Limits and Continuity

Vanderbilt Mathematics

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All of calculus is built on the technical tools of limits. Without limits, there are no derivatives and there are no integrals. In other words, no calculus. Without calculus, most modern science and technology would not be possible. Calculus is one of the most influential ideas of humankind, as it can be used to model the motion of celestial bodies, the behavior of the stock market, the growth of bacteria, the radioactive decay of elementary particles, or any quantity that depends on another quantity.

The basic idea of a limit is very simple: as the input of a function gets closer and closer to a certain number, we ask: what number is the output getting closer and closer to? Is it getting close to anything at all?

Another very important idea in calculus is the notion of a continuous function. Lots of functions in real life are continuous (or at least can be approximated by a continuous function), for instance your body temperature as a function of time. (Is your height a continuous function of time?) Again, the basic idea is simple: a function is continuous if we can draw its graph without lifting our pencil off of the paper.

These kinds of intuitions are important. But they are not enough because they are not precise. In mathematics we give formal definitions of our terms so that we can communicate our ideas unambiguously and so we can argue with confidence that certain statements are true or false. The power of mathematics lies in its ability to give yes or no answers to concrete questions. These answers can often be used to understand models, or approximations, of real-life problems.

If you open almost any textbook on calculus, the definition of a limit will look like this: if $f : \mathbb{R} \to \mathbb{R}$ is a function, we say that

$$\lim_{x \to a} f(x) = L$$

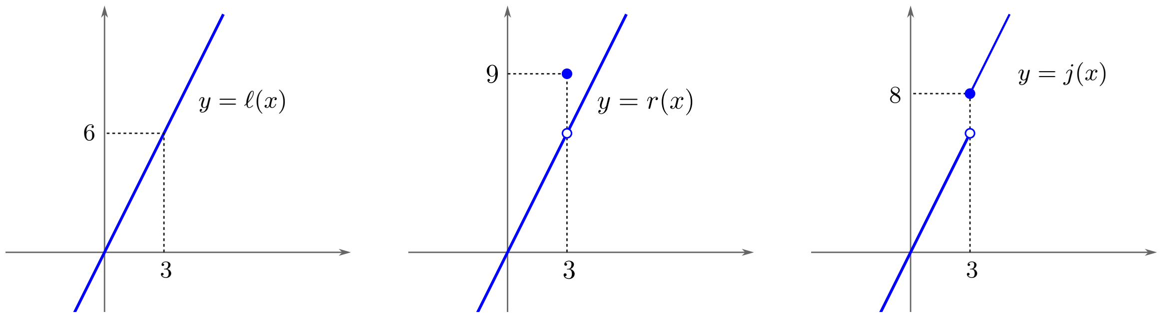
(or, the limit of $f(x)$ as x approaches a equals L) if

for all $\epsilon > 0$ *there is a* $\delta > 0$ *so that whenever* $0 < |x - a| < \delta$
it is true that $|f(x) - L| < \epsilon$.

And we say that $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

There is so much to unpack here, and we will not try to unpack this definition here. Instead, we will give precise definitions of limits and continuity, but using different language. These notes are intended for students learning the definition of continuity for the first time, say, in a first-year Calculus course. As the reader moves on in mathematics, we encourage them to learn the ϵ–δ definition (and its generalizations) and how it relates to the definition here.



1 Continuity

The basic intuition about continuity is that a function is continuous if we can draw the graph without lifting our pencil off of the paper. More specifically, a function $f(x)$ is continuous at $x = a$ if we can draw the part of the graph of $f(x)$ near $(a, f(a))$ without lifting our pencil off the paper. As we will see, this intuition works well for the simplest of examples, but we will quickly find examples that confound this intuition.

There is another way to think about continuity that foreshadows our formal definition a little better. You are already familiar with the idea of zooming in on a photo. Another intuitive idea explanation of continuity is: no matter how much we zoom in on the graph of $f(x)$ at the point $(a, f(a))$ we do not see a break in the graph. This still is not mathematically precise, and still does not apply to all examples, but it moves us in the right direction.

Before diving into the definition of continuity, we try to understand what it is by looking at a number of examples.

1.1 Examples

We will now introduce five different functions and discuss their continuity, or lack thereof, at various places. You should make friends with the five functions, as they will reappear a number of times in these notes.

Example 1: a linear function. To start with a simple example, consider $\ell(x) = 2x$. Clearly we can draw a line like this without lifting our pencil off of the paper. So we should believe that $\ell(x)$ is a continuous function at all real numbers a in the domain. In the case where a function is continuous at all real numbers a, we simply say that the function *is* continuous.

Example 2: a removable discontinuity. Let us consider now the function

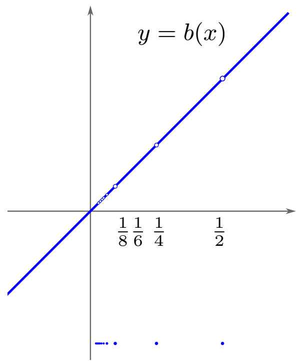
$$r(x) = \begin{cases} 2x & x \neq 3 \\ 9 & x = 3 \end{cases}$$

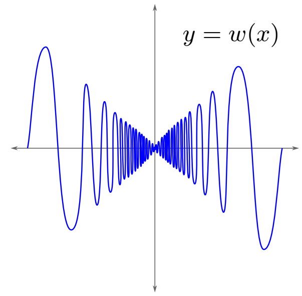
It is pretty clear from the picture that we would not be able to draw the graph without lifting up our pencil. Somehow, we have to draw the point $(3, 9)$. So it seems like a no-brainer that $r(x)$ is not continuous at $x = 3$ (although it is continuous everywhere else!). We say that $r(x)$ has a discontinuity at $x = 3$. We call this a removable discontinuity because we can remove the discontinuity—meaning, we can make the function continuous—by changing the value of $r(x)$ at a single value of x.

Example 3: a jump discontinuity. We consider now the function

$$j(x) = \begin{cases} 2x & x < 3 \\ 2x + 2 & x \geq 3 \end{cases}$$

Again, this is a discontinuous function, because we need to jump up from 6 to 8 when we get to $x = 3$. This time, there is no way to redefine $j(3)$ in order to make a continuous function. So this





discontinuity is in a sense worse than the last one. It is still true that j is continuous at every x not equal to 3.

Example 4: a bouncy function. We define a function as follows:

$$b(x) = \begin{cases} -1/2 & x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \\ x & \textit{otherwise} \end{cases}$$

Here, the dot-dot-dot means "and so on."

Like the previous examples, we can see that $b(x)$ is discontinuous at $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$ But there is a more refined question we can ask: is $b(x)$ continuous at 0? This is more mysterious. The question is: can we draw part of the graph near zero without lifting our pencil? The answer is no, because no matter how much we zoom in on the picture near $x = 0$, there will be points on the graph with $y = -1/2$. Our intuition is still working here, but hopefully you agree that it is starting to become less tenable to use the intuition only.

Example 5: A wiggly function. This is our most complicated function yet:

$$w(x) = \begin{cases} x \sin(\frac{\pi}{2x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Again, our picture is only an approximation; this function has infinitely many wiggles as we get closer to the origin. Can we draw this graph without lifting our pencil? Well, since there are infinitely many wiggles near the origin, it's not clear that we can ever draw this graph at all, let alone without lifting our pencil! We could try to adjust our intuitive notion of continuity so that we have infinitely many pencils(?), but we would be getting into murky waters. Instead, we will take the cue to make an honest, formal mathematical definition of continuity.

Preview of things to come. Hopefully you are convinced that we need a proper definition of continuity. We will create the definition in several steps, by describing three concepts:

- collapsing collections,
- wrappers, and
- images of intervals.

We discuss these concepts in separate sections, before putting them together to define continuity.

Exercises

Draw graphs of the following functions (perhaps use a computer). Are they continuous, or are there values of x where they are not continuous? For now, make your best guess. We'll revisit these below.

1. $f(x) = 5$
2. $f(x) = x^{2/3}$
3. $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$
4. $f(x) = \begin{cases} x+3 & x \leq 1 \\ -2x+6 & x \geq 1 \end{cases}$
5. $f(x) = \begin{cases} x^2 & x \leq 2 \\ x+2 & x \geq 2 \end{cases}$
6. $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

1.2 Collapsing collections

As above, the intuition behind continuity is that we are zooming in on one part of the graph, namely, the graph of the numbers near a. The ideas in this section are the first step to making this idea more formal. Before giving the definition of a collapsing collection, we present some examples.

Example 1: Using more and more decimal places. Let's consider a function $f(x)$ and the input $a = 3$. In the spirit of zooming in on the graph of $f(x)$, we want to consider the points $(x, f(x))$ where x is close to 3. Here is a list of open intervals consisting of real numbers closer and closer to 3:

$$(2.9, 3.1) \qquad (2.99, 3.01) \qquad (2.999, 3.001) \qquad \cdots$$

Hopefully you can see where we are going with this. What is the next term in the list? The 10th term? Can you find a formula for the nth term of the list? Here are some properties of the above list of intervals:

1. each interval of the collection is an open interval,
2. each interval is contained in the previous one,
3. 3 is in every interval, and
4. 3 is the only number in every interval.

The last item is the most striking. Can you convince yourself of this? One important point to emphasize is that none of the numbers used to define the intervals is equal to 3; they are all a little more or a little less than 3. Even the number 3.0000000000000000001 is not equal to 3 (why?).

To understand, you might imagine that you have a photo of the real numbers on your phone. You zoom in around the number 3, getting closer and closer to the number 3. As you zoom in, you are seeing numbers closer and closer to 3. But somehow the zooming never stops (unlike on your phone)! We call this as the Archimedean property of the real numbers.

Example 2: Using reciprocals of larger and larger numbers. Again let us consider the number $a = 3$. Here is another list of open intervals we might consider:

$$(2, 4) \qquad (5/2, 7/2) \qquad (8/3, 10/3) \qquad (11/4, 13/4) \qquad \cdots$$

Can you see the pattern? What about if we write the list like this:

$$(3-1, 3+1) \qquad (3-\tfrac{1}{2}, 3+\tfrac{1}{2}) \qquad (3-\tfrac{1}{3}, 3+\tfrac{1}{3}) \qquad (3-\tfrac{1}{4}, 3+\tfrac{1}{4}) \qquad \cdots$$

What are the next few terms in the list? This collection has the same four properties as before. Make sure you believe all four statements before moving on!

Collapsing collections of open intervals. You hopefully have a pretty good guess for what our next definition will look like.

Definition. For a real number a, a collapsing collection of open intervals for a is a list of intervals so that

1. each interval in the list is an open interval,
2. each interval is contained in the previous one,
3. a is in every interval, and
4. a is the only number in every interval.

Optional side note. In the definition it is actually not important that the intervals come in order. We could just as well have a collection of intervals with the property that if we take any two of the intervals, one is contained in the other (a mathematician might say that any pair of intervals is nested). This justifies the use of the terminology "collapsing collection" instead of "collapsing list."

Example 3: Collapsing collections for an arbitrary number a. As in the previous section, we can do all this with an unspecified number a:

$$(a-1,a+1) \qquad (a-\tfrac{1}{2},a+\tfrac{1}{2}) \qquad (a-\tfrac{1}{3},a+\tfrac{1}{3}) \qquad (a-\tfrac{1}{4},a+\tfrac{1}{4}) \qquad \cdots$$

By changing the number a, we obtain infinitely many different collapsing collections of open intervals, one collection for each number a.

Collapsing collections of closed intervals. We will also have use for collapsing collections of closed intervals. This is defined in the exact same way as collapsing collections of open intervals. We just replace the word "open" with "closed."

Definition. For a real number a, a collapsing collection of closed intervals for a is a list of intervals so that

1. each interval in the list is an closed interval,
2. each interval is contained in the previous one,
3. a is in every interval, and
4. a is the only number in every interval.

Here is one example:

$$[-\tfrac{1}{2},\tfrac{1}{2}] \qquad [-\tfrac{1}{3},\tfrac{1}{3}] \qquad [-\tfrac{1}{4},\tfrac{1}{4}] \qquad \cdots$$

Convince yourself that this is a collapsing collection of closed intervals for the number 0. What about the next one?

$$[1,1] \qquad [1,1] \qquad [1,1] \qquad \cdots$$

This is a strange one, but we will need it in some examples. The closed interval $[1,1]$ has one number, namely 1. So this is a collapsing collection of closed intervals.

With more rigor. In mathematics, we like to prove that statements are true (or not true). While it should be intuitively clear that our examples of collapsing collections are indeed collapsing collections, we will explain here how a mathematician might write down a formal rigorous proof. On a first pass through Calculus, the reader may or may not want to read the paragraphs that we have labeled with the words "rigor" or "fine print." We hope that some readers will be interested, and so are providing these explanations for completeness.

Consider for instance the collection of open intervals

$$(a-1,a+1) \qquad (a-\tfrac{1}{2},a+\tfrac{1}{2}) \qquad (a-\tfrac{1}{3},a+\tfrac{1}{3}) \qquad (a-\tfrac{1}{4},a+\tfrac{1}{4}) \qquad \cdots$$

Let us check that this really is a collapsing collection. Of the four properties of a collapsing collection of open intervals, the only property that is not immediately clear is the last one, namely, that a is the only number in all of the intervals. Let us prove this formally. Suppose that b is a number that is not equal to a. Say that b is larger than a (the argument for the case where b is smaller is similar). Then $b-a$ is the distance from a and b on the number line, and

$$\frac{1}{b-a}$$

is a real number. There is a whole number N that is positive and larger than $1/(b-a)$. But then we can verify that b is not in the interval

$$(a-\tfrac{1}{N},a+\tfrac{1}{N})$$

from the collapsing collection. Indeed, we have

$$\begin{aligned} b &= a+(b-a) \\ &> a+1/N. \end{aligned}$$

since b is larger than a, this tells us that b is not in the interval. To summarize, we took any number b that was not equal to a and showed that it is not in all of the intervals in the collapsing collection. It follows that a is the only number in all of the intervals, which is the desired fourth property.

Again, we have just explained in details something that should be fairly intuitive. This kind of argument is a good warm-up for more serious arguments we will do below (and you might do later in your career!).

Exercises

1. Write down the first few intervals in a collapsing collection of open intervals for the number -1.

2. Write down the first few intervals in a collapsing collection of open intervals for the number 0.0001.

3. Is this list of intervals a collapsing collection? Explain.

$$\begin{gathered}(-1.01,1.01)\\(-1.001,1.001)\\(-1.0001,1.0001)\\(-1.00001,1.00001)\ldots\end{gathered}$$

4. Is this list of intervals a collapsing collection? Explain.

$$\begin{gathered}(0.9,1.1)\\(1.99,2.01)\\(2.999,3.001)\\(3.9999,4.0001)\\(4.99999,5.00001)\\\vdots\end{gathered}$$

5. Is the list of intervals of the form $(-1/2^n,1/2^n)$ a collapsing collection? Here are the first few intervals on the list:

$$(-\tfrac{1}{2},\tfrac{1}{2}) \qquad (-\tfrac{1}{4},\tfrac{1}{4}) \qquad (-\tfrac{1}{8},\tfrac{1}{8}) \qquad \cdots$$

6. Describe a collapsing collection of open intervals for π.

1.3 Images of intervals

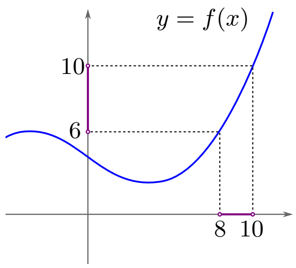
We are already familiar with the idea of applying a function to a number: we just plug the number into the function. For example, if $f(x)=x^2$. Then when we apply f to the number -3, we obtain

$f(-3) = 9$. In the mathematical parlance, we say that 9 is the *image* of -3 under the function $f(x) = x^2$. We can also say that 9 is the f-image of -3.

In this section, we introduce the idea of applying a function to an entire interval. We can also say that we are finding the image of the interval under a given function.

Definition. The image of an interval (a, b) under a function $f(x)$, also called the f-image of (a, b) is the collection of all outputs $f(c)$ where c is a number in (a, b).

In other words, we plug every number in the interval, and record all of the resulting outputs. An effective way to understand these outputs is from the graph.



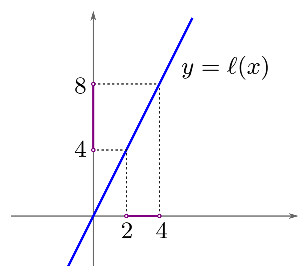
As the picture suggests, the f-image of $(8, 10)$ is $(6, 10)$; we can also write this as $f(8, 10) = (6, 10)$. The method for computing f-images is: (1) we start at points of the interval $(8, 10)$ on the x-axis, (2) we travel vertically to the graph of f, and (3) we then travel horizontally to the y-axis. We follow this procedure for all points in $(8, 10)$ and record all of the outputs.

We continue with some concrete examples, our some of our favorite functions.

Example 1: a linear function. Let us start with our simple example, consider $\ell(x) = 2x$. And let us apply the function ℓ to the interval $(2, 4)$. Again, we write this is $\ell(2, 4)$. What is the output? Let us draw the picture:

We can also plug in a few numbers

x	$\ell(x)$
2.01	4.02
3	6
3.99	7.98



From the picture and from the table, we can guess that

$$\ell(2, 4) = (4, 8)$$

This is indeed correct. For any input a we have $2 < a < 4$ and so $4 < 2a < 8$. This means that $\ell(2, 4)$ must be a subset of $(4, 8)$. We want to check that we get all of $(4, 8)$. If we take any number b in $(4, 8)$, then $b/2$ is in $(2, 4)$. This means that we can plug $b/2$ into ℓ and get the output b. This is just what we wanted.

Let us preview our definition of a continuity by applying $\ell(x)$ to smaller and smaller open intervals containing the number 3.

(a, b)	$\ell(a, b)$
$(2.9, 3.1)$	$(5.8, 6.2)$
$(2.99, 3.01)$	$(5.98, 6.02)$
$(2.999, 3.001)$	$(5.998, 6.002)$
$(2.9999, 3.0001)$	$(5.9998, 6.0002)$

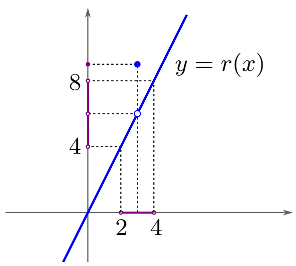
We can see that as we apply $\ell(x)$ to smaller and smaller intervals containing 3, we obtain smaller and smaller intervals containing $\ell(3) = 6$. This example is central to our definitions of limits and continuity.

From this example, we might think that the way to find $\ell(2, 4)$ is to just apply ℓ to the two numbers defining the input interval. That will not be true in general, as we will see.

Example 2: a removable discontinuity. Let us consider again the function

$$r(x) = \begin{cases} 2x & x \neq 3 \\ 9 & x = 3 \end{cases}$$

Let us evaluate $r(x)$ on the open interval $(2, 4)$. We draw the picture:



Something strange happens here: we see that 6 is not an output of $r(x)$. This is because the only input that can give us the output of 6 would be 3, but we have defined $r(3) = 9$. From the picture and this discussion, we can hopefully see that

$$r(2, 4) = (4, 6) \cup (6, 8) \cup \{9\}$$

Here we have the union of two intervals together with one more number, namely 9. Here the union symbol $\cup$ should be thought of as "and." In other words the r-image of $(2, 4)$ is $(4, 6)$ with $(6, 8)$ and the number 9.

What if we try the same thing we tried in Example 1? Let us apply the function $r(x)$ to smaller and smaller intervals containing the number 3.

(a, b)	$r(a, b)$
$(2.9, 3.1)$	$(5.8, 6) \cup (6, 6.2) \cup \{9\}$
$(2.99, 3.01)$	$(5.98, 6) \cup (6, 6.02) \cup \{9\}$
$(2.999, 3.001)$	$(5.998, 6) \cup (6, 6.002) \cup \{9\}$

In this case we are not obtaining smaller and smaller intervals around $r(3) = 9$. Nor are we obtaining smaller and smaller intervals around 6, which is where the white dot is in the picture. As we will see, this is exactly the reason why $r(x)$ is not continuous at $x = 3$.

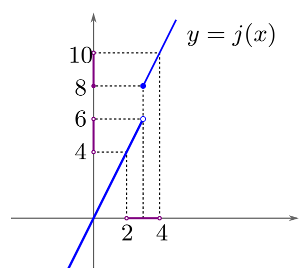
Example 3: a jump discontinuity. Let us now revisit our function $j(x)$:

$$j(x) = \begin{cases} 2x & x < 3 \\ 2x + 2 & x \geq 3 \end{cases}$$

Let us apply this function to the interval $(2, 4)$.

We can see from the picture that the answer is

$$j(2, 4) = (4, 6) \cup [8, 10).$$



This is different from what we saw above. Let us try our technique of applying the function $j(x)$ to smaller and smaller intervals containing 3.

(a, b)	$r(a, b)$
$(2.9, 3.1)$	$(5.8, 6) \cup [8, 8.01)$
$(2.99, 3.01)$	$(5.98, 6) \cup [8, 8.02)$
$(2.999, 3.001)$	$(5.998, 6) \cup [8, 8.002)$

No matter how small we make the input interval, the output has the jump from 6 to 8 that we see in the graph. Again, this is a signal that $j(x)$ is not continuous at $x = 3$.

With more rigor. So how would a mathematician prove formally that, say

$$\ell(2, 4) = (4, 8).$$

Hopefully it is clear from the picture that this is the case. But perhaps we want to write an argument that does not rely on pictures.

We will give the argument in two steps. First, we will check that $\ell(2,4)$ is contained in $(4,8)$ and also that it contains $(4,8)$. In symbols, we would write:

$$\ell(2,4) \subseteq (4,8) \quad \text{and} \quad \ell(2,4) \supseteq (4,8).$$

In other words, statements say that every number in $\ell(2,4)$ is in $(4,8)$ and vice versa. This means the sets of number are the same (think about this!).

For the containment $\ell(2,4) \subseteq (4,8)$ we take an arbitrary number a in $(2,4)$ and check that $\ell(a)$ is in $(2,4)$. To say that a is in $(2,4)$ is to say that $2 < a < 4$. From this we obtain

$$\ell(a) = 2a < 8 \quad \text{and} \quad \ell(a) = 2a > 4$$

Since $4 < \ell(a) < 8$, this is saying that $ell(a)$ is in $(4,8)$ as desired. But since we said at the start that a was any number in $(2,4)$, this means that we have showed that all numbers in $(2,4)$ have their ℓ-image in $(4,8)$, exactly what we wanted.

The other containment is perhaps simpler. Let y be any number in $(4,8)$. We want to show that y is in the ℓ-image of $(2,4)$. We notice that

$$\ell(y/2) = y$$

and (similar to above) we have that $y/2$ is a number in $(2,4)$. So this tells us that every number in $(4,8)$ is in the ℓ-image.

Exercises

1. Find the image of $(-1,1)$ under $f(x) = x^2$.
2. Find the image of $(1,2)$ under $f(x) = x^2$.
3. Find the image of $(0,\pi)$ under $f(x) = \sin(x)$.
4. Find the image of $(3,6)$ under $r(x)$ (above).
5. What is the image of $(0,1)$ under the constant function $f(x) = -1$?
6. Let $f(x)$ be a function. If (a,b) is contained in (c,d), is it necessarily true that $f(a,b)$ is contained in $f(c,d)$?

1.4 Wrappers

We now move to the third step in our quest to define continuity.

Definition. For a set of real numbers, we define the wrapper to be the smallest closed interval containing the set.

The terminology is meant to evoke the idea of shrink wrapping: if you shrink wrap an object in real life, you expect to obtain the tightest possible container of that object. Again, examples.

Example 1: Intervals. We start by investigating the wrappers of intervals. First, for the closed interval $[0,1]$ the wrapper is

$$[0,1]$$

In other words, $[0,1]$ is its own wrapper. Here are some more examples of intervals and their wrappers:

interval	wrapper
$(0,1)$	$[0,1]$
$[0,1)$	$[0,1]$
$(0,1]$	$[0,1]$

Hopefully this is all believable. Convince yourself that we can replace the $(0,1)$ with (a,b), etc.

Example 2: Unions of intervals. We now proceed to looking at unions of intervals. We first consider the union

$$(0,1) \cup (2,3)$$

What is the wrapper? We might be tempted to think that the answer is

$$[0,1] \cup [2,3]$$

But this does not satisfy our definition, since it is not an interval. The correct wrapper for $(0,1) \cup (2,3)$ is

$$[0,3]$$

Indeed, $[0,3]$ is a closed interval containing $(0,1) \cup (2,3)$, and any smaller interval, such as $[0,2.999]$ fails to contain $(0,1) \cup (2,3)$. (You should find a specific number that is in $(0,1) \cup (2,3)$ but not $[0,2.999]$.)

We leave it to the reader to find more examples of unions of intervals and to figure out what their wrappers are.

Example 3: Reciprocals. Consider the collection of real numbers of the form $1/n$ for $n = 1,2,3,\ldots$. In other words, this is the set of numbers

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots$$

What is the wrapper of this set? We see that the closed interval $[0,1]$ contains the set. But is it the smallest interval that contains it? Can we move 1 to the left, or 0 to the right? We definitely shouldn't move the 1 to the left—for instance the interval $[0,0.9]$—because then the interval would not contain the number 1. If we move the 0 to the right—for instance the interval $[0.1,1]$—then the interval does not contain the numbers $\frac{1}{11}, \frac{1}{12}, \ldots$. Hopefully you are convinced at this point that the wrapper for this set is the interval

$$[0,1]$$

With more rigor. Let us argue formally that the wrapper of the open interval (a,b) is the closed interval $[a,b]$. Say that the wrapper of (a,b) is $[a',b']$. We want to prove that $a' = a$ and $b' = b$. We cannot have $a' < a$, because then we could make a smaller closed interval containing (a,b) by replacing a' with a. Similarly, we cannot have that $b' > b$. So now we have $a \le a' \le b' \le b$. If $a \ne a'$, then there is a real number between a and a', for example $(a+a')/2$. This number is in (a,b) but not $[a',b']$, which is against the rules for a wrapper. So it must be that $a = a'$. Similarly $b = b'$. That's it!

Some fine print. We found the wrappers of a few specific sets of real numbers. But how do we know that there always is a wrapper? What if there is a set of real numbers where there is no smallest interval containing it? There is a vast world of sets of real numbers. How can we make a statement about all of them?

It turns out that wrappers always exist. For the purposes of these notes, we take it as a given (or an axiom) that wrappers always exist (what is the wrapper of an unbounded set, like the set of natural numbers $1,2,3,\ldots$?). If you want to know more about this, look up the definitions of the infimum and supremum of a set of real numbers (and/or take a class in analysis!).

Exercises

1. What is the wrapper of $(1,2) \cup (2,3)$?
2. What is the wrapper of $\{3,4,7\}$?
3. What is the wrapper of $(1,2) \cup \{0,10\}$?
4. What is the wrapper of the set

$\left\{\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \dots\right\}$?

5. What is the wrapper of the set

$\{0.9, 0.99, 0.999, \dots\}$?

6. What is the wrapper of the set $\left\{e, \pi, \sqrt{2}\right\}$?

1.5 The definition of continuity

We are finally ready to give our formal definition of continuity!

> **Definition.** We say that $f(x)$ is *continuous at a* if there is a collapsing collection of open intervals for a so that the wrappers of the f-images form a collapsing collection of closed intervals for $f(a)$. If $f(x)$ is continuous at all real numbers, we say that f is continuous.

That is a mouthful! Let's break it down. We can think of our definition as describing a four step process for checking if a function $f(x)$ is continuous at $x = a$:

Step 1. Choose a collapsing collection of open intervals for a.

Step 2. Find the f-images of the intervals in the collection from Step 1.

Step 3. Find the wrappers of the intervals from Step 2.

Step 4. Check if $f(a)$ is defined and if the sets from Step 3 form a collapsing collection for $f(a)$.

In short, the definition says that a function f is continuous if, when we take smaller and smaller intervals containing a, and apply f, we obtain smaller and smaller sets containing $f(a)$.

Some fine print. Implicit in Step 4 is that if the sets from Step 3 do not form a collapsing collection then f is not continuous. This is true, but it is not obvious from the way we stated things. Of course we want "not continuous" to be the opposite of "continuous." But the opposite of a statement "if there is a collapsing collection with a certain property" would look like "no collapsing collections have the certain propery" For example the opposite of "there is a person named Lily" is "no person is named Lily." Checking if there is a person named Lily is easy: just find one. Checking that there is no person named Lily is much harder: you have to check all the people! At the end of this section, we will explain why this issue is not a problem. To show that a function is not continuous, we really do need to check just one collapsing collection. This is surprising, and special to this definition (as the Lily example shows!).

Example 1: a linear function. Let us start with our simple example, consider $\ell(x) = 2x$. We want to show that $\ell(x)$ is continuous at $x = 3$. As per Step 1 above, we start by choosing a collapsing collection of open intervals for 3. Here is one such choice:

$$(2.9, 3.1) \qquad (2.99, 3.01) \qquad (2.999, 3.001) \qquad \cdots$$

We move on to Step 2. The ℓ-images of these intervals are:

$$(5.8, 6.2) \qquad (5.98, 6.02) \qquad (5.998, 6.002) \qquad \cdots$$

In fact, we already did this calculation above (see what we did there?)!

For Step 3, we find the wrappers of these intervals. We remember that the wrapper of an open interval is the closed interval with the same bounds. Specifically, we find:

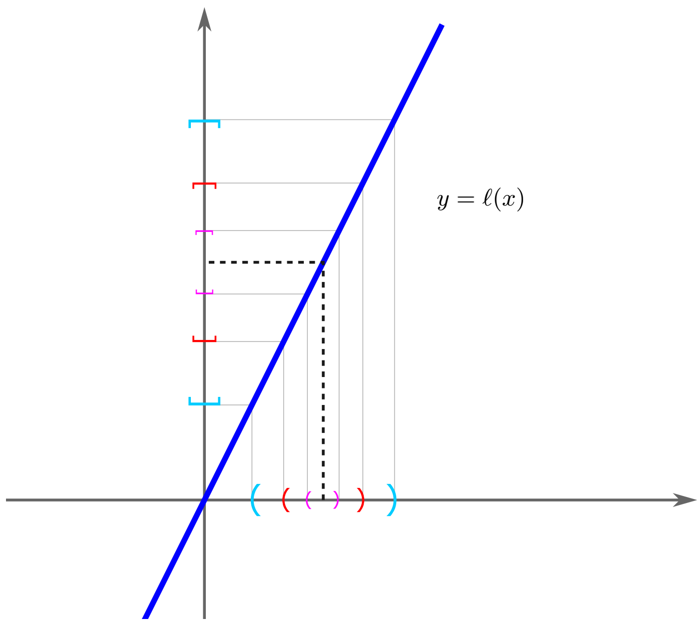
$$[5.8, 6.2] \qquad [5.98, 6.02] \qquad [5.998, 6.002] \qquad \cdots$$

Finally, for Step 4 we need to check two things: (1) the number 6 is in all of these intervals and (2) the number 6 is the only number in all of the intervals. The first one is true because for each of these intervals, the left endpoint is less than 6 and similarly the right endpoint is greater than 6.

For the second one, notice that if you take any number, for example 6.000001, then there will be a set on the list of outputs that does not contain it, for example the 7th set in the list:

$$[5.9999998, 6.0000002]$$

That does it! The function $\ell(x)$ is continuous at $x = 3$. Here is a picture of (the beginning of) a collapsing collection (on the x-axis) and the corresponding wrappers of the ℓ-images (on the y-axis). As the picture suggests, as the intervals on the x-axis get smaller and smaller, the intervals on the y-axis do as well. (Artistic note: as the intervals get smaller, we drew the corresponding parentheses smaller. When color is not an option, this is one way to indicate which parentheses match up with which other ones.)



Example 2: a removable discontinuity. Let us consider now our function

$$r(x) = \begin{cases} 2x & x \neq 3 \\ 9 & x = 3 \end{cases}$$

We said before that this function is not continuous at $x = 3$. Let us verify this with our definition. We follow the same steps as above. If we use the same collapsing collection of open intervals, the list of r-images is

$$(5.8, 6) \cup (6, 6.2) \cup \{9\} \qquad (5.98, 6) \cup (6, 6.02) \cup \{9\} \qquad (5.998, 6) \cup (6, 6.002) \cup \{9\} \qquad \cdots$$

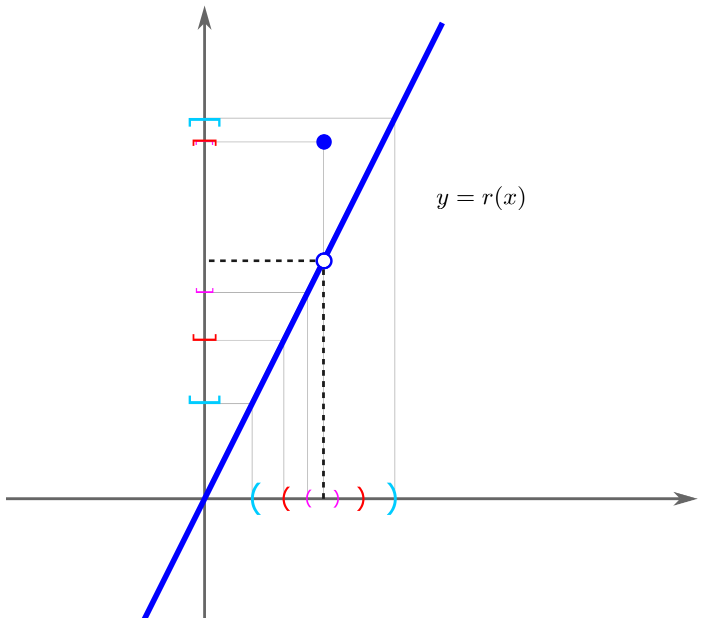
Again, we did this calculation already. The wrappers of these set of numbers are

$$[5.8, 9] \qquad [5.98, 9] \qquad [5.998, 9] \qquad \cdots$$

This is not a collapsing collection, since there are two numbers, 6 and 9, that are in all of the wrappers on the list. Actually the entire interval $[6, 9]$ is contained in each one of these wrappers. This verifies that $r(x)$ is not continuous at $x = 3$.

The following picture illustrates the situation. We see in the picture that the tops of the wrappers on the y-axis get "stuck" on $r(3) = 9$ (the top endpoints of the second and third wrappers are the same). This is precisely what causes the definition of continuity to fail.

This finally explains our terminology: the function $r(x)$ is discontinuous, but if we redefine it at one value of x, namely $x = 3$, we remove the discontinuity.



Example 3: a jump discontinuity. Let us turn to our function

$$j(x) = \begin{cases} 2x & x < 3 \\ 2x+2 & x \geq 3 \end{cases}$$

If we use the same collapsing collection of open intervals, the j-images are

$$(5.8, 6) \cup [8, 8.2)$$
$$(5.98, 6) \cup [8, 8.02)$$
$$(5.998, 6) \cup [8, 8.002)$$
$$(5.9998, 6) \cup [8, 8.0002)$$

and the wrappers of these are

$$[5.8, 8.2] \qquad [5.98, 8.02] \qquad [5.998, 8.002] \qquad \cdots$$

Again, this is not a collapsing collection (can you see why?), and so $j(x)$ is not continuous at $x = 3$.

Example 4: a bouncy function. Let's remember our bouncy function:

$$b(x) = \begin{cases} -1/2 & x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \\ x & otherwise \end{cases}$$

We already said that $b(x)$ is not continuous at $x = 0$. To verify this, we use the collapsing collection of open intervals

$$(-1/2, 1/2) \qquad (-1/3, 1/3) \qquad (-1/4, 1/4) \qquad \cdots$$

The b-images of these intervals are:

$$(-1/2, 1/2) \cup \{-1/2\}$$
$$(-1/3, 1/3) \cup \{-1/2\}$$
$$(-1/4, 1/4) \cup \{-1/2\} \quad \cdots$$

and the wrappers of these are

$$[-1/2, 1/2] \qquad [-1/2, 1/3] \qquad [-1/2, 1/4] \qquad \cdots$$

In this case, the entire interval $[-1/2, 0]$ is contained in all of these wrappers. So again, this is not a collapsing collection, and $b(x)$ is not continuous at $x = 0$.

Example 5: a wiggly function. Finally, let's look at the wiggly function

$$w(x) = \begin{cases} x\sin(\frac{\pi}{2x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We want to show that, surprisingly, $w(x)$ *is* continuous at $x = 0$. Again, this should be surprising, because there is no reasonable way to draw infinitely many wiggles with a pencil. This task might seem daunting—and it is challenging—but we have all the tools we need to do this (and I believe in you!).

First, we must remember that the sine function only takes outputs between -1 and 1. Therefore:

$$x\sin(\tfrac{2}{\pi x})$$

can only take values between $-x$ and x. Indeed, since $|\sin(\frac{2}{\pi x})|$ is always less than or equal to 1, the product $x\sin(\frac{2}{\pi x})$ has absolute value less than or equal to x:

$$\left|x\sin(\tfrac{2}{\pi x})\right| \leq |x| \left|\sin(\tfrac{2}{\pi x})\right| \leq |x| \cdot 1 = |x|$$

It follows that if we take an open interval such as

$$(-1/10, 1/10)$$

then the image under $w(x)$ is somewhere in

$$[-1/10, 1/10]$$

Indeed, if we take a positive number less than 1/10 and multiply it by a number between -1 and 1, we obtain a number in this interval. In actuality, the image might be even smaller than $[-1/10, 1/10]$. So the wrapper of the w-image of $(-1/10, 1/10)$ is *no bigger than* $[-1/10, 1/10]$. And the wrapper of $(-1/100, 1/100)$ is *no bigger than* $[-1/100, 1/100]$. We can see from this reasoning that the wrappers of the w-images of the standard collapsing collection is itself a collapsing collection for 0, which is $w(0)$. Therefore, $w(x)$ is continuous at $x = 0$.

We left one detail hanging here: why is 0 in all of the wrappers of all of the w-images of the collapsing collection for 0? We will leave this to the exercises.

With more rigor. We gave a fairly thorough explanation of why $\ell(x)$ is continuous. We will take a moment here to give a formal proof. This means we need to show that $\ell(3) = 6$ is the only number in the wrappers of the f-images of the intervals in whatever collapsing collection of open intervals we start with. We will place emphasis on the fourth step, since we have discussed the rigor behind the first three steps in the previous sections.

First, we will start with a different collapsing collection of open intervals from the one used above, namely:

$$(3-1, 3+1) \qquad (3-\tfrac{1}{2}, 3+\tfrac{1}{2}) \qquad (3-\tfrac{1}{3}, 3+\tfrac{1}{3}) \qquad (3-\tfrac{1}{4}, 3+\tfrac{1}{4}) \qquad \cdots$$

The ℓ-images of these are

$$(6-2, 6+2) \qquad (6-\tfrac{2}{2}, 6+\tfrac{2}{2}) \qquad (6-\tfrac{2}{3}, 6+\tfrac{2}{3}) \qquad (6-\tfrac{2}{4}, 6+\tfrac{2}{4}) \qquad \cdots$$

and the wrappers of these are

$$[6-2, 6+2] \qquad [6-\tfrac{2}{2}, 6+\tfrac{2}{2}] \qquad [6-\tfrac{2}{3}, 6+\tfrac{2}{3}] \qquad [6-\tfrac{2}{4}, 6+\tfrac{2}{4}] \qquad \cdots$$

Finally, we arrive at the fourth step, which is to show that 6 is the only number in all of these intervals. As in some of our previous arguments, let y be a number that is not equal to 6. We may assume that $y > 6$ (the case $y < 6$ is essentially the same). Similar to the argument at the end of Section 1.2 we consider the number $2/(y-6)$ and choose a number N so that $N > 2/(y-6)$. It follows that $(y-6) > 2/N$. We now have that

$$\begin{aligned} y &= 6 + (y-6) \\ &> 6 + 2/N, \end{aligned}$$

and so y is not in the interval $[6 - \frac{2}{N}, 6 + \frac{2}{N}]$. This is what we wanted to show.

We encourage you to carry out the argument for the collapsing collection of open intervals that we started with, namely

$$(2.9, 3.1) \qquad (2.99, 3.01) \qquad (2.999, 3.001) \qquad \cdots$$

To get you started, here is a way to write the terms of this sequence in a single formula:

$$(3 - 10^{-n}, 3 + 10^{n}) \qquad n = 2, 3, 4 \ldots$$

Good luck!

Back to the fine print. To verify that our four step process works even when the function is not continuous, we want to convince ourselves of the following: if there is a single collapsing collection of open intervals for a, and the f-images do not form a collapsing collection for $f(a)$, then the function $f(x)$ is not continuous. This discussion is technical and not required for the basic understanding of the material. We encourage the reader to skip this on a first tour through the subject.

Say that the collapsing collection in the previous paragraph is

$$(a_1, b_1) \qquad (a_2, b_2) \qquad (a_3, b_3) \qquad \cdots$$

We have assumed that the wrappers of

$$f(a_1, b_1) \qquad f(a_2, b_2) \qquad f(a_3, b_3) \qquad \cdots$$

do not form a collapsing sequence of closed intervals. This means that either these wrappers do not form a collapsing collection of closed intervals, or they do form a collapsing collection but do not collapse to $f(a)$. These are two different cases we need to address.

In the first case, there are two points numbers c and d in all of these wrappers. This means that, no matter how far down the list we go, the wrapper of $f(a_i, b_i)$ contains c and d. Now we want to take some other collapsing collection of open intervals for a, say:

$$(a_1', b_1') \qquad (a_2', b_2') \qquad (a_3', b_3') \qquad \cdots$$

we want to show that the wrappers of

$$f(a_1', b_1') \qquad f(a_2', b_2') \qquad f(a_3', b_3') \qquad \cdots$$

do not form a collapsing collection. We will show that c and d are in all of these wrappers. For any given (a_i', b_i') there is a large j so that (a_j, b_j) is contained in (a_i', b_i') (why?). This means that $f(a_i', b_i')$ is at least as big as $f(a_j, b_j)$. But since we assumed c and d are in the wrapper for $f(a_j, b_j)$, they are also in the wrapper for $f(a_i', b_i')$. We have showed that c and d are in all the wrappers of the $f(a_i', b_i')$ and so that $f(a_i', b_i')$ do not form a collapsing collection, as desired.

The second case is very similar. If $f(a)$ is not in all of the wrappers of the $f(a_i, b_i)$ then by a similar argument it is not in the corresponding wrappers $f(a_i', b_i')$, and so the $f(a_i', b_i')$ do not form a collapsing collection for $f(a)$.

Loose wrappers: a (complicated?) simplification. The hardest part of showing that a function is continuous is finding the images of the intervals in a collapsing collection of open sets. What if we said this was not actually required to give a proof that a function is continuous?

For any set of real numbers A, we say that a loose wrapper for A is any closed interval that contains A. Loose wrappers are much easier to find that wrappers, because they are not required to be minimal.

Here is how loose wrappers make our life simpler in practice. Say we have the interval $(2.9, 3.1)$ and we want to find a loose wrapper for $f(2.9, 3.1)$. For any number x in $(2.9, 3.1)$ we know that

$$8.41 = 2.9^2 < f(x) < 3.1^2 = 9.61$$

This proves that $[8.41, 9.61]$ is a loose wrapper for $f(2.9, 3.1)$; there is no way any $f(x)$ with x in $(2.9, 3.1)$ can be outside of this interval. What we do not have to check is that $f(2.9, 3.1)$ is the entire interval $(8.41, 9.61)$, nor do we have to worry whether or not $[8.41, 9.61]$ is the smallest closed interval containing $(8.41, 9.61)$ (which it is).

How exactly does this help with continuity? Here is an alternate definition that uses loose wrappers:

> **Definition.** We say that $f(x)$ is *continuous at a* if there is a collapsing collection of open intervals for a, and some choice of loose wrappers for the f-images that form a collapsing collection of closed intervals for $f(a)$. If $f(x)$ is continuous at all real numbers, we say that f is continuous.

With this definition, the steps for checking if a function $f(x)$ is continuous at $x = a$ are:

Step 1. Choose a collapsing collection of open intervals for a.

Step 2. Find loose wrappers of the f-images of the intervals from Step 1.

Step 3. Check if $f(a)$ is defined and if the sets from Step 3 form a collapsing collection for $f(a)$.

The catch is this: in the original definition, the process always works and tells you whether the function is continuous or not. Here, if the process does not work, then it could be that our loose wrappers we chose are, well, too loose. If there is a collection of loose wrappers that form a collapsing collection for $f(a)$ then $f(x)$ is continuous at $x = a$. If there is no such collection of loose wrappers for the giving collapsing collection of open intervals, then the function is not continuous. In summary, this definition is recommended for showing that a function is continuous, but less for showing that a function is not continuous.

Exercises

1. Check all of the details of the five examples in this section.
2. Show $f(x) = x^2$ is continuous at $x = 3$.
3. Show $f(x) = x^2$ is continuous at $x = 0$.
4. Show a constant function like $f(x) = 5$ is continuous.
5. Show the function $f(x) = x^3 - 1$ is continuous at $x = 1$.
6. Consider the function

   $$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   Sketch of the graph of $f(x)$. Is $f(x)$ continuous at $x = 0$?
7. Use loose wrappers to show that $f(x) = x^2$ is continuous.

2 Limits

As above, when we write

$$\lim_{x \to a} f(x) = L$$

we read this as: the limit of $f(x)$ as x approaches a is equal to L. Again, intuitively this means that as we choose inputs x closer and closer to a, the outputs $f(x)$ get closer and closer to L. As with the notion of continuity, we want to put the idea of a limit on firm theoretical footing. Fortunately, we have already established most of the tools we will need.

2.1 Examples of limits

Example 1: a linear function. To start with a simple example, consider $\ell(x) = 2x$. We can see from the graph, or from algebra, that as the input x gets closer and closer to 3, the output $\ell(x)$ gets closer to 6. In symbols, this is:

$$\lim_{x\to 3} 2x = 6$$

More generally, for any real number a, as the input x gets closer to a, the output $\ell(x)$ gets closer to $2a$. In symbols:

$$\lim_{x\to a} 2x = 2a$$

This is just what one would expect. Limits are easy! What could possibly go wrong?!

Example 2: a removable discontinuity. Let us consider now the function

$$r(x) = \begin{cases} 2x & x \neq 3 \\ 9 & x = 3 \end{cases}$$

Since this function looks the same as the last function when x is away from 3, a lot of the limits are the same, for instance:

$$\lim_{x\to 5} r(x) = 10$$

The question staring us in the face is: what is the limit of $r(x)$ as x approaches 3? Is it 6? Is it 9? Both? Neither? Something else entirely?

You can make an argument for the limit being 9. Because if x is getting closer to 3, then we might as well let x be equal to 3. And $r(3)$ is equal to 9. That would be a reasonable conclusion.

There is also an argument for the limit being 6. If we take x to be closer and closer to 3, for example, 3.1, then 3.01, then 3.001... then the outputs are getting closer to 6.

If both of these answers are reasonable, how do we know which one is right? Can we have two different correct answers? That is why we have definitions! Instead of saying that

$$\lim_{x\to a} f(x) = L$$

means that as we choose inputs x closer and closer to a, the outputs $f(x)$ get closer and closer to L, we instead take the limit to mean: as we choose inputs x closer and closer to *a and not equal to a*, the outputs $f(x)$ get closer and closer to L. It is because of that added phrase that the answer is 6 and not 9:

$$\lim_{x\to 3} r(x) = 6$$

Let's look at a similar example, our jump discontinuity function.

Example 3: a jump discontinuity. Let's look at our function

$$j(x) = \begin{cases} 2x & x < 3 \\ 2x+2 & x \geq 3 \end{cases}$$

Again we ask: what is the limit of $j(x)$ as x approaches 3? Is it 6? Is it 8?

Similar to the last example, we choose inputs x closer and closer to 3 *and not equal to* 3. If we choose inputs getting closer from the right side, the limit looks like 8. But, if we choose inputs getting closer from the left side, the limit looks like 6.

So either this function has two limits (should that be allowed?) or it does not have any limit. Indeed, there is no single number L that both sets of numbers are getting closer to.

The correct answer is that

$$\lim_{x\to 3} j(x) \text{ does not exist}$$

And this is exactly what our working definition of a limit told us. So maybe we have the right definition!

Unfortunately, as we think of more and more complicated functions, we realize that this latest version of the definition is still not enough.

Example 4: a bouncy function. Let's consider the bouncy function:

$$b(x) = \begin{cases} -1/2 & x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \\ x & otherwise \end{cases}$$

What is the limit of $b(x)$ as x approaches 0? In other words:

$$\lim_{x\to 0} b(x) = ?$$

It is reasonable to say that the limit is $-1/2$, based on the following table:

x	$b(x)$
$1/2$	$-1/2$
$1/4$	$-1/2$
$1/6$	$-1/2$
$1/8$	$-1/2$

We see that, as the input is getting closer to 0, and not equal to 0, the output is getting close to $-1/2$. In fact all of these outputs are $-1/2$!

But let's consider these two tables:

x	$b(x)$
$1/3$	$1/3$
$1/5$	$1/5$
$1/7$	$1/7$
$1/9$	$1/9$

x	$b(x)$
$-1/2$	$-1/2$
$-1/4$	$-1/4$
$-1/6$	$-1/6$
$-1/8$	$-1/8$

From both of the last two tables, it seems reasonable to say that the limit is 0. Again, the correct answer is that the limit does not exist. We could try to adjust our definition of limits by saying that

$$\lim_{x\to a} f(x) = L$$

means that as we choose *any sequence of* inputs x closer and closer to a and not equal to a, the outputs $f(x)$ get closer and closer to L. This is actually pretty close to being correct... but let's look at our last example.

Example 5: A wiggly function. We now go back to our wiggly function:

$$w(x) = \begin{cases} x\sin(\frac{\pi}{2x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

If we plug in the numbers $1/2,\ 1/3,\ 1/4,\ \ldots$ we obtain the outputs

$$0,\ -1/3,\ 0,\ 1/5,\ 0,\ -1/7,\ 0,\ \ldots$$

It is true that

$$\lim_{x \to 0} w(x) = 0$$

But is it really right to say that the outputs are closer to 0 as the inputs are getting closer to 0? It seems that the function gets close (actually, all the way) to 0, and then farther away, then close, then farther away...

We could try to further amend our working definition so that it applies to this example. But once we start using so many words, there is more and more room for ambiguity. And even with an improved definition, how will we know if we have dealt with all of the issues that might arise with stranger and stranger functions? That is why we are led to make an honest, formal mathematical definition of a limit. It turns out that we only need one slight modification from what we already have: punctured collapsing collections.

Exercises

Draw graphs of the following functions. Determine if the limits exist at the given values of x, and if so what the limits are.

1. $f(x) = 5$ as $x \to 3$
2. $f(x) = x^{2/3}$ as $x \to 0$
3. $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ as $x \to 0$
4. $f(x) = \begin{cases} x+3 & x \leq 1 \\ -2x+6 & x \geq 1 \end{cases}$ as $x \to 1$
5. $f(x) = \begin{cases} x^2 & x \leq 2 \\ x+2 & x \geq 2 \end{cases}$ as $x \to 2$
6. $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ as $x \to 0$

2.2 Punctured collapsing collections

As mentioned, for the definition of a limit, we only need to slightly modify one of our tools. The modification is being made to deal with the issue that, when finding a limit at $x = a$, we want to consider real numbers that are near a, but not equal to a.

If we have an open interval, such as $(2, 4)$, then we can "puncture" the interval by removing one number. For instance, if we puncture $(2, 4)$ at the number 3, we obtain

$$(2, 3) \cup (3, 4)$$

This set is the union of two intervals. It is the interval $(2, 4)$ "minus" the number 3.

We are already familiar with examples of collapsing collections of open intervals, such as

$$\begin{gathered} (2.9, 3.1) \\ (2.99, 3.01) \\ (2.999, 3.001) \quad \ldots \end{gathered}$$

The way we puncture the collection is that we remove a single number from all of the intervals, namely, the number that is in all of them. If we do this we obtain the punctured collapsing collection of open intervals:

$$\begin{gathered} (2.9, 3) \cup (3, 3.1) \\ (2.99, 3) \cup (3, 3.01) \\ (2.999, 3) \cup (3, 3.001) \quad \ldots \end{gathered}$$

This is the same as before, just without the number 3. For a real number a, a punctured collapsing collections of open intervals might look like

$$(a-1,a)\cup(a,a+1)$$
$$(a-\tfrac{1}{2},a)\cup(a,a+\tfrac{1}{2})$$
$$(a-\tfrac{1}{3},a)\cup(a,a+\tfrac{1}{3}) \quad \ldots$$

That's it for the new tools we need for the definition of a limit.

Exercises

1. Write down the first few sets in a punctured collapsing collection of open intervals for the number -1.
2. Write down the first few intervals in a punctured collapsing collection of open intervals for the number 0.0001.
3. Write down the first few sets in a punctured collapsing collection of open intervals for a number a, different from the one given above.

2.3 The definition of a limit

We are ready to give our formal definition of the limit of a function.

> **Definition.** We say that
> $$\lim_{x\to a} f(x) = L$$
> if when we apply f to a punctured collapsing collection of open intervals for a, and take the wrappers of these sets, we obtain a collapsing collection of closed intervals for L.

As with the definition of continuity, there is a little bit of ambiguity in the way we wrote the definition. Do we mean that we apply f to all punctured collapsing collections, or just one? Again, these statements end up being the same: if it works for one such collection, it works for all of them (the proof is the same as in the continuity case above). As with continuity, that is good news, because it means to find a limit, we only need to check our favorite punctured collapsing collection.

In short, the definition says that if we take smaller and smaller intervals containing a, then apply f, we obtain smaller and smaller sets containing L. As with continuity, we can think of our definition as giving a four step process:

Step 1. Choose a punctured collapsing collection of open intervals for a.

Step 2. Apply the function f to the punctured intervals from Step 1.

Step 3. Find the wrappers of the sets found in Step 2.

Step 4. Check if the wrappers from Step 3 form a collapsing collection for L.

Example 1: a linear function. Let us start with our simple example, consider $\ell(x) = 2x$. We want to show that

$$\lim_{x\to 3} \ell(x) = 6$$

As per Step 1 above, we start by choosing a punctured collapsing collection of open intervals for 3. Here is one such choice:

$$(2.9,3)\cup(3,3.1)$$
$$(2.99,3)\cup(3,3.01)$$
$$(2.999,3)\cup(3,3.001)$$
$$(2.9999,3)\cup(3,3.0001)$$

We move on to Step 2. The ℓ-images are

$$(5.8, 6) \cup (6, 6.2)$$
$$(5.98, 6) \cup (6, 6.02)$$
$$(5.998, 6) \cup (6, 6.002)$$
$$(5.9998, 6) \cup (6, 6.0002)$$

For Step 3, we find the wrappers of these sets:

$$[5.8, 8.2] \qquad [5.98, 8.02] \qquad [5.998, 8.002] \qquad [5.9998, 8.0002] \qquad \cdots$$

Finally, for Step 4 we need to check two things: (1) the number 6 is in all of these intervals and (2) the number 6 is the only number in all of the intervals. We already did this when we showed that $\ell(x)$ is continuous at $x = 3$.

Example 2: a removable discontinuity. Let us consider now the function

$$r(x) = \begin{cases} 2x & x < 3 \\ 2x + 2 & x \geq 3 \end{cases}$$

We said before that

$$\lim_{x \to 3} r(x) = 6$$

To prove this, we follow the same four steps as in Example 1. There is almost no difference! This is because we used punctured open intervals instead of open intervals. In doing so, we ignore what the function is doing at $x = 3$. Convince yourself that this works!

Example 3: a jump discontinuity. Let us turn to our function

$$j(x) = \begin{cases} 2x & x < 3 \\ 2x + 2 & x \geq 3 \end{cases}$$

Using the usual punctured collapsing collection of open intervals and find the j-images:

(a, b)	$j(a, b)$
$(2.9, 3) \cup (3, 3.1)$	$(5.8, 6) \cup (8, 8.01)$
$(2.99, 3) \cup (3, 3.01)$	$(5.98, 6) \cup (8, 8.02)$
$(2.999, 3) \cup (3, 3.001)$	$(5.998, 6) \cup (8, 8.002)$

The wrappers of these images are

$$[5.8, 8.2] \qquad [5.98, 8.02] \qquad [5.998, 8.002] \qquad [5.9998, 8.0002] \qquad \cdots$$

The limit cannot be anything! This is for the simple reason that there is more than one number in all of these intervals. In fact, every number in $[6, 8]$ is in all of the intervals, as we saw in our discussion of the (dis)continuity of $j(x)$. This agrees with our statement earlier that the limit is not defined.

Example 4: a bouncy function. Remember that our bouncy function is

$$b(x) = \begin{cases} -1/2 & x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \\ x & otherwise \end{cases}$$

We will leave the details of this one as an exercise. But we will give some hints. We already said that the limit does not exist at $x = 0$. So we want to model our reasoning after the last example. We choose any collapsing collection of open intervals at 0. What we want to show is that if we apply f to the intervals in this collection and take the wrappers, then there is more than one number in all of the wrappers. Can you find a number (or infinitely many numbers) that are in all of the wrappers?

Example 5: A wiggly function. Finally, let's look at

$$w(x) = \begin{cases} x\sin(\frac{\pi}{2x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We want to show that

$$\lim_{x\to 0} w(x) = 0$$

Hopefully we are seeing the pattern that the arguments in this section are eerily similar to the arguments in the section on continuity. We leave it to you to fill in the details here!

Exercises

1. Think about all of the details of the five examples in this section.
2. Explain why $\lim_{x\to 3} x^2 = 9$.
3. Explain why $\lim_{x\to 0} x^2 = 0$.
4. What is $\lim_{x\to 1} 5$?
5. Explain why $\lim_{x\to 1} x^3 - 1$.
6. Consider the function

$$f(x) = \begin{cases} x & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

Does $\lim_{x\to 0} f(x)$ exist? What about $\lim_{x\to 1} f(x)$?

2.4 One-sided limits

In this section we give an alternate approach to limits of functions. Specifically, we will first define left-limits and right-limits, written as

$$\lim_{x\to a^-} f(x) \quad \text{and} \quad \lim_{x\to a^+} f(x).$$

We can refer to both at the same time as one-sided limits. With these in hand, we will say that

$$\lim_{x\to a} f(x) = L$$

if the two one-sided limits are both equal to L. Besides giving us another way to formulate the definition of a limit, one-sided limits are also useful in their own right. For example, suppose we have a function

$$f : [a, b] \to \mathbb{R}$$

meaning the the domain (inputs) are in the interval $[a, b]$ and the co-domain (potential outputs) are real numbers. In this case the limit

$$\lim_{x\to a} f(x) = L$$

does not make sense, but the right-sided limit

$$\lim_{x\to a^+} f(x)$$

does.

The basic idea of a one-sided limit is straightforward: as we take inputs closer and closer to a from one side (either the right or the left), we are asking what number (if any) the outputs are getting closer to. Consider, for example, the function

$$j(x) = \begin{cases} 2x & x < 3 \\ 2x + 2 & x \geq 3 \end{cases}$$

We already said in Section 2.3 that the limit of this function does not exist. But the one-sided limits do exist! Can you see what they are? We will return to this below.

The mechanics of one-sided limits are essentially the same as for the regular (two-sided) limits we have already covered. Actually, they are a little bit simpler. Instead of punctured collapsing collections, where the intervals are punctured (that is, they have one point removed), we can use left- and right-sided collapsing collections, which consist of open intervals (with no points removed).

Definition. For a real number a, a right-sided collapsing collection of open intervals for a is a list of intervals so that

1. each interval in the list is an open interval
2. each interval is contained in the previous one,
3. a is the left endpoint of each interval, and
4. there are no numbers in all of the intervals.

An example of a right-sided collapsing collection for $x = 3$ is

$$(3, 3.1) \qquad (3, 3.01) \qquad (3, 3.001) \qquad \cdots$$

Be sure to check all four of the above properties, especially the last one!

We can of course define left-handed collapsing collections in the same way. In this case, the number a should be the right endpoint of each interval.

Definition of right-sided limit. We say that

$$\lim_{x \to a^+} f(x) = L$$

if when we apply f to a right-sided collapsing collection of open intervals for a, and take the wrappers of these sets, we obtain a collapsing collection of closed intervals for L.

We can define left-sided limits in the same way, just with left-sided collapsing collections. In both cases, we can think of the definition as giving a four step process, as with the definitions of continuity and limits.

Let us apply the definition of a left-sided limit in the case of our function $j(x)$. What is

$$\lim_{x \to 3^-} j(x) \quad ?$$

We use the standard left-sided collapsing collection

$$(2.9, 3) \qquad (2.99, 3) \qquad (2.999, 3) \qquad \cdots$$

The images of these intervals under $j(x)$ are

$$(5.8, 6) \qquad (5.98, 6) \qquad (5.998, 6) \qquad \cdots$$

The wrappers of these images are

$$[5.8, 6] \qquad [5.98, 6] \qquad [5.998, 6] \qquad \cdots$$

And we can see that this is a collapsing collection of closed intervals for 6. We can thus conclude that

$$\lim_{x \to 3^-} j(x) = 6.$$

Now, you should use the definition of the right-sided limit to check that

$$\lim_{x\to 3^+} j(x) = 8.$$

The calculation is similar, but there is something different about it. What is it?

Now that we have one-sided limits in place, we can give our alternate formulation of the definition of a limit.

Definition. We say that

$$\lim_{x\to a} f(x) = L$$

if the one-sided limits

$$\lim_{x\to a^-} f(x) \quad \text{and} \quad \lim_{x\to a^+} f(x)$$

both exist and are equal to L.

We encourage the reader to verify that this definition is equivalent to our first definition of a limit.

Exercises

1. COMING SOON!

2.5 Continuity from limits

As we alluded to earlier, we can use our newfound definition of a limit to give a simple definition of continuity.

Definition. We say that $f(x)$ is continuous at a if

$$\lim_{x\to a} f(x) = f(a)$$

If f is continuous at every number a in the domain, we say that $f(x)$ is continuous.

The equality in the definition really means two things: the limit exists, and it is equal to $f(a)$. We can further say that $f(x)$ is continuous (in general) if this is true for every input a.

It shouldn't be too surprising that we can use the definition of a limit to give a definition of continuity, since the definitions are very similar, and the definition of a limit is slightly more involved. Let's use this new definition of continuity on our five favorite examples and to verify they are, or are not, continuous for the same choices of a we used before.

The five examples. First, our linear function $\ell(x)$ is continuous at 3, since we already checked that

$$\lim_{x\to 3} \ell(x) = \ell(3) = 6$$

Next, our removable discontinuity function $r(x)$ is not continuous at 3 since

$$\lim_{x\to 3} r(x) \neq r(3) = 9$$

The jump discontinuity function $j(x)$ is not continuous at 3 since the limit of $j(x)$ is not defined at 3. Similarly, the bouncy function $b(x)$ is not continuous at 0 since the limit of $b(x)$ is not defined at 0. Finally, the wiggly function $w(x)$ is continuous at 0 since

$$\lim_{x\to 0} w(x) = 0 = w(0)$$

We encourage the reader to show that these functions are continuous for other values of a.

Exercises

1. Use the new definition to explain why $f(x) = x^2$ is continuous at $x = 3$.
2. Use the new definition to explain why $f(x) = x^2$ is continuous at $x = 0$.
3. Use the new definition to explain why $f(x) = x^2$ is continuous (at all x).
4. Use the new definition to explain why a constant function like $f(x) = 5$ is continuous.

3 Properties of limits and continuity

In math, a typical phenomenon is that we work very hard to define our concepts formally—as we have just done for limits and continuity—and then we use the definitions to introduce tools. Often, the purpose of the tools is to free ourselves from having to use the definitions we worked so hard on. This all might seem counterintuitive (or maybe it seems like nonsense). Hopefully we can illustrate this idea in this section. In different subsections, we develop the following tools:

1. if f and g are continuous, then $f + g$, fg, and f/g are continuous,
2. if f and g are continuous, then $f \circ g$ is continuous,
3. the squeeze theorem: if f and h are continuous, and $f \leq g \leq h$, and $f(a) = h(a)$, then g is continuous at a

Each of these statements has an analogous statement for limits. As we go through these three tools for continuity, we will explain the analogues for limits.

3.1 Sums, products, and quotients

Given two functions $f(x)$ and $g(x)$, we can define three new functions $f + g$, fg, and f/g by the formulas

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ (fg)(x) &= f(x)g(x)\\ (f/g)(x) &= f(x)/g(x)\end{aligned}$$

Note that the function $(f/g)(x)$ is only defined for x with $g(x) \neq 0$. (There are other functions we could have listed, such as $f - g$, $1/f$, and cf where c is a real number; why didn't we list these?)

Theorem 3.1. *Suppose that $f(x)$ and $g(x)$ are continuous functions and let c be a real number. Then $f + g$ and fg are continuous, and f/g is continuous at all x with $g(x) \neq 0$.*

We can give a more refined version of the theorem, for example: if $f(x)$ and $g(x)$ are continuous at $x = a$, then $(f + g)(x)$ is continuous at $x = a$.

We will need to use the definition of continuity to prove this theorem. But once we do this, we get for free that many functions are continuous. For instance, if we know that $f(x) = x$ is continuous, then we immediately get that

$$(ff)(x) = f(x)f(x) = x^2$$

is continuous. From there we can show—without using the definition of continuity directly—that all polynomials are continuous (how?). That is very powerful! We'll delve more into this below.

We will prove just the first part of the theorem and leave the rest to the exercises, since they are all similar. Specifically, we will show that $f + g$ is continuous, given that f and g are continuous.

To start, we choose an arbitrary real number a. We will show that $f+g$ is continuous at a. Since a is an arbitrary number, this means that $f+g$ is continuous at every real number, and so it is continuous (think about this!).

We are assuming that $f(x)$ and $g(x)$ are continuous. So in particular they are both continuous at $x=a$. By the definition of continuity, the wrappers of the f- and g-images of the intervals

$$(a-1, a+1) \quad (a-1/2, a+1/2) \quad (a-1/3, a+1/3) \quad (a-1/4, a+1/4) \quad \cdots$$

form collapsing collections of closed intervals for $f(a)$ and $g(a)$. The thing to show now is that the wrappers of the sets

$$(f+g)(a-1, a+1) \quad (f+g)(a-1/2, a+1/2) \quad (f+g)(a-1/3, a+1/3) \quad (f+g)(a-1/4, a+1/4) \quad \cdots$$

form a collapsing collection for $(f+g)(a) = f(a)+g(a)$. This should be intuitive, but lets give a formal argument.

As usual, there are two things to check. First, since a is in all the intervals $(a-1/n, a+1/n)$ it follows that $(f+g)(a)$ is in all the $(f+g)(a-1/n, a+1/n)$ (and hence in all the wrappers). We need to check that $(f+g)(a)$ is the only number in all of these wrappers.

We make the following important observation.

> *Key observation.* If we go far enough down the list of intervals $(a-1/n, a+1/n)$, we can make the wrappers of $f(a-1/n, a+1/n)$ and $g(a-1/n, a+1/n)$ as small as we want.

Here's what this means. Say that wrappers of the f- and g-images of the $(a-1/n, a+1/n)$ are written as $[c_n, d_n]$ and $[c'_n, d'_n]$. The key observation means that, by going far enough down the list, we can make d_n and d'_n as close to $f(a)$ and $g(a)$ as we want.

Let z be some number bigger than $(f+g)(a)$ (we can use basically the same argument if z were smaller). We want to show that z is not in all the wrappers of the $(f+g)(a-1/n, a+1/n)$. Say that $z = (f+g)(a)+e$ for some number e. By the previous paragraph, if we go far enough down the list, then we can make $d_n < f(a)+e/3$ and $d'_n < g(a)+e/3$. This means that the f- and g-images of the numbers in $(a-1/n, a+1/n)$ are all less than $f(a)+e/3$ and $g(a)+e/3$, respectively. But this means that the $(f+g)$-images of the numbers in $(a-1/n, a+1/n)$ are all less than $z-e/3$ (why?). But this means that z is not in the wrapper of $(f+g)(a-1/n, a+1/n)$, and we are done!

And the version for limits. As advertised, we now give a version of Theorem 3.1 for limits.

Theorem 3.2. *Suppose that $f(x)$ and $g(x)$ are functions and let a be a real number. Suppose that*

$$\lim_{x\to a} f(x) = L \quad \textit{and} \quad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a}(f+g)(x) = L+M, \qquad \textit{and} \qquad \lim_{x\to a}(fg)(x) = LM.$$

Also, if $M \neq 0$ then

$$\lim_{x\to a}(f/g)(x) = L/M.$$

Theorem 3.2 is very similar in spirit to Theorem 3.1. The idea of the proof is basically the same, and so we leave this an an exercise to the reader.

Exercises

1. Show that all polynomials are continuous functions.

2. Show that if $f(x)$ is continuous and there is no x so that $f(x) = 1$, then
$$g(x) = \frac{1}{1 - f(x)}$$
is also continuous.

3. Fill in the details for the second and third statements of Theorem 3.1, that is, the versions for fg and f/g.

4. Prove Theorem 3.2.

3.2 Compositions

The composition of two functions $f(x)$ and $g(x)$ is the function
$$(f \circ g)(x) = f(g(x)).$$
We can think of $f \circ g$ as the function obtained by plugging g into f. Or in other words, to find $(f \circ g)(x)$ we first find the g-image of x and then find the f-image of $g(x)$.

What is function composition?. The composition $(f \circ g)(x)$ is a way of combining the functions $f(x)$ and $g(x)$, just as in the previous section. Compared to $f + g$, fg, and f/g, the composition $f \circ g$ is more subtle. So let's look at a couple of examples, to get used to the idea. Say that
$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1.$$
To find $(f \circ g)(x)$, we plug $g(x)$ into $f(x)$:
$$(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2.$$
In other words, given a number x, the function $f \circ g$ first adds one to x and then squares the result. What about $g \circ f$? Is that the same as $f \circ g$, or different? Let's see:
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1.$$
This is a different function!

An example of function composition. Where do compositions come up in real life? The answer is: any time you apply one operation to a number, and then apply another operation. For instance, suppose I have an object at room temperature, 72 degrees Fahrenheit. I heat it at a rate of two degrees per second. I can make a function that describes the temperature of the object as a function of time (specifically, the number of seconds that elapse):
$$g(x) = 2x + 72.$$
Now suppose I want to know the temperature in Celsius. We have the conversion function:
$$f(x) = 5x - 160/9.$$
(Let's check: $f(212) = 100$ and $f(32) = 0$). If I compose the two functions, we get the function
$$(f \circ g)(x) = (10/9)x + 200/9.$$
This gives us a function that describes the temperature in Celsius as a function of time. You still might be asking: what is the point of this? Why can't I just use the formula $(10/9)x + 200/9$ instead of thinking about compositions? The answer is that this reflects the way we (or at least I) think about things in real life. We like to break complicated procedures into manageable chunks. This makes the procedