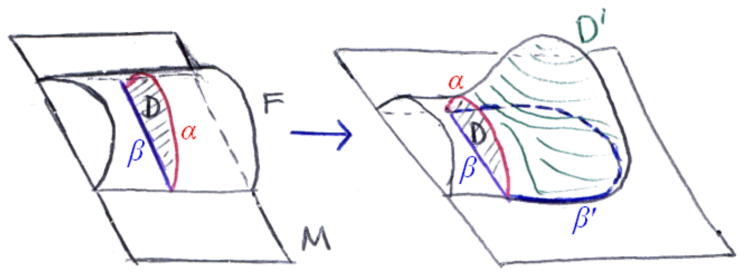

Figure 2: D is a trivial surgery bigon. Note that D' is not properly embedded in M but contained entirely in F.

Recall that a two-sided simple closed curve $\alpha \subset F^2$ is *essential* if α does not bound a disk on either side (Figure 3(a)). A sphere $S \subset M^3$ is *essential* if it does not bound a three-ball on either side (Figure 3(b)). If M is irreducible then a disk $(D, \partial D) \subset (M, \partial M)$ is *essential* if ∂D is essential in ∂M (Figure 3(c)).

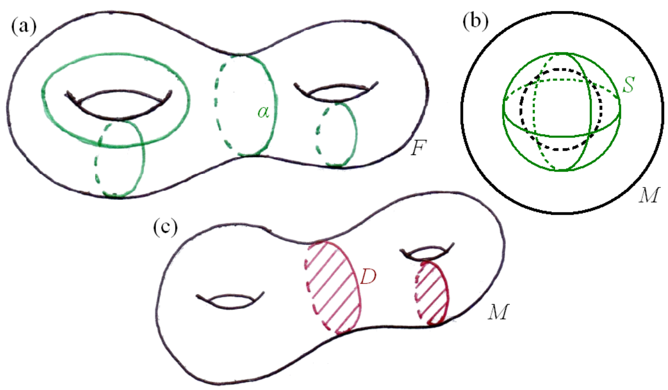

Figure 3: (a) All the green curves here are essential. (b) Here S is essential in M. (c) These disks are essential in M.

Definition 19.3. Suppose that $S \subset M$ is a properly embedded, connected, two-sided surface that is not a disk or a sphere. We say S is *essential* if it is incompressible and boundary incompressible.

Definition 19.4. If all surgery disks are trivial, we call F *incompressible*; similarly, if all surgery bigons are trivial, call F *boundary incompressible.*

Exercise 19.1. Suppose $S \subset M$ is an essential surface. Show that $\partial S \subset \partial M$ is essential.

Proposition 19.1. *If $S \subset D^2 \times S^1$ is essential then S is isotopic to $D^2 \times \{z\}$ for some $z \in S^1$.*

Proof. Let $\mu_z = \partial D^2 \times \{z\}$. We call μ_z the meridian curves. Abusing notation, let $D = D^2 \times \{1\}$. Then by Exercise 19.1, ∂S is essential so we may isotope components of ∂S so that all are either equal to meridian curves, or are transverse to all meridian curves, as in Figures 4(a) and (b).

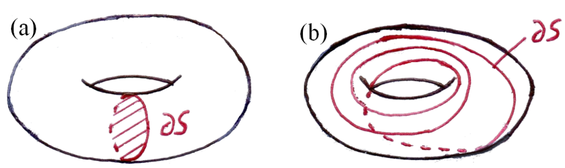

Figure 4: (a) Here the component of ∂S is meridian curve. (b) Here ∂S is transverse to all meridian curves.

Thus, we may assume that ∂S is transverse to μ_1, and via isotopy relative to ∂M, we may assume that S is transverse to D. Then $S \cap D$ is a collection of arcs and loops, as in Figure 5.

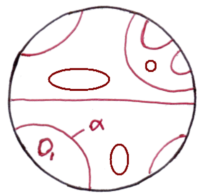

Figure 5: $S \cap D$ is a collection of arcs and loops.

We proceed as follows:

Step 1: First suppose $\alpha \subset D \cap S$ is an innermost loop, so α bounds a disk $D_1 \subset D$ such that $D_1 \cap S = \partial D_1$. So D_1 is a surgery disk for S and thus, as S is incompressible, there is a disk $E \subset S$ with $\partial E = \partial D_1 = \alpha$, as in Figure 6(a). So $D_1 \cup E$ is a 2–sphere. As $D \times S^1$ is irreducible, $D_1 \cup E$ bounds a 3–ball B, so there is an isotopy supported in $n(B)$ moving E past D_1; see Figure 6(b). This gives an isotopy of S, reducing $|S \cap D|$. So without loss of generality, we may assume that $D \cap S$ consists only of arcs.

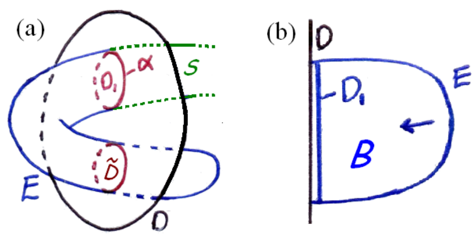

Figure 6: (a) $E \cup D_1$ bounds a 3–ball B, so (b) we may isotope E through $n(B)$ past D_1 to reduce $|S \cap D|$.

Step 2: Now suppose $\alpha \subset D \cap S$ is an outermost arc. So α cuts off from D a surgery bigon D_1. Since S is boundary incompressible, α cuts off a bigon E from S. Let $\gamma = E \cap \partial(D \times S^1)$ and $\beta = D_1 \cap \partial(D \times S^1)$. See Figure 7.

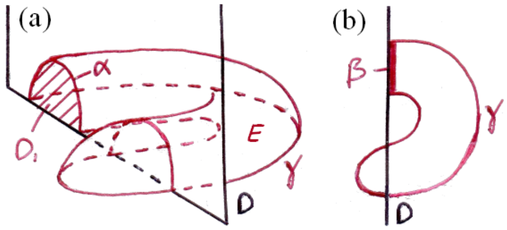

Figure 7: (a) α cuts a surgery bigon D_1 from D and E from S. (b) A plan view of (a).

Notice that $D_1 \cup E$ is a disk, with $D_1 \cap E = \alpha$. Thus $D_1 \cup E$ lifts to $\widetilde{D \times S^1} \cong D \times \mathbb{R}$, as in Figure 8.

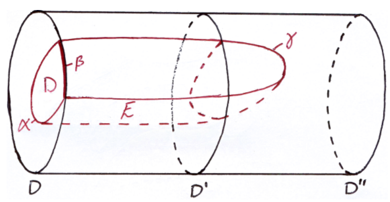

Figure 8: $D_1 \cup E$ lifts to $\widetilde{D \times S^1} \cong D \times \mathbb{R}$.

Let $h : D \times \mathbb{R} \to \mathbb{R}$ be projection to the second factor, and notice that:

$$h(\partial_+\gamma) = h(\partial_-\gamma)$$

as $\partial_\pm\gamma \in \partial D$. So by Rolle's theorem, $(h|\gamma)'$ has a zero, so γ is not transverse to μ_z for some $z \in S^1$, giving a contradiction. Thus without loss of generality, we may assume $S \cap D = \varnothing$.

Step 3: Next, define $B = (D \times S^1) - n(D)$. This is a 3–ball, and $S \subset B$. Pick any component $\delta \subset \partial S$. So δ divides ∂B into disks C and C'. So push C, say, into B, keeping ∂C inside of S. This gives a disk in the interior of B. See Figure 9. If $C \cap S \neq \partial C$, then we may isotope S, as in Step 1, to reduce $|S \cap C|$. So C gives a surgery disk for S. Thus S is a disk.

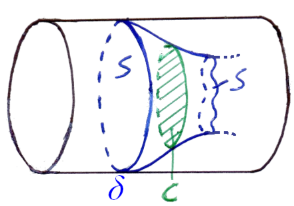

Figure 9: Push C into B (keeping ∂C inside of S) to get a disk in the interior of B.

Finally, Alexander's theorem implies that S is isotopic to $D \times \{z\}$ for some $z \in S^1$, fixing δ pointwise. $\square$

Note. All surgery disks for S^2 are trivial, and all surgery disks and bigons for $\mathbb{D}^2$ are trivial, hence they are excluded from the statement of Proposition 19.1.

Definition 19.5. Suppose $(\alpha, \partial\alpha) \subset (A^2, \partial A^2)$ is an arc in an annulus. It is *trivial* if it cuts a bigon off of A, and *essential* otherwise. See Figure 10.

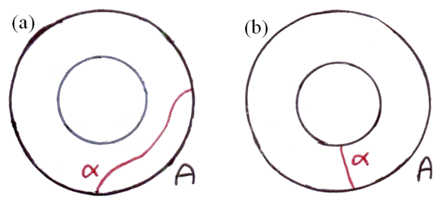

Figure 10: (a) A trivial arc. (b) An essential arc.

Lecture 20

Exercise 20.2. Suppose $F \subset M$ is two-sided and incompressible. Suppose $D \subset M$ is a surgery bigon for F and suppose F_D is the result of surgery. Show that $F_D \subset M$ is incompressible.

Exercise 20.3. Deduce from the above that if $\rho : T \to F$ is an I–bundle then $\partial_h T$ is boundary incompressible.

Lemma 20.2 (1.10 in Hatcher)**.** *Suppose that $S \subset M$ is a connected, two-sided, incompressible surface, and M is irreducible. Suppose S admits a boundary compressing bigon D with $\partial D = \alpha \cup \beta$, $\alpha = D \cap S$, $\beta = D \cap \partial M$ and β is contained in a torus component $T \subset \partial M$. Then S is a boundary parallel annulus.*

Proof. By Exercise 19.1, $\partial S \cap T$ is essential in T. Let $A = T - n(\partial S)$, so A is a collection of annuli. So $\beta \subset A$ is either trivial or essential, as in Figure 11(a).

Case 1: Suppose that $\beta \subset A$ is trivial. So β cuts a bigon E off of A. Then $D \cup E$ is a disk. Isotope $D \cup E$, keeping $\partial(D \cup E)$ in S, to get a surgery disk for S; see Figure 11(b).

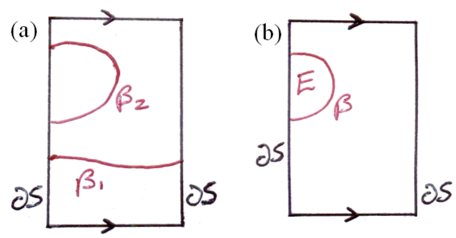

Figure 11: (a) β_1 is essential while β_2 is trivial. (b) Trivial arcs define a surgery bigon for S.

Since S is incompressible, $D \cup E$ cuts a disk D' out of S, and hence D was a trivial surgery bigon, as in Figure 12.

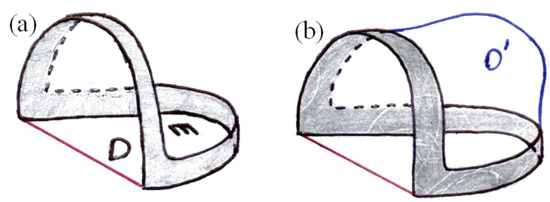

Figure 12: $D \cup E$ cuts a disk D' from S and so D is trivial.

Case 2: Suppose β is essential in A. If ∂B is contained in a single component of ∂S, then S is one-sided, giving a contradiction. To see this, we can orient β and ∂S so that both intersections have positive sign, as in Figure 13.

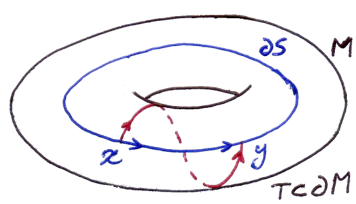

Figure 13: We can orient β and ∂S so that both intersections have positive sign.

Then following α we find that S is one-sided, as in Figure 14.

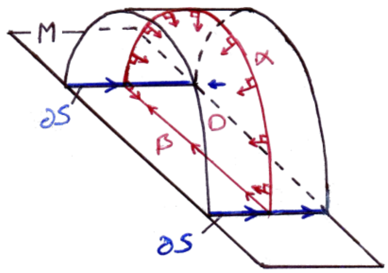

Figure 14: Carrying the orientation along α gives a different orientation to carrying along ∂S, a contradiction.

So we have that β connects distinct components of ∂S, as in Figure 15.

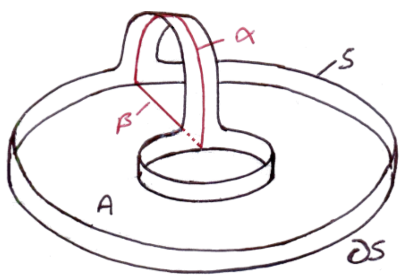

Figure 15: β connects distinct components of ∂S.

Boundary compress S along D to get S_D. Note that S_D is incompressible, by Exercise 20.2, and that S_D has a trivial boundary component, so S_D is a disk. To see this, say ∂S_D bounds E in T. So isotope E into E' in M, keeping ∂E in S_D, as in Figure 16.

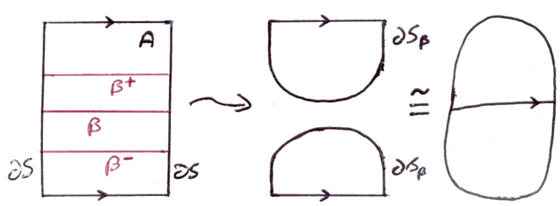

Figure 16: Cutting along β gives two components of ∂S_β, and the identification gives a trivial curve in ∂S_D.

Since S_D is incompressible, $\partial E'$ must cut a disk out of S_D, so S_D is a disk. Since M is irreducible, S_D is boundary parallel; in fact it is parallel to the original E.

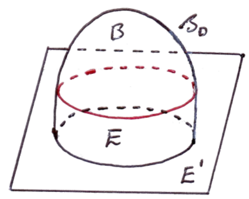

Figure 17: S_D is boundary parallel.

So S_D cuts a 3–ball B out of M. Letting $V = B \cup N(D)$, this is a solid torus, giving a parallelism of S with the annulus A, as in Figure 18. $\square$

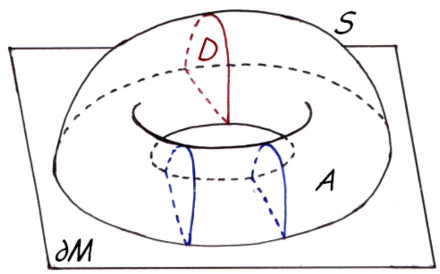

Figure 18: S is boundary parallel to the annulus A.

Definition 20.6. Suppose that $(M, \mathcal{F})$ is Seifert fibred. Then we say that a properly embedded surface $S \subset M$ is *vertical* if S is a union of fibres, and it is *horizontal* if S is transverse to the fibres. We make the same definitions for $S \subset T$ for an I–bundle $\rho : T \to F$.

Exercise 20.4. All essential surfaces $S \subset T$, where $\rho : T \to F$ is an I–bundle, are isotopic to either vertical or horizontal surfaces.

Lecture 21

Lemma 21.3 (1.11 in Hatcher)**.** *Suppose that $(M, \mathcal{F})$ is compact, connected and irreducible. Supppose $S \subset M$ is essential. Then after a proper isotopy, S is either vertical or horizontal.*

Proof. Let $Z := \{\alpha_i\}_{i=1}^k$ be the set of singular fibres of $\mathcal{F}$; if M has no singular fibres, and $\partial M = \varnothing$, then let $\{\alpha_1\}$ be a single generic fibre. Let $M_0 = M - n(Z)$. Let $B = M/S^1$ and let $B_0 = M_0/S^1$. Note that $\partial B_0 \neq \varnothing$. In fact B_0 is B with neighbourhoods of cone points removed, as in Figure 19.

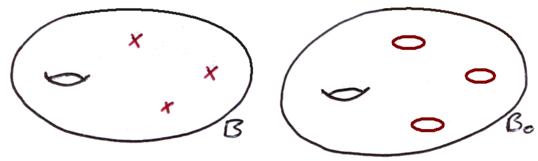

Figure 19: B_0 is B with neighbourhoods of cone points removed.

Example 21.1. If $M = V_{p,q}$ then Z is just the central fibre. Then $M_0 = \mathbb{A}^2 \times S^1$ and $B_0 = \mathbb{A}^2$; See Figure 20.

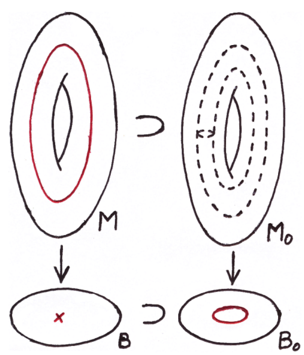

Figure 20: $M_0 = \mathbb{A}^2 \times S^1$ and $B_0 = \mathbb{A}^2$.

Choose a system of arcs in B_0 cutting B_0 into a disk, i.e. as in Figure 21.

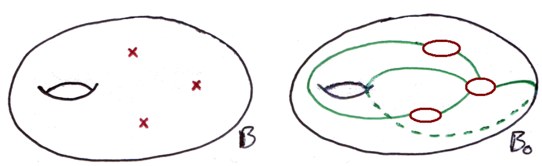

Figure 21: We may choose a system of arcs cutting B_0 into a disk.

Let $A \subset M_0$ be the vertical annuli above this system of arcs. So $M_0 - n(A) =: M_1$ is a solid torus, fibred by $\mathcal{F}|M_1$, with all fibres generic. Given an essential surface S, all components of ∂S are essential in ∂M.

(i) We may isotope them to all be vertical or horizontal with respect to the fibring $\mathcal{F}|\partial M$.

(ii) Isotope S (relative to ∂S) so that S meets Z transversely, and so meets $n(Z)$ in horizontal disks. Define $S_0 = S \cap M_0$, and make S_0 intersect A transversely. Consider the arcs and loops of $S_0 \cap A$, as in Figure 22.

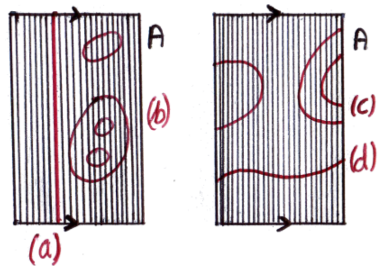

Figure 22: (a) An essential loop. (b) Trivial loops. (c) Trivial arcs. (d) An essential arc.

(iii) If there is a trivial loop, then there is an innermost such. Now, using incompressibility of S and irreducibility of M, there is an isotopy of S reducing $|S \cap A|$ as usual. So without loss of generality, there are no trivial loops.

(iv) Suppose $\beta \subset S \cap A$ is an outermost trivial arc and let D be the bigon cut our of A by β. If $\partial\beta \subset \partial M$ then D is a surgery bigon for S, but as in Proposition 19.1, ∂S is either contained in or transverse to $\mathcal{F}|\partial M$, giving a contradiction. To see this, since S is boundary incompressible, there is a bigon E contained in S, as in Figure 23.

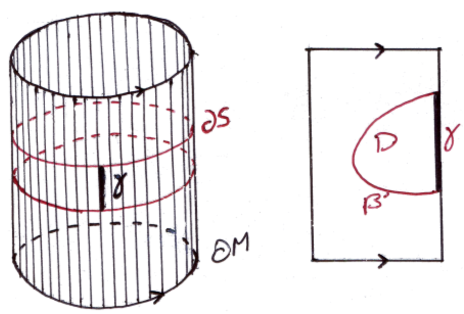

Figure 23: γ is parallel to the fibres.

So letting $\partial E = \beta \cup \gamma'$, we find that γ' is not transverse to $\mathcal{F}|\partial M$. On the other hand, if $\partial\beta \subset \partial M_0 - \partial M$, then a baseball move across D reduces $|S \cap (Z)|$ by two. Now without loss of generality, every component of $S \cap A$ is either horizontal or vertical.

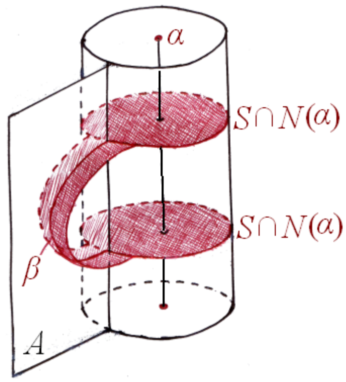

Figure 24: A baseball move across α reduces $|S \cap Z|$ by 2.

(v) Define $S_1 = S_0 \cap M = S_0 - n(A)$. So $\partial S_1 \subset M_1$ is completely horizontal or completely vertical. We may assume that S_1 is incompressible in M_1. Thus S_1 is either a collection of horizontal meridian disks, or a collection of boundary parallel annuli. If S_1 contains an annulus with slope that of the meridian, then S_1 is compressible. If S_1 contains an annulus $B \subset S_1$ with ∂B horizontal, then we see a surgery bigon with vertical boundary. So do a baseball move and return to case (iv).

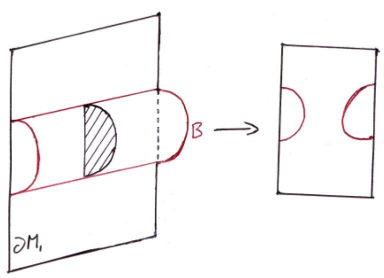

Figure 25: If S_1 contains an annulus B with ∂B horizontal, we may do a baseball move and reduce to case (iv).

So S_1 is now a collection of horizontal meridian disks, or a collection of boundary parallel vertical annuli. It follows that S_0, and so S, is either horizontal or vertical. $\square$

Remark. Vertical surfaces are easy to classify. They are orientable or not, and the base is I or S^1.

Base Orbifold	I	S^1	
	A^2	T^2	orientable
	M^2	K^2	non-orientable

Please let me know if any of the problems are unclear or have typos.

Exercise 7.1. Suppose that M, N are three-manifolds, $F, G \subset \partial M, \partial N$ are components of their boundaries, and $\phi, \phi' \colon F \to G$ are isotopic homeomorphisms. Define $Y = M \cup_\phi N$ and $Y' = M \cup_{\phi'} N$. Show that Y is homeomorphic to Y'.

Exercise 7.2. Using Exercise 5.3 or otherwise, classify essential curves in the Klein bottle K^2.

Exercise 7.3. Let $V = D \times S^1$ be a solid torus and let T be the orientation I–bundle over K^2. Suppose that $\phi \colon \partial V \to \partial T$ is a homeomorphism. Then $M_\phi = V \cup_\phi T$ is called a *prism manifold*. Show that prism manifolds are double covered by lens spaces (or $S^2 \times S^1$).

Exercise 7.4. [Hard] Show that $S^2 \times S^1$ and also the quarter-turn space are prism manifolds.

Exercise 7.5. Suppose that $(M, \mathcal{F})$ is a oriented three-manifold equipped with a Seifert fibering. Show that the singular fibers are isolated and lie in the interior of M. Thus, if M is compact then there are only finitely many singular fibers.

Exercise 7.6. Find the Seifert fibering of the knot complement X_K where $K = K(p, q)$ is a torus knot. Compute the number of singular fibers, their multiplicities, and the base orbifold.

Exercise 7.7. Find a Seifert fibering of the lens space $L(p, q)$. Compute the number of singular fibers and multiplicities as well as the base orbifold of your fibering.

Exercise 7.8. Consider the two tetrahedra shown Figure 1. Does the given triangulation determine a three-manifold? If it does not, give a reason. If it does, recognize the manifold.

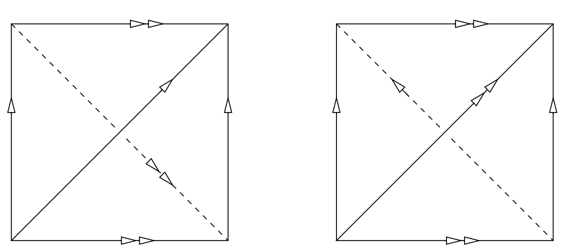

Figure 1: The front faces on the left are glued to the back faces on the right; likewise the back faces on the left are glued to the front faces on right. In this example, the face pairings are determined by the edge identifications but this does not hold in general.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Anna Lena Winstel
Assisted by Matthew Pressland and David Kitson

March 21, 2011

Lecture 22

Notation. Suppose F is not orientable. Let $F \tilde{\times} I$ denote the orientation I–bundle over F. Likewise define $F \tilde{\times} S^1$.

Exercise 22.1. Show that $P^2 \tilde{\times} I$ is homeomorphic to $P^3 - \operatorname{int}(B^3)$.

We now discuss orbifolds.

Definition 22.1. We say that $B = (S, Z)$ is an 2–*orbifold* if S is a surface and $Z \subset \operatorname{int}(S)$ is a finite set such that for every $z \in Z$ we have an *order* $p_z \in \mathbb{Z}_+$. We call Z the *singular set*. A point $z \in Z$ is a *cone point* if $p_z > 1$.

Example 22.1. A surface is an orbifold with $Z = \varnothing$.

Example 22.2. The square pillow case, $S^2(2, 2, 2, 2)$, shown in Figure 1, is an orbifold.

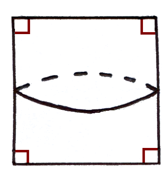

Figure 1: A picture of the square pillow case $S^2(2, 2, 2, 2)$.

Definition 22.2. If S is a surface with a triangulation T then we define the *Euler characteristic* of S to be $\chi(S) = V - E + F$ where V denotes the number of vertices, E the number of edges and F the number of triangles (faces).

Exercise 22.2. Show that χ stays unchanged under the Pachner moves. Figure 2 shows the Pachner moves. Since any two triangulations of a fixed closed surface are related by Pachner moves, the Euler characteristic is independent of the choice of T.

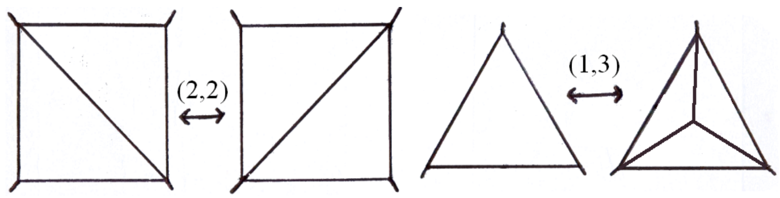

Figure 2: The Pacher moves.

Example 22.3. You can see by the triangulation shown in Figure 3(a) that $\chi(S^2) = 4-6+4 = 2$. Similarly, Figure 3(b) shows that $\chi(T^2) = 1-3+2 = 0$.

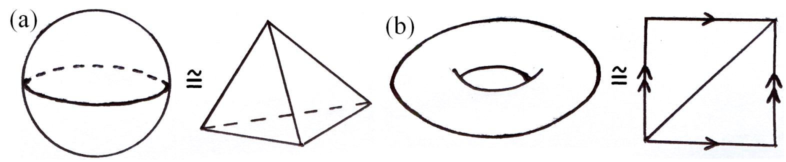

Figure 3: (a) A triangulation of a 2–sphere. (b) A triangulation of the 2–dimensional torus.

Definition 22.3. We define the *Euler characteristic* of an orbifold via

$$\chi_{\mathrm{orb}}(B) = \chi(S) + \sum_{z \in Z} \left(\frac{1}{p_z} - 1 \right).$$

Example 22.4. $\chi_{\mathrm{orb}}(S^2(2,2,2,2)) = 2 + 4(1/2-1) = 0.$

Exercise 22.3. List all 2–orbifolds B so that $\chi_{\mathrm{orb}}(B) = 0$.

Exercise 22.4. What can you say about B so that $\chi_{\mathrm{orb}}(B) > 0$?

Orbifold Covers

Example 22.5. The map from $D \subset \mathbb{C} \to D$ which sends z to z^n is an orbifold map of order n. In Figure 4, $n = 3$.

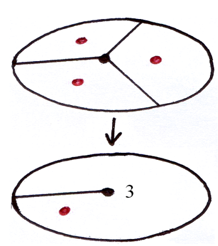

Figure 4: The map $z \mapsto z^3$ from $D \subset \mathbb{C}$ to itself is a three-fold cover.

Definition 22.4. If C, B are 2–orbifolds then $\varphi\colon C \to B$ is a *cover* if

1. $\varphi^{-1}(Z_B) = Z_C$,
2. $\varphi|(C - Z_C)\colon C - Z_C \to B - Z_B$ is a d–fold cover and
3. for every point $z \in Z_B$, we have $d/p_z = \sum_{y \in \varphi^{-1}(z)} 1/p_y$.

Note that φ restricted to any regular neighbourhood of a point $z \in Z_C$ is modelled on the example $z \longmapsto z^n$.

Example 22.6. The quotient of T^2 via the 180° rotation shown in Figure 5 is a degree two orbifold cover.

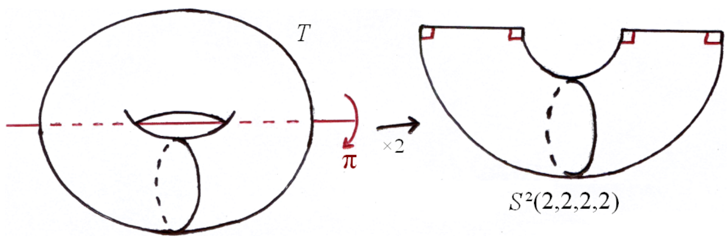

Figure 5: The quotient map of the 2–dimensional torus via the 180° rotation.

Exercise 22.5. Show that if $\varphi\colon C \to B$ is a d–fold orbifold cover then $\chi_{\mathrm{orb}}(C) = d \cdot \chi_{\mathrm{orb}}(B)$. As warm-up, show that if $\varphi\colon T \to S$ is a d–fold cover of surfaces then $\chi(T) = d \cdot \chi(S)$.

Exercise 22.6. List all 2–fold covers of $S^2(2, 2, 2, 2)$.

The following question is known as the *Hurwitz problem* and still open in general: Given B, C such that $\chi_{\mathrm{orb}}(C)/\chi_{\mathrm{orb}}(B) \in \{2, 3, 4, \ldots\}$ does there exists a d–fold cover?

Example 22.7. For $n \geq 2$, $S^2(n)$ is a *bad orbifold*, meaning it is not covered by a surface. Hence $S^2(n)$ is not covered by S^2. You can also see this because $2/(2 + (1/n - 1)) \notin \mathbb{N}$.

We now return to our original topic, horizontal surfaces. Suppose that $S \subset (M, \mathcal{F})$ is horizontal. As in the proof of Lemma 21.4, we may form $M \supset M_0 \supset M_1$ and $S \supset S_0 \supset S_1$. Let λ be any generic fibre and $d = |S \cap \lambda|$, so S_1 is a collection of d horizontal disks. Recall that Z is the set of all singular fibres. Thus $S \cap (N(Z))$ is also a collection of disks. Then S is formed by gluing horizontal disks along horizontal loops in $\partial N(Z)$ and horizontal arcs in A. Thus the quotient $\rho\colon M \to M/S^1 = B$ restricts to S to give a d–fold cover $\rho\colon S \to B$. So

$$\chi(S) = d \cdot \chi_{\mathrm{orb}}(B) = d \cdot \left(\chi(B) + \sum_{z \in Z} \left(\frac{1}{p_z} - 1 \right) \right).$$

Proof. See Hatcher. $\square$

Lecture 23

To answer the question of a student, we will expand the definition of a *boundary compression*.

Definition 23.5. Suppose $S \subset \partial M$ is a subsurface. Then we say S is *boundary compressible* if there is a bigon D with $\partial D = \alpha \cup \beta$ so that $D \cap S = \alpha$, $D \cap \overline{\partial M - S} = \beta$ and α does not cut a bigon out of S. Say that S is *boundary incompressible* if no such bigon exists.

Now we continue our discussion of horizontal surfaces. Suppose that $S \subset (M, \mathcal{F})$ is two-sided, horizontal and connected. Then we get the following corollary of Proposition 21.3 (1.11 in Hatcher).

Corollary 23.1. *The manifold $M - n(S)$ is an I–bundle.*

Proof sketch. Recall that S_1 was a collection of horizontal disks in $M_1 \cong D \times S^1$. So $n(S_1)$ cuts M_1 into cylinders foliated by intervals. The vertical sides of these solid cylinders glue to give the desired I–bundle. $\square$

Let $\rho\colon M - n(S) \longrightarrow F$ be the I–bundle map. Then there are two cases.

1. The manifold $M - n(S)$ is connected. So $M - n(S) \cong S \times I$ and thus $\partial_h(M - n(S)) = S \sqcup S$ and so $F \cong S$ and we find that M is an S–bundle over S^1. See Figure 6.

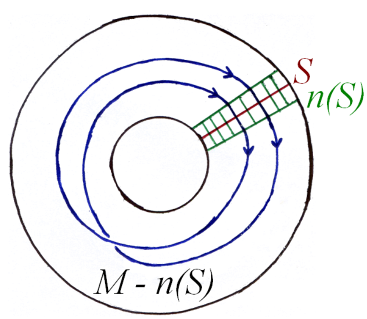

Figure 6: A picture of $M - n(S)$ as an S–bundle over S^1. The blue curve represents a generic fibre.

So the I–fibres in $N(S)$ and in $M - n(S)$ glue to give the Seifert fibring, $\mathcal{F}$. I.e., there is a *monodromy* (a homeomorphism $\varphi\colon S \longrightarrow S$) such that $M \cong S \times I/(x,1) \sim (\varphi(x), 0) =: M_\varphi$ and finally $S/\varphi \cong B$. The monodromy is *periodic* of period $d = |S \cap \lambda|$, i.e. $\varphi^d = \mathrm{Id}_S$. See Figure 7.

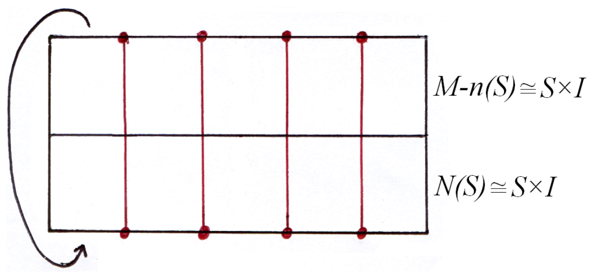

Figure 7: $M = (M - n(S)) \cup N(S) \cong M_\varphi$. Here φ has periodicity 4.

Example 23.8. Let φ be the hyperelliptic involution on the 2–torus shown in Figure 5. This is periodic.

Example 23.9. Glue the cube as shown in Figure 8 and note that planes parallel to the xy–plane glue to give tori.

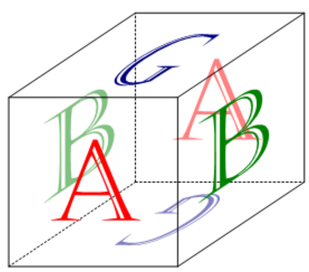

Figure 8: A cube with face pairings. The front and back are glued by the identity as are the left and right face. The bottom and top face are glued together by a 180° rotation.

Note that intervals parallel to the z–axis glue to give circles, 4 of length 1 and the rest of length 2.

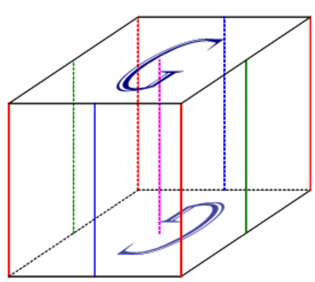

Figure 9: A picture of the different circles achieved by gluing intervals parallel to the z–axis. The gluings of the vertical faces are the same as in Figure 8 and are omitted.

All of the singular fibres in Figure 9 have length one, while all other vertical circles have length two. All other vertical circles have length 2. So $B \cong S^2(2,2,2,2)$ is the base orbifold, double covered by double covered by any horizontal surface, all of which are tori. See Figure 10.

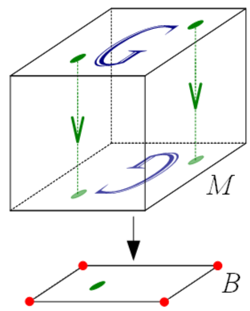

Figure 10: The base orbifold is a copy of the square pillow case: $B \cong M/S^1 \cong S^2(2,2,2,2)$, and double covered by T.

2. If $M - n(S)$ has two components then each is a twisted I–bundle over F and these glue to $N(S) \cong S \times I$ giving a *semibundle* (also called a *fibroid*). See Figure 11.

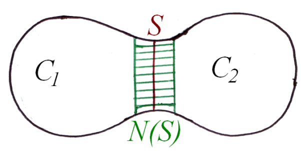

Figure 11: A picture of the two twisted I–bundles over F.

So letting T_1 and T_2 be the two I–bundles, we obtain M by gluing T_1 and T_2 to $N(S)$ and find involutions $\tau_i \colon S \longrightarrow S$ such that $T_i = S \times I/(x,0) \sim (\tau_i(x),0)$. Here the homeomorphism $\varphi = \tau_1 \circ \tau_2$ is again *periodic*.

Example 23.10. As an exercise, we showed that $P^3 - \mathrm{int}(B^3) = P^2 \tilde{\times} I$. Here $\partial_h T \cong S^2$ and the involution τ is the antipodal map. So if we consider $T_1 \cup_S T_2$ where $T_i \cong P^2 \tilde{\times} I$, we find that $P^3 \# P^3$ is Seifert fibred. Check that $\tau_1 \circ \tau_2 = \tau^2 = \varphi = \mathrm{Id}_S$ and so it is periodic.

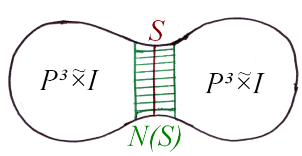

Figure 12: A picture of the gluing of $T_1 \cup_S T_2$.

Example 23.11. Consider the cube with face pairings given in Figure 3. Notice that the intervals parallel to the x–axis also define a Seifert fibring with $B = K^2$, the Klein bottle, and all fibres are generic, as in Figure 13(a). The planes $y = 1/4$ and $y = 3/4$ define a 2–torus $S \subset M$ and $M - n(S)$ has two components, both homeomorphic to $K \tilde{\times} I$.

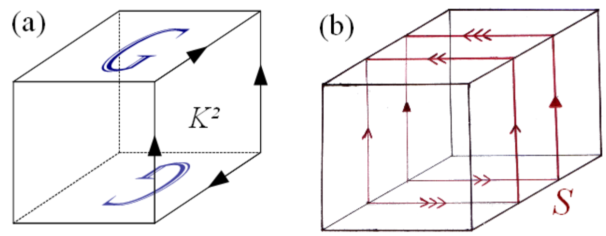

Figure 13: (a) Intervals parallel to the x–axis give a fibring with $B = K^2$. (b) Both components of $M - n(S)$ are homeomorphic to $K^2 \tilde{\times} I$.

Exercise 23.7. Check that these planes give a 2–torus with the claimed properties. Find the involutions τ_1, τ_2.

Lecture 24

Recall that every essential arc in $A^2 \cong S^1 \times I$ is isotopic to $\{\text{pt}\} \times I$, as in Figure 14.

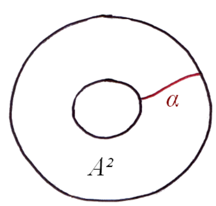

Figure 14: An essential arc in an annulus.

Exercise 24.8. Classify up to isotopy the essential arcs and loops in $\#_3 D^2$, the pair of pants.

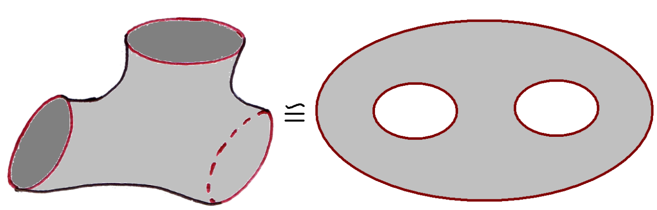

Figure 15: Two diagrams of the Pair of Pants.

Recall that if $X = X_K$ where $K = K_{p,q}$ is the (p,q)–torus knot then $B = X/S^1$ is the orbifold $D^2(p,q)$.

Exercise 24.9. Classify essential arcs and loops in $D(p,q)$. Deduce that the only essential vertical annulus in X is $A = V_K \cap W_K$. (Care is required if p or q is equal to 2, as then X contains a vertical Mobius band.)

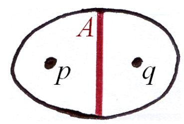

Figure 16: A diagram of $D^2(p,q)$. Note that the A here is the projection of the annulus into the orbifold.

Exercise 24.10. Use orbifold Euler characteristic to show that any horiontal surface $S \subset X$ has $\chi(X) \leq p + q - pq < 0$ as $p, q \geq 2$ and $p \neq q$. Deduce that X is atoroidal and A is the unique essential annulus in X, up to isotopy.

Exercise 24.11 (Harder)**.** Use Exercise 24.10 to prove that

$$g(K_{p,q}) = \frac{(p-1)(q-1)}{2}$$

where $g(K)$ is the minimal genus of a spanning surface for K.

Furthermore, X is a surface bundle over S^1 with monodromy of order pq. To show this, let S be the minimal spanning surface and consider $X - n(S)$.

Aside. To answer the question of a student, we define the Euler characteristic of an n–manifold.

Definition 24.6. We define $\chi(M^n)$ by taking a finite triangulation of M and setting $\chi(M) = \sum_{k=0}^{n}(-1)^k|T^{(k)}|$ where $|T^{(k)}|$ denotes the number of k-simplices in the image $\|T\|$.

Proposition 24.2 (1.12 in Hatcher)**.** *Suppose $(M, \mathcal{F})$ is compact, connected and Seifert fibred. Then M is irreducible or M is homeomorphic to one of $S^2 \times S^1, S^2 \tilde{\times} S^1$ or $P^3 \# P^3$.*

Proof. Suppose $S \subset M$ is an essential 2–sphere. Following the proof of Proposition 23.1 (1.11 in Hatcher) with surgery of essential surfaces replacing isotopy of essential spheres, we find an essential 2–sphere S' such that S' is vertical or horizontal. Since S' is not A^2, T^2, M^2 or K^2, we find S' must be horizontal.

1. If S' is non-separating, then $M - n(S')$ is homeomorphic to $S^2 \times I$. So $M \cong S^2 \times S^1$ or $S^2 \tilde{\times} S^1$.

2. If S' separates, then it is an exercise to show that $M \cong P^3 \# P^3$. $\square$

Proposition 24.3 (1.13 in Hatcher)**.** *Let $(M, \mathcal{F})$ be as above. Then*

1. *every horizontal 2–sided surface is essential and*

2. *every vertical 2–sided surface is essential except for tori bounding fibred solid tori and boundary parallel annuli cutting off fibred solid tori.*

Proof. Suppose that D is a surgery disk or bigon for $S \subset M$.

1. Suppose S is horizontal. By the previous discussion, $M - n(S)$ is an I–bundle and D gives a surgery for $\partial_h(M - n(S))$. But the horizontal boundary of an I–bundle is always essential.

   Exercise 24.12. The horizontal boundary of an I–bundle is always essential.

2. Suppose S is vertical. So D gives a surgery in $M' \subset M - n(S)$ where M' is the component of $M - n(S)$ containing D. Suppose D is essential. Since $D \subset M'$ is essential, D must be vertical or horizontal, hence horizontal. Let $B = M'/S^1$.

   Exercise 24.13. Show that B is a disk with at most one orbifold point. Hint: use that $d \cdot \chi_{\text{orb}}(B) = \chi(D) = 1$.

   Thus M' is a solid torus. If D was a bigon, then, as $D \cap \partial M = D \cap \partial M'$ is a single arc, the fibring of M' is the trivial fibring, so $M' \cong V_{1,0}$. $\square$

Please let me know if any of the problems are unclear or have typos.

Exercise 8.1. Let $P \tilde{\times} I$ be the orientation I–bundle over the projective plane. Show that $P \tilde{\times} I$ is homeomorphic to $P^3 \# B^3$, a punctured projective space.

Exercise 8.2. Suppose that $S \subset V$ is a surface properly embedded in a handlebody V. Show that S is compressible, is boundary compressible, is a disk or is a sphere.

Exercise 8.3. [Easy] Suppose that $S \subset M$ is a connected incompressible surface (not a disk). Show that if $\partial S \neq \emptyset$ then every component of ∂S is essential in ∂M.

Exercise 8.4. Suppose that $F \subset M$ is incompressible. Suppose that the bigon D is a boundary compression for F. Show that F_D, the surgery of F along D, is again incompresible.

Exercise 8.5. Suppose that $\rho\colon T \to F$ is an I–bundle. Show that $\partial_h T$ and $\partial_v T$ are boundary incompressible. [If $\partial F \neq \emptyset$ and if $\partial_h T$ is not connected then the components are, individually, boundary compressible. Likewise, the "1/2–section" (zero-section) is boundary compressible in T.]

Exercise 8.6. [Hard] Suppose that $\rho\colon T \to F$ is an I–bundle. Show that any essential surface $S \subset T$, that is not a disk, may be isotoped to be either vertical or horizontal. [Hint: there is a proof modelled on the proof of Proposition 1.11 from Hatcher.]

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Matthew Pressland
Assisted by Anna Lena Winstel and David Kitson

May 11, 2011

Lecture 25

Lemma 25.1 (1.14 in Hatcher)**.** *Let $A \subset (M, \mathcal{F})$ be an essential annulus. Then A can be properly isotoped to be vertical with respect to $\mathcal{F}$, possibly after changing $\mathcal{F}$ if M is $T \times I$, $T \mathbin{\tilde{\times}} I$, $K \times I$ or $K \mathbin{\tilde{\times}} I$.*

Proof. Since A is essential, it may be isotoped to be vertical or horizontal. Suppose A is horizontal. So $M - n(A)$ is an I–bundle with annuli as horizontal boundary components.

(i) If $M - n(A)$ is connected, then $M - n(A) \cong A \times I$. So

$$M = {}^{A \times I}\Big/_{(x,1) \sim (\varphi(x), 0)} =: M_\varphi,$$

as in Figure 1.

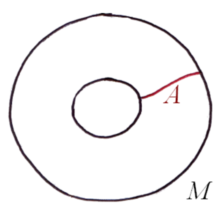

Figure 1: $M = (A \times I)/((x,1) \sim (\varphi(x), 0))$.

But there are only four possibilities for φ, up to isotopy: the identity, reflections switching or preserving the boundary components, and the rotation given by composing these reflections. See Figure 2.

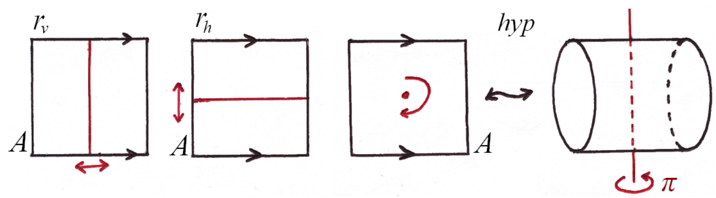

Figure 2: The three non-trivial possibilities for φ.

Exercise 25.1. Show that $\mathrm{MCG}(A) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Here $\mathrm{MCG}(S)$ is the mapping class group of S, the group of homeomorphisms of S, up to isotopy.

These four maps give the four exceptions.

Exercise 25.2. Check this.

(ii) If $M - n(A)$ has two components, as in Figure 3, then $M - n(A) \cong \mathbb{M}^2 \tilde{\times} I \sqcup \mathbb{M}^2 \tilde{\times} I$.

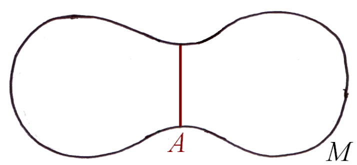

Figure 3: $M - n(A)$ may have two components.

Note that $\mathbb{M}^2 \tilde{\times} I$ is a cube with a pair of opposite faces glued by a π twist, shown in Figure 4.

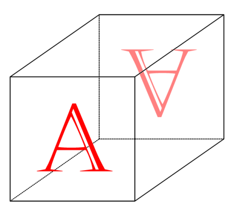

Figure 4: A picture of $\mathbb{M}^2 \tilde{\times} I$.

Exercise 25.3. Find the Möbius bands in this cube.

It is again an exercise to show that all four gluings give $K \tilde{\times} I$ with base orbifold $D^2(2, 2)$. $\square$

Note. We have an exact sequence of groups: $S^1 \to K \tilde{\times} I \to D^2(2,2)$

$$\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(K^2) & \longrightarrow & D_\infty & \longrightarrow & \mathbf{1} \\
\mathbf{1} & \longrightarrow & \left\langle a^2 \right\rangle & \longrightarrow & \left\langle a, b \mid a^2 = b^2 \right\rangle & \longrightarrow & \left\langle a, b \mid a^2 = b^2 = 1 \right\rangle & \longrightarrow & \mathbf{1}
\end{array}$$

coming from the long exact sequence for the Seifert fibering. See Theorem 4.41 page 276 of Hatcher's *Algebraic Topology* for more details.

Lemma 25.2 (1.15 in Hatcher)**.** *Let $(M, \mathcal{F})$ be as above. Then the slopes of $\mathcal{F}|\partial M$ are determined by M only, unless M is $V_{p,q}$ or one of the four exceptions above.*

Proof. If $\partial M = \varnothing$ then we have nothing to prove. If $B = M/S^1$ has no essential arcs, then $B = D^2(p)$.

Exercise 25.4. Check this.

Then $M \cong D \times S^1$ and we are done. So let $\alpha \subset B$ be an essential arc. See Figure 5.

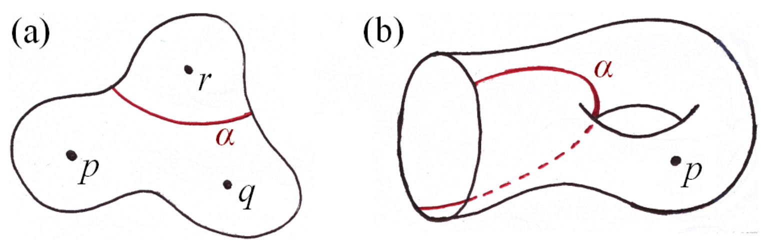

Figure 5: Two examples of essential arcs in (a) where $B = D^2(p, q, r)$ with $p, q, r > 1$, and (b) where $B = T^2 \# D^2(p)$.

Let $A \subset M$ be the vertical annulus above α. In this case:

(i) A is essential by Lemma 1.13 in Hatcher.

(ii) A is vertical in any fibering of M, with exceptions as above, by Lemma 1.14 in Hatcher.

So ∂A is determined by M alone, and we are done. $\square$

Remark. Note that in the above we used the fact that solid Klein bottles are not Seifert fibered spaces.

Exercise 25.5. Show that the solid Klein bottle can be partitioned as a disjoint union of circles. Show, nonetheless, that the solid Klein bottle cannot be Seifert fibered.

Exercise 25.6. Show that $K \times I$ contains a solid Klein bottle, yet is still a Seifert fibered space.

Lemma 25.3 (1.16 in Hatcher)**.** *Suppose M is connected, compact, orientable, irreducible and atoroidal. Suppose $A \subset M$ is an essential annulus with ∂A contained in torus components of ∂M. Then M admits a Seifert fibering.*

Proof. Let M, A be as above. Let T be the components of ∂M meeting A. Let $N = N(A \cup T)$. So there are three cases:

(i) A meets two boundary components, T_1 and T_2, as in Figure 6.

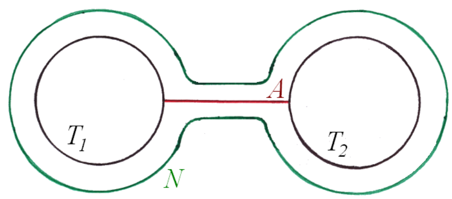

Figure 6: A meets two boundary components.

(ii) A meets a single boundary component without twisting, as shown in Figure 7.

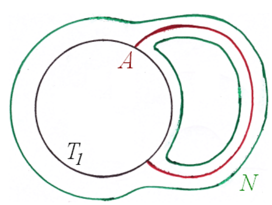

Figure 7: A meets a single boundary component without twisting.

(iii) A meets a single boundary component with a twist, as shown in Figure 8.

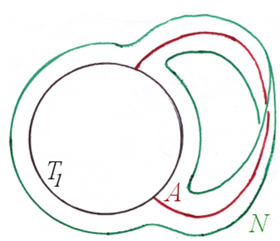

Figure 8: A meets a single boundary component with a twist.

Note. Note that Figures 6, 7 and 8 give a cross section of N. For example in Figure 6, the entirety of N is shown in Figure 9. Unfortunately the neighborhood N, in the third situation, does not embed in $\mathbb{R}^3$.

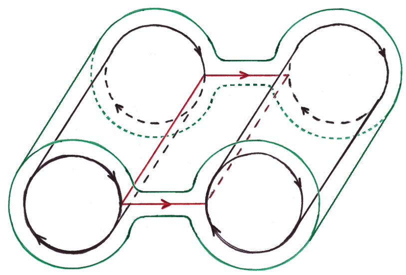

Figure 9: The whole of N in case (i), of which Figure 6 is a cross section. The front and back faces and edges are identifed.

Note that $N(A)$ and $N(T)$ are Seifert fibered, and we may glue these fibrings to get a fibering of N. Fix F, a component of $\partial N - \partial M$. In other words, a component of the *frontier* of N in M. Note that $F \cong \mathbb{T}^2$.

(i) Suppose that F compresses in M via a disk $(D, \partial D) \subset (M, F)$. Since A is essential we may arrange via an isotopy to have $A \cap D = \varnothing$. So we may assume that $D \cap N = \partial D$; thus F compresses to the "outside" of N. So F_D is a 2–sphere bounding a ball $B \subset M$. Note that $N \subset B$ is a contradiction as $\partial M \cap \partial N \neq \varnothing$. So $X = B \cup N(D)$ is a solid torus attached to F.

(ii) Suppose F is boundary parallel. Say $M - n(F)$ contains X, with $X \cong F \times I$ the parallelism. Since A is essential, we find that $X \cap N = F$, as $N \subset F$ leads to a contradiction.

So the fibering on N extends to a fibering on $N \cup X$. We do the same for all components of $\partial N - \partial M$. □

Exercise 25.7. Read the proof of Theorem 1.9 in Hatcher.

Lecture 26

We now state the Poincaré conjecture, proved by Perelman, following a program of Hamilton.

Poincaré Conjecture. *Suppose M^3 is closed and simply connected. Then M is homeomorphic to S^3.*

Recall that closed means that M is compact and $\partial M = \varnothing$. Simply connected means that M is connected and $\pi_1(M) = \{\mathbf{1}\}$. Note that the equivalent statement in dimension two follows from the classification of surfaces and the Seifert-van Kampen theorem. In dimensions greater than three, the conjecture was solved previously by (among others) Smale, Stallings, and for dimension four, Freedman.

Remark. Poincaré originally conjectured that if $H_1(M, \mathbb{Z}) = 0$ then $M = S^3$. He then gave a counterexample to this, called the *Poincaré homology sphere*. Let D be the dodecahedron and let $P = D/\sim$, where we glue opposite faces with a $1/10$ right-handed twist, as in Figure 10.

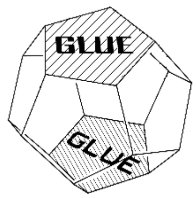

Figure 10: The Poincaré homology sphere. This diagram is adpated from one in The Shape of Space by J. Weeks.

Exercise 26.8. Let $\Gamma = \pi_1(P)$. Give a presentation of Γ and check that $\Gamma^{\mathrm{ab}} = 0$.

Exercise 26.9. What if we use a $5/10$ twist?

Remark. If we use a $3/10$ twist we get the *Seifert-Weber dodecahedron space*. See Figure 11.

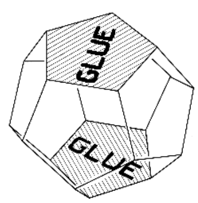

Figure 11: The Seifert-Weber dodecahedron space. This diagram is adapted from one in The Shape of Space by J. Weeks.

Definition 26.1. We say a knot $K \subset S^3$ is *spanned* by a surface $F \subset S^3$ if F is embedded and two-sided away from ∂F, and $\partial F = K$. In other words, the boundary of F wraps exactly once about K. See Figure 12a. Equivalently, $S \subset X_K$ is a *spanning surface* for K if it is two-sided, embedded, $|\partial S| = 1$ and the following holds. Let $N = N(K)$ and let $(D, \partial D) \subset (N, \partial N)$ be a meridian disk. Let $\mu = \partial D$. Then the transverse intersection $\mu \cap \partial S$ is a single point. See Figure 12b.

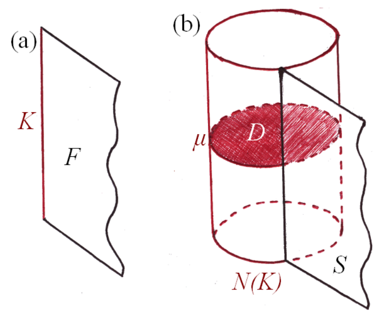

Figure 12: Diagrams of equivalent definitions of the spanning surface.

Recall that a knot K is the *unknot* if K is isotopic to a round circle.

Theorem 26.4. *Suppose $K \subset S^3$ is a knot. The following are equivalent:*

(i) *K is the unknot.*

(ii) *K is spanned by a disk E.*

(iii) *$X_K = S^3 - n(K)$ is a solid torus.*

(iv) *$\pi_1(X_K) \cong \mathbb{Z}$.*

See Figure 13.

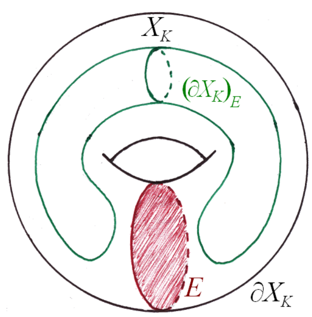

Figure 13: Illustration of Theorem 26.4.

Proof.

(i) $\Longrightarrow$ (ii) Use ambient isotopy.

(ii) $\Longrightarrow$ (iii) Use irreducibility of X_K and the fact that $(\partial X_K)_E \cong S^2$. Note that $E \subset X_K$ is essential as $\partial E \cap \mu$ is a point. So if $(\partial X_K)_E$ bounds a 3–ball B, we have $B \cup N(E) \cong E \times S^1$ is a solid torus.

(iii) $\Longrightarrow$ (i) This follows from Exercises 2.2 and 6.6.

(iii) $\Longrightarrow$ (iv) Since $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, we have $\pi_1(X_K) \cong \pi_1(S^1) = \mathbb{Z}$.

(iv) $\Longrightarrow$ (iii) We must show that if M is irreducible, $\partial M = \mathbb{T}^2$ and $\pi_1(M) \cong \mathbb{Z}$, then $M \cong D \times S^1$. This requires Dehn's Lemma. □

Exercise 26.10. Deduce (iv) $\Longrightarrow$ (iii) from the following lemma.

Dehn's Lemma (Papakyriakopoulos, 1957)**.** *Suppose $\alpha \subset \partial M$ is a simple closed curve, bounding a singular disk in M. Then α bounds an embedded disk in M.*

Loop Theorem. *Suppose F is a component of ∂M, and $i_* \colon \pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential simple closed curve $\alpha \subset F$ such that $[\alpha] = \mathbf{1} \in \pi_1(M)$.*

This leads nicely to the following conjecture.

Simple Loop Conjecture. *If $i \colon F \looparrowright M$ is a two-sided map, and i_* is not injective, then there is an essential simple loop in the kernel.*

This has been proved by Gabai if M is a surface, and by Hass if M is Seifert fibered.

Exercise 26.11. Prove the simple loop conjecture when F is two-sided and properly embedded in M.

Lecture 27

Disk Theorem. *Suppose that $F \subset \partial M$ is a component, and $i_* \colon \pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential disk $(D, \partial D) \subset (M, F)$.*

Exercise 27.12. Show that the Disk Theorem is implied by the Loop Theorem and Dehn's Lemma.

The Disk Theorem is the first "promotion" theorem, among many others. For example we have the following:

Sphere Theorem. *Suppose M is an orientable 3–manifold with $\pi_2(M)$ nontrivial. Then there is an embedded 2–sphere $S \subset M$ such that $[S] \neq \mathbf{1} \in \pi_2(M)$.*

In general we assume that there is an essential map $(F, \partial F) \looparrowright (M, \partial M)$. The corresponding promotion theorem gives us an embedding. For example, F could be a disk or sphere (due to Papakyriakopoulos), a projective plane (due to Epstein), an annulus or torus, or indeed any F with $\chi(F) \geq 0$.

We now discuss hierarchies. Suppose that $M_0 = M$, suppose that $S_i \subset M_i$ is a properly embedded two-sided surface, and define:

$$M_{i+1} := M_i - n(S_i).$$

So we have a sequence of manifolds:

$$M_0 \xrightarrow{S_0} M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} M_n.$$

Definition 27.2. Call a sequence $\{M_i, S_i\}$ a *partial hierarchy* if every S_i is essential in M_i.

Note. Some authors only require S_i to be incompressible.

The following example demonstrates why we require the S_i to be essential.

Example 27.1. Take annuli in V_2, the genus 2 handlebody, as in thr right hand side of Figure 14, and glue them to give $M_0 \cong V_2$. Let S_0 be the single annulus given by the image of the two annuli under the gluing map. Then cutting along S_0 gives $M_1 \cong V_2$, so we could continue the process indefinitely.

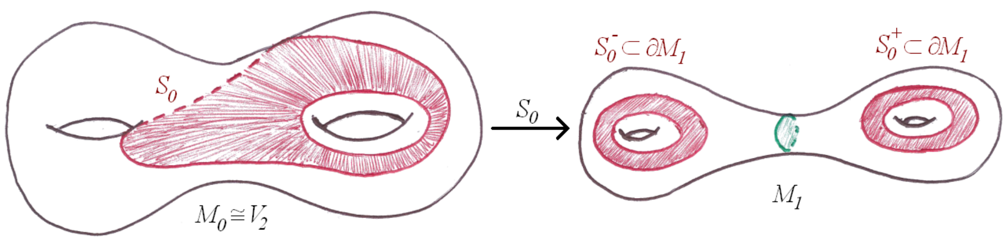

Figure 14: Note that S_0 is inside M_0, not on the boundary (although $\partial S_0 \subset \partial M_0$).

Equivalently, one can think of V_2 as $\big(T^2 - n(\big(B^2\big)\big) \times I$, as in Figure 15.

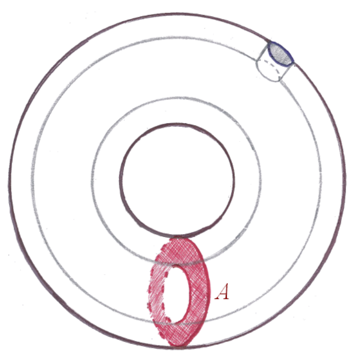

Figure 15: Another way to look at V_2.

Cut along A to get the pair of pants $\times I$, as in Figure 16.

Figure 16: Let $F = T - \text{int}(D)$ be a once-holed torus. Let G be a pair of pants. Cutting $F \times I$ along a vertical annulus gives a copy of $G \times I$. As $F \times I \cong G \times I$ this could lead to an infinite hierarchy, were we to allow non-essential surfaces.

Definition 27.3. If M_n is a collection of 3–balls, then the partial hierarchy is simply called a *hierarchy*.

Example 27.2. Let $M_0 = \mathbb{T}^3$, thought of as the unit cube in $\mathbb{R}^3$ with face pairings. Let $S_0 \subset M_0$ be the image of the xy–plane, so $S_0 \cong T^2$. Then $M_1 \cong T \times I$. Let S_1 be the image of the yz–plane, so $S_1 \cong A^2$, and $M_2 \cong D \times S^1$. Let S_2 be the image of the zx–plane, a disk. Then $M_3 \cong B^3$. See Figure 17.

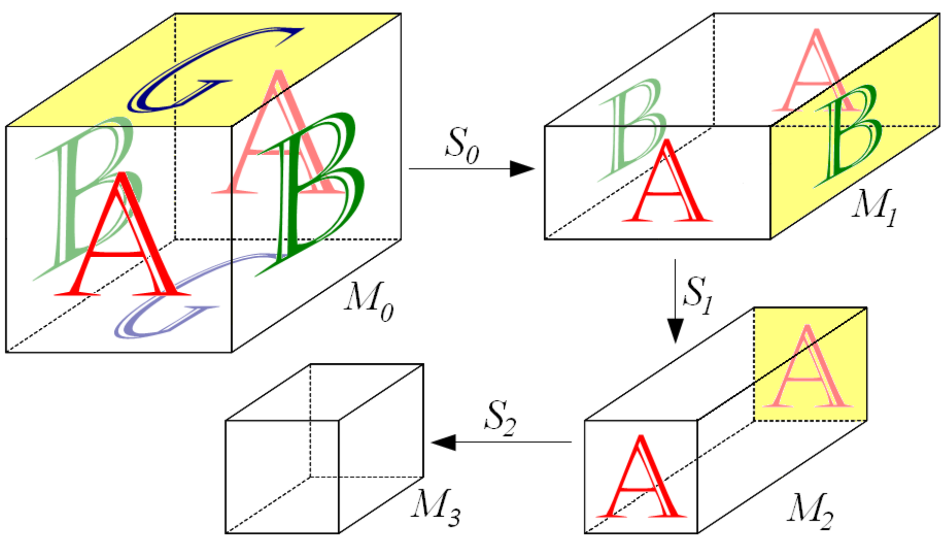

Figure 17: A hierarchy of length three for the three-torus.

Example 27.3. Let $M_0 = X_K$, where K is the (p, q)–torus knot, as shown in Figure 18, and let $S_0 = A$, the unique essential annulus. Then $X_K \xrightarrow{A} V_K \sqcup W_K = M_1$. Now letting S_1 be a pair of meridian disks, one in each of V_K and W_K, we find that $M_2 \cong B_1^3 \sqcup B_2^3$. See Figures 19 and 20.

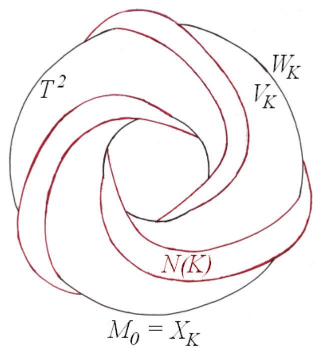

Figure 18: The (p, q)–torus knot complement, M_0.

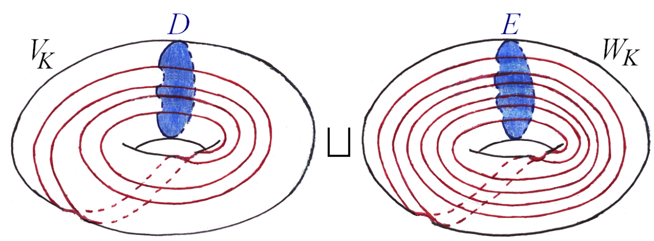

Figure 19: Compressing disks for V_K and W_K.

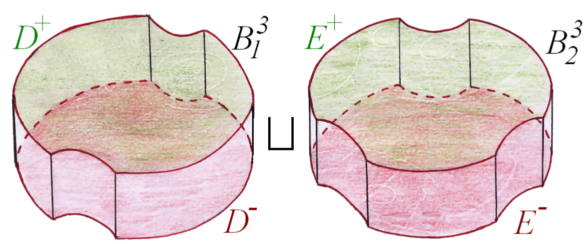

Figure 20: The final stage of the heirarchy.

Definition 27.4. If M is compact, orientable and irreducible, and $S \subset M$ is properly embedded, two-sided and essential, then M is called *Haken*.

Theorem 27.5. *If M is compact, orientable, irreducible and $\partial M \neq \varnothing$, then either M is a 3–ball or M is Haken.*

This theorem is implied by the following:

Theorem 27.6. *If M is compact, orientable and irreducible, and if*

$$\operatorname{rank}(H_1(M, \mathbb{Z})) \geq 1$$

then M is Haken.

Please let me know if any of the problems are unclear or have typos.

Exercise 9.1. Suppose that $\rho\colon X \to F$ is an I–bundle. Show that X is atoroidal if it is not homeomorphic to $T^2 \tilde{\times} I$.

Exercise 9.2. List all compact, connected 2–orbifolds B with $\chi_{\rm orb}(B) = 0$. Challenge: can you do the same when the orbifold Euler characteristic is positive?

Exercise 9.3. Find all orbifold double covers of $D^2(2,2)$ and of $S^2(2,2,2,2)$.

Exercise 9.4. Suppose that S is a horizontal surface in a Seifert fibered space M. Let α be any generic fiber and set $d = |S \cap \alpha|$. Let $B = M/S^1$. Prove that $\chi(S) = d \cdot \chi_{\rm orb}(B)$.

Exercise 9.5. Show that lens spaces are atoroidal.

Exercise 9.6. Let $P = \#_3 D^2$ be a *pair of pants*. Classify, up to proper isotopy, all essential loops and arcs in P.

Exercise 9.7. Define the *solid Klein bottle* to be $V = D \tilde{\times} S^1 = D \times I/(z,1) \sim (\bar{z},0)$; that is, we glue by a reflection. Show that V admits a partition into circles yet is not a Seifert fibered space, according to Hatcher's definition.

Exercise 9.8. Suppose that $K = K_{p,q}$ is a (p,q)–torus knot, with $|p|, |q| > 1$. Let $X = X_K$ be the knot exterior. Using the Seifert fibering prove the following statements.

- Any horizontal surface has negative Euler characteristic.
- There are only two essential vertical 2–sided surfaces in X: the boundary parallel torus and a separating annulus A so that $X - n(A)$ is a pair of solid tori.

Deduce that ∂X is incompressible, X is atoroidal, and $A \subset X$ is the unique essential annulus as stated in Lecture 17. It follows that the numbers $|p|, |q|$ are invariants of the homeomorphism type of X.

Exercise 9.9. Take K, X as in Exercise 9.8. Show that there is a horizontal surface $F \subset X$ with the following properties.

- The surface F *spans* K. That is, the boundary ∂F is a single curve, meeting the meridian of $N(K)$ is a single point.
- The surface F has genus $g(F) = (p-1)(q-1)/2$.
- There is no orientable spanning surface for K with lower genus.

Since F is non-separating and horizontal it follows that X is an F–bundle over the circle, with periodic monodromy.

MA4J2 Three Manifolds

Lectured by Dr Saul Schleimer
Typeset by Anna Lena Winstel
Assisted by Matthew Pressland and David Kitson

May 18, 2011

Lecture 28

Definition 28.1. Suppose M, N are 3–manifolds and $D \subset \partial M$ and $E \subset \partial N$ are disks. Let $\varphi \colon D \longrightarrow E$ be an orientation reversing homeomorphism. Then we define the *boundary connect sum* of M and N to be $M \#_\partial N := M \sqcup N/\varphi$. See Figure 1.

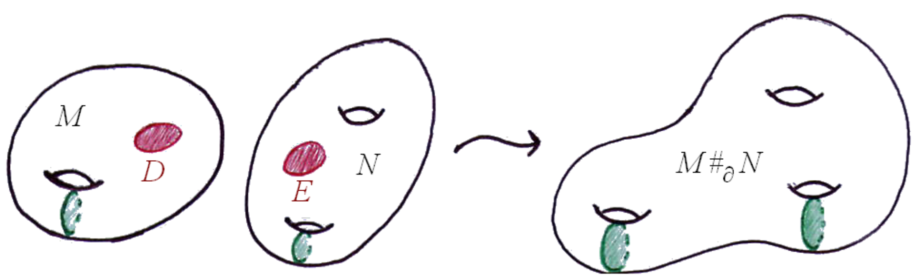

Figure 1: An example of the boundary connect sum.

Recall that φ only matters up to isotopy.

Definition 28.2. Suppose V is a handlebody and $F = \sqcup F_i$ is a collection of closed orientable surfaces, none of which is a two-sphere. Then $C := V \#_\partial (\#_\partial F_i \times I)$ is a *compression body*. We define the *inner boundary* $\partial_- C = \sqcup_{F_i \times \{0\}} C$ and the *outer boundary* $\partial_+ C = \partial C - \partial_- C$.

Example 28.1. See Figure 2.

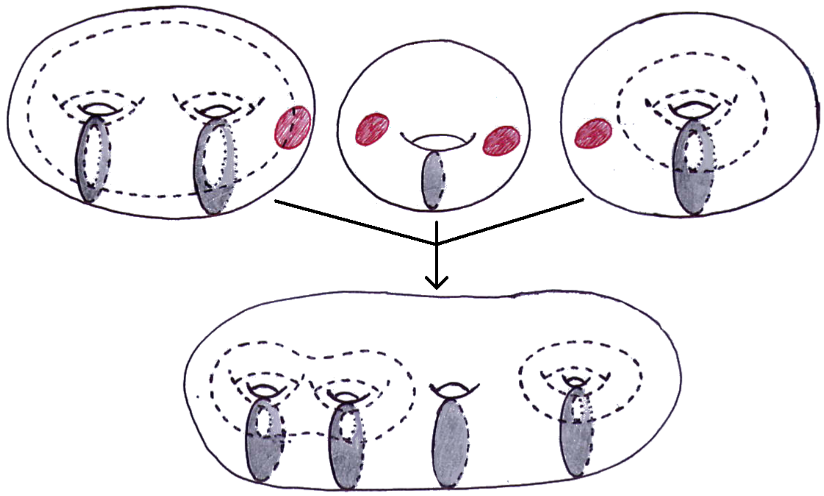

Figure 2: Another example of the boundary connect sum. Note that the third grey surface is a disk while the others are all annuli.

Exercise 28.1. Show that $\#_\partial$ is associative, commutative and B^3 is the unit.

Exercise 28.2. Show that the essential surfaces in C are

- essential disks compressing $\partial_+ C$,
- components of $\partial_- C$ and
- annuli meeting both $\partial_+ C$ and $\partial_- C$.

Example 28.2. See Figure 3.

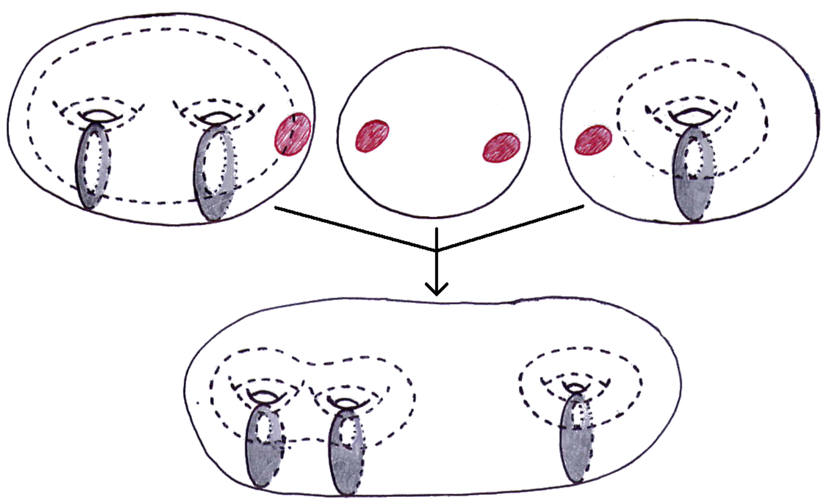

Figure 3: An example of the boundary connect sum.

Now we demonstrate the existence of short hierarchies, following Jaco. Suppose that M_0 is Haken and additionally that ∂M_0 is incompressible. Let $S_0 \subset M_0$ be a maximal collection of disjoint, non-parallel, closed, incompressible, two-sided surfaces in M_0 none of which are spheres. Since M_0 is Haken, S_0 is non-empty and it is finite by Haken-Kneser finiteness. See Figure 4.

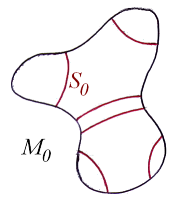

Figure 4: $S_0 \subset M_0$ is non-empty and finite. It is convenient to take $\partial M_0 = \varnothing$.

Aside. Note that closed incompressible surfaces, which are not spheres, are essential.

Note that every component $N \subset M_1 := M - n(S_0)$ has boundary with genus ≥ 1. So N contains some essential surface by Theorem 27.5. Let $S_1 \subset M_1$ be a maximal collection of disjoint, nonparallel, two-sided, essential surfaces in M_1: these are the green lines in Figure 5. Again, S_1 cuts every component of M_1 and S_1 is finite by Haken-Kneser finiteness in the bounded case. See the addedum to Exercise 5.5. Define $M_2 := M_1 - n(S_1)$ and let C be any component of M_2.

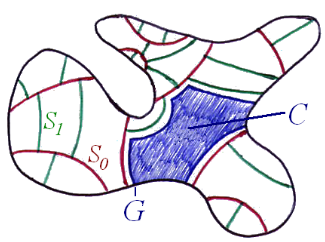

Figure 5: The component C contains an essential surface.

Proposition 28.1. *The component C is a compression body.*

Proof. Suppose that some component $G \subset \partial C$ is compressible into C. So let G_i, D_i be a sequence where $G_0 = G$ and D_i compresses G_i in the same direction as D_0, into C. Define $G_{i+1} = (G_i)_{D_i}$. So we get a sequence

$$G_0 \xrightarrow{D_0} G_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{n-1}} G_n.$$

See Figure 6.

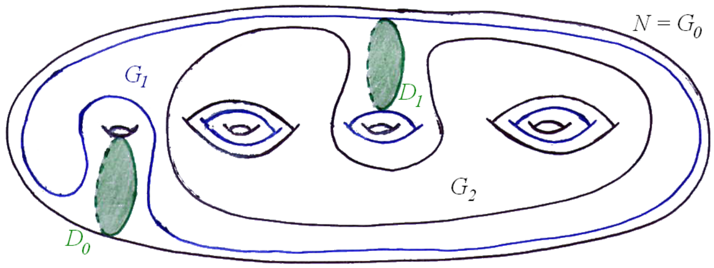

Figure 6: The first few terms in the sequence (G_i, D_i).

Note that G_{i+1} may be disconnected, as in Figure 7.

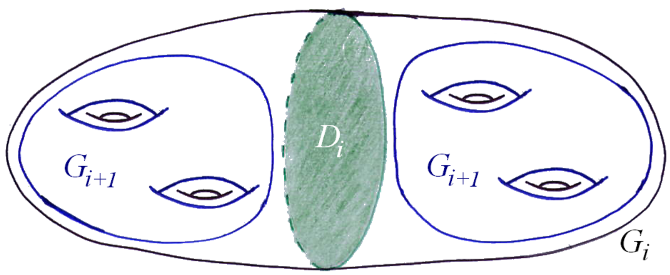

Figure 7: G_{i+1} may be disconnected.

Claim. If some component of G_n is a 2–sphere then it bounds a 3–ball in C.

Proof sketch. M is irreducible, thus C is irreducible as well. □

So cap off such 2–spheres, deleting them from G_n.

Claim. The closed surface G_n is incompressible in M.

Proof. As G_n is last in the sequence, G_n cannot compress into C. So suppose E is a surgery disk for G_n in the other direction. See Figure 8.

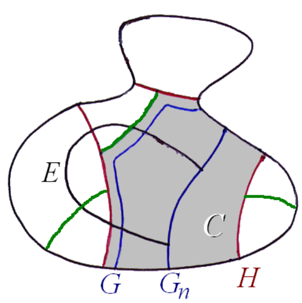

Figure 8: E is a compressing disk for G_n in the other direction. C is all of the grey area.

Then we can do the following: Isotope E off of S_0, then off of S_1 and then off of $\{D_i\}$. It follows that E is a surgery disk for G_n in the compression body cobounded by G_0 and G_n. Thus G_n is the inner boundary of this compression body and so is essential. Thus E is trivial, as desired. $\square$

To finish the proposition, deduce that the components of G_n are parallel to components $H \subset S_0$ since G_n is essential, closed and disjoint from S_0 (as it lies in C). Again see Figure 8. $\square$

Now let $S_2 \subset M_2$ be a collection of essential disks, cutting all compression bodies into products. Let $S_3 \subset M_3$ be a collection of vertical annuli (one per product). Finally $S_4 \subset M_4$ is a collection of disks cutting all handlebodies into 3–balls, as in Figure 9. This proves the existence of *short hierarchies.*

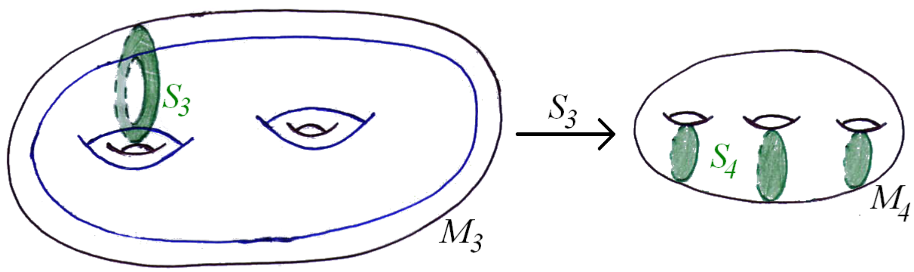

Figure 9: S_3 is a collection of vertical annuli; cut along these annuli to get a collection of handlebodies. Then cutting along S_4 gives a collection of 3-balls.

Lecture 29

In this lecture, we again follow Lackenby.

Definition 29.3. A *boundary pattern* P for M^3 is a trivalent graph embedded in ∂M. We allow P to be the empty set, to be disconnected and to have simple closed curves as components.

Example 29.3. Trivalent graphs in $S^2 = \partial B^3$ are patterns for B^3. See Figure 10.

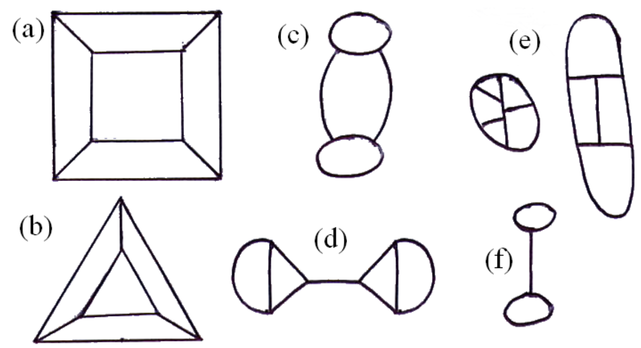

Figure 10: Six examples of trivalent graphs in S^2. Note that (e) is a disconnected pattern.

Suppose (M, P) is a manifold equipped with a boundary pattern. Suppose $S \subset M$ is properly embedded and ∂S is transverse to P. So ∂S misses the vertices of P and intersects the edges of P transversely. Let $N = M - n(S)$ and let

$$Q = (P - n(S)) \cup \partial S^+ \cup \partial S^-.$$

So Q is a pattern for N and we write $(M, P) \xrightarrow{S} (N, Q)$. See Figure 11.

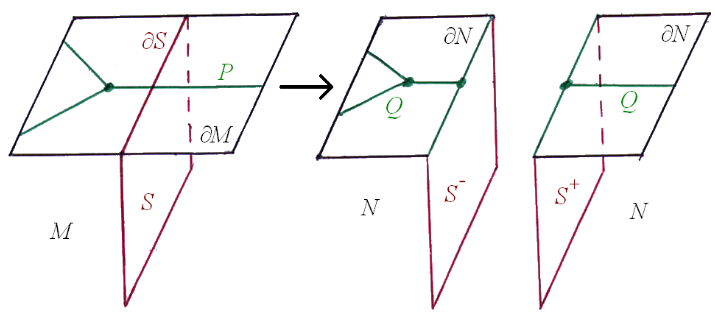

Figure 11: A picture of the cutting.

Definition 29.4. Let P be a boundary pattern for M. Then we call P *essential* if for any $(D, \partial D) \subset (M, \partial M)$ with ∂D transverse to P and $|\partial D \cap P| \leq 3$ we have

- a disk $E \subset \partial M$ such that $\partial E = \partial D$ and
- the intersection $E \cap P$ contains at most one vertex of P and contains no cycles of P.

Exercise 29.3. Verify that if P is essential then we get the implications shown in Figures 12 to 15:

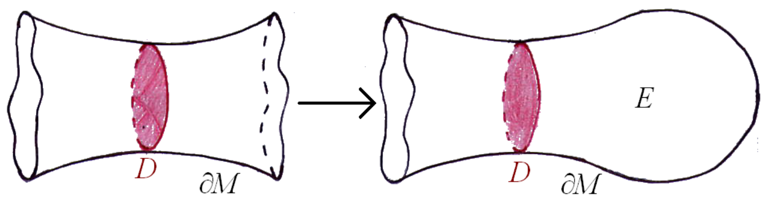

Figure 12: The case $\partial D \cap P = \varnothing$.

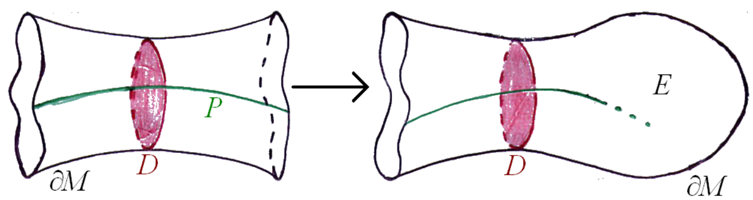

Figure 13: The case $|\partial D \cap P| = 1$ is not possible.

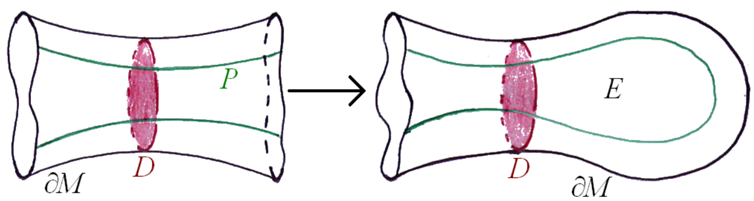

Figure 14: The case $|\partial D \cap P| = 2$.

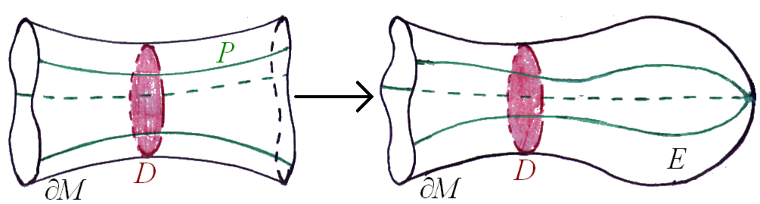

Figure 15: The case $|\partial D \cap P| = 3$.

Exercise 29.4. Analyse the examples of (B^3, P) given above. Which are, and which are not, essential?

Exercise 29.5. Give necessary and sufficient conditions for P to be an essential pattern for B^3.

Example 29.4. If $M_0 = \mathbb{T}^3 = I^3/\!\sim$ then $S_0 = \{z = 0\}$ is an essential torus, $S_1 = \{x = 0\} \subset M_1$ is an essential annulus and $S_2 = \{y = 0\} \subset M_2$ is an essential disk. We can see this in Figure 16.

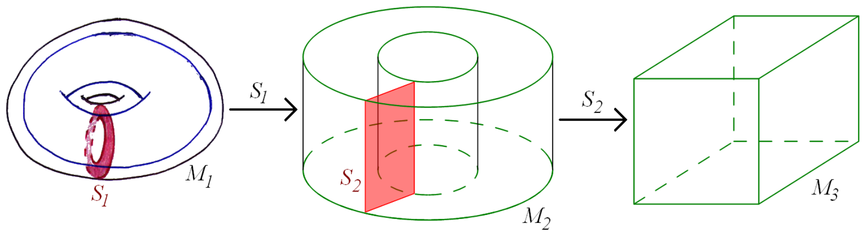

Figure 16: Pictures of these cuttings with boundary patterns. For M_3, P_3 is the 1–skeleton of the cube.

Definition 29.5. Let $P \subset \partial M$ be a pattern. We say P is *homotopically essential* if the following condition hold. For any map $f\colon (D, \partial D) \longrightarrow (M, \partial M)$ (which need not be an embedding) transverse to P, we define $Z = Z_f = \partial D \cap f^{-1}(P)$. If $|Z| \leq 3$ then there is a homotopy $H\colon D \times I \longrightarrow M$ such that

- for all t: $H_t|Z = f|Z$,
- $H_0 = f$,
- $H_1(D) \subset \partial M$ and finally
- $H_1(D)$ contains at most one vertex of P and contains no cycles of P.

Exercise 29.6. If P is homotopically essential, then P is essential.

Theorem 29.1 (9.1 in Lackenby)**.** *If P is essential, then it is homotopically essential.*

We will indicate a proof, using *special hierarchies*, in the next lecture.

Exercise 29.7. Theorem 29.1 implies the Disk Theorem. As a hint, recall that we allow $P = \varnothing$.

Lecture 30

We pause to give another example of a hierarchy.

Example 30.5. Consider the knot $K \subset S^3$ shown in Figure 17: the $(1, 1, -3)$–pretzel knot. The surface shown is a spanning surface for K. This is one of the two so-called *checkerboard surfaces* for this diagram of K.

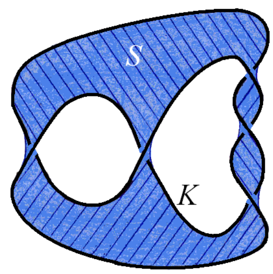

Figure 17: A diagram of the $(1, 1, -3)$–pretzel and S, one of its two checkerboard surfaces.

Near a twist we see a half-twisted band, as in Figure 18.

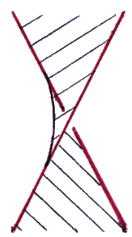

Figure 18: A half twisted band.

Let $N = N(K)$ be a regular neighbourhood and write $X = X_K = S^3 - n(K)$. See Figure 19. Let S_0 be the remains of the spanning surface in X.

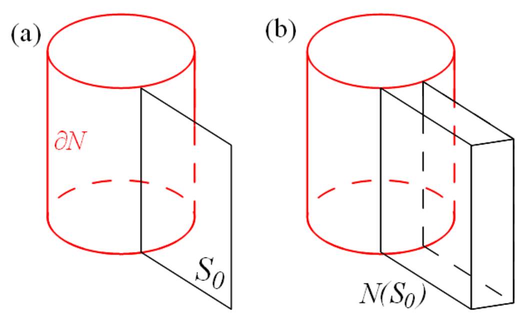

Figure 19: (a) A picture of $N(K)$, S_0 and (b) $N(S_0)$.

Let $M_0 = X$ and cut M_0 along S_0 to get M_1. Thus, as M_1 is a genus two handlebody, we find that $\partial S_0^{\pm}$ gives a pattern to ∂M_1, shown in Figure 20.

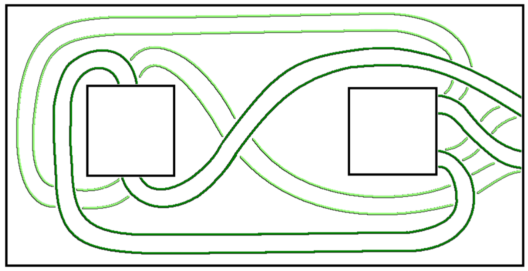

Figure 20: A pattern to ∂M given by $\partial S^{\pm}$. Note that M_1 is the handlebody on the outside.

The two components of P in ∂M_1 cobound an annulus, the remains of ∂N. We take S_1 to be the union of a pair of disks as in Figure 21.

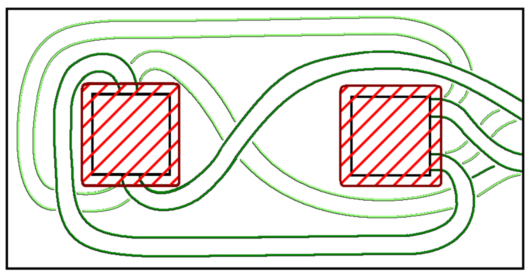

Figure 21: The essential surface S_1 in M_1, consisting of two disks which meet ∂M_1 in two loops around the holes.

Now cut along S_1 to get $M_2 \cong B^3$.

Exercise 30.8. Show that (M_2, P_2) is homeomorphic to the pattern shown in Figure 22.

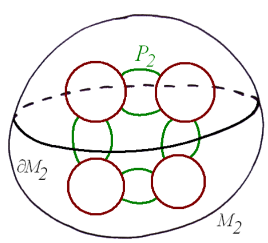

Figure 22: A 3–ball with a pattern.

Exercise 30.9. Show that $P_2 \subset \partial M_2$ is essential. Figure 23 may be helpful.

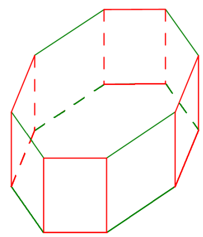

Figure 23: $(M_2, P_2) \cong \mathrm{Oct} \times I$ where Oct denotes an octagon.

Claim. The surface $S_0 \subset X$ is essential.

Proof. Suppose $(D, \partial D) \subset (X, S_0)$ is a surgery disk. So consider $D \cap S_1 \subset D$. This is a collection of simple loops and arcs.

1. Suppose α is an innermost loop. Then α bounds E in D. So $(E, \alpha) \subset (M_2, \partial M_2)$ and $\alpha \cap P_2 = \varnothing$ which implies that we may isotope E past S_1, reducing $|S_1 \cap D|$. See Figure 24.

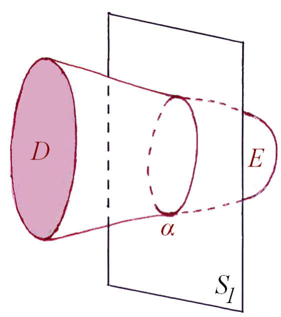

Figure 24: We may isotope E past S_1, reducing $|S_1 \cap D|$.

2. Suppose $\alpha \subset D$ is an outermost arc of $S_1 \cap D$. So α cuts off a bigon E. So $(E, \partial E) \subset (M_2, \partial M_2)$ is a bigon and $\partial E \cap P_2$ is exactly two points. But (M_2, P_2) is essential and we continue as usual.

So we may assume that $D \cap S_1 = \varnothing$. So $(D, \partial D)$ embeds in $(M_2, \partial M_2)$ with $\partial D \cap P_2 = \varnothing$. Since M_2 is a ball we find that D is parallel to a disk $D' \subset S_0$. So S_0 is incompressible. Now by Lemma 20.2 (1.10 in Hatcher) S_0 is boundary incompressible. It is also possible to directly prove that by repeating the proof using bigons. See Figure 25. $\square$

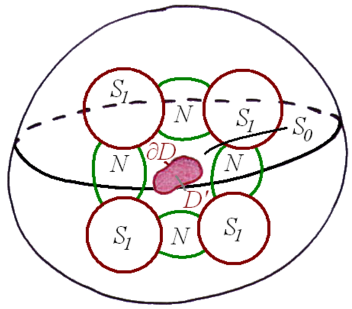

Figure 25: D is parallel to a disk $D' \subset S_0$.

We now give the ideas necessary to prove Theorem 29.1. We need a few more definitions.

Definition 30.6. Suppose $S \subset (M, P)$ is properly embedded and suppose $P \subset \partial M$ is an essential pattern. A surgery bigon D for S is a *pattern surgery* if $|\beta \cap P| \leq 1$ where $\partial D = \alpha \cup \beta$ and $\alpha = \partial D \cap S$. Say D is *trivial* if α cuts a bigon E out of S with $\partial E = \alpha \cup \gamma$ and $|\gamma \cap P| \leq 1$. Otherwise call D a *pattern compression.*

Definition 30.7. If S is essential and all pattern surgeries are trivial, we call S *pattern essential.* See Figure 26.

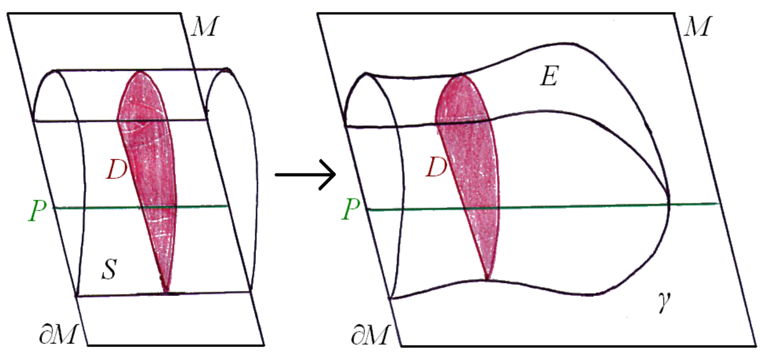

Figure 26: A picture of what it means to be pattern essential.

Definition 30.8. A *special hierarchy* is a sequence $(M_i, P_i) \xrightarrow{S_i} (M_{i+1}, P_{i+1})$ where all P_i are essential and all S_i are pattern essential. We do not allow S_i to be a sphere.

Proposition 30.2. *If $S \subset (M, P)$ is essential we may isotope S to be pattern essential.*

Proof. Exercise. $\square$

Using the above one can show the following two propositions which imply Theorem 29.1.

Proposition 30.3. *If P is a pattern for $M \cong B^3$ and is essential, then P is homotopically essential.* $\square$

Proposition 30.4. *If $(M, P) \xrightarrow{S} (N, Q)$ are all essential and $Q \subset \partial N$ is homotopically essential, then P is homotopically essential in M.* $\square$

Please let me know if any of the problems are unclear or have typos.

Exercise 10.1. Classify the Seifert fiberings of $T \times I$, up to isotopy. (The classification up to homeomorphism is much simpler.)

Exercise 10.2. [Hard] Show that the Pachner moves (also called *bisteller flips*) do not change the quantity $\sum_{k=0}^{n}(-1)^k|T^{(k)}|$.

Exercise 10.3. [Part of the proof of Proposition 1.13] Suppose that $(M, \mathcal{F})$ is a Seifert fibered space and $T \subset \partial M$ is a torus. Suppose that $(D, \partial D) \subset (M, T)$ is a compressing disk. Deduce that the orbifold $B = M/S^1$ is a disk with at most one cone point. Deduce $M \cong D \times S^1$.

Exercise 10.4. Suppose that T_i, for $i = 0, 1$, are copies of $M^2 \tilde{\times} I$. Let $A_i = \partial_h T_i$. Show that for any homeomorphism $\phi\colon A_0 \to A_1$ the manifold $T_0 \cup_\phi T_1$ is homeomorphic to $K^2 \tilde{\times} I$. Classify Seifert fiberings on $K \tilde{\times} I$, up to isotopy.

Exercise 10.5. [Reading exercise] Read the proof of the uniqueness statement in Theorem 1.9 in Hatcher's notes.

Exercise 10.6. Let D be the dodecahedron. Let $P = D/\!\sim$ be the space obtained by gluing opposite faces via $1/10^{\text{th}}$ right-handed rotation. Show that the result is a three-manifold. Give a presentation of $\pi_1(P)$. Check that $H_1(P, \mathbb{Z})$ is trivial. [What manifold do you obtain if you instead use rotation by $1/2$? The manifold obtained via $3/10^{\text{th}}$ rotation is harder to understand.]

Exercise 10.7. Suppose that M is irreducible, connected, $T \subset \partial M$ is a torus, and $\pi_1(M) \cong \mathbb{Z}$. Prove that $M \cong D \times S^1$.

Exercise 10.8. Suppose that F is a properly embedded 2–sided surface in M^3. Suppose that $\Gamma = \ker(\pi_1(F) \to \pi_1(M))$ is nontrivial. Then there is an essential, simple loop in Γ.

Exercise 10.9. Show that Exercise 10.8 is false if we remove the two-sided hypothesis.

Please let me know if any of the problems are unclear or have typos.

Exercise 11.1. Suppose that F is a closed connected orientable surface other than the two-sphere. Give a hierarchy for $F \times I$. Give two distinct hierarchies for $F \times S^1$.

Exercise 11.2. Suppose that C is a compression body. Show that any essential surface in C is either

- a compressing disk for $\partial_+ C$,
- a component of $\partial_- C$, or
- an annulus that meets both $\partial_\pm C$.

Exercise 11.3. Suppose that S is essential in M, a Haken three-manifold. Let $N = M - n(S)$. Let $F \subset \partial N$ be an component and let G be the result of maximally compressing F into N, always in the same direction, and then discarding two-sphere components. Show that G is incompressible in M.

Exercise 11.4. Suppose that Q is a regular n–gon in the plane. Let V be the vertices of Q. Let $B = Q \times I$ be a three-ball, with boundary pattern $P = (\partial Q cross\{0, 1\}) \cup (V \times I)$. Show that, if $n > 3$, that P is an essential boundary pattern. Next, classify pattern-essential surfaces in (B, P).

Exercise 11.5. Deduce the Disk Theorem from Theorem 9.1 in Lackenby's notes.

Exercise 11.6. Let $K \subset S^3$ be the 5_2 knot, as shown in Figure 1. Let $S \subset X = X_K$ be the shaded surface shown in the figure. Check that S is orientable. As done in class, let $M_1 = X - n(S)$. Check that M_1 is a handlebody, and carefully draw the boundary pattern P_1. Now cut along disks to get M_2. Check that M_2 is a three-ball, and carefully draw the resulting pattern P_2. Using the above or otherwise prove that that S is essential.

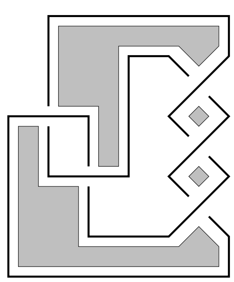

Figure 1: The 5_2 knot, with Seifert surface shaded.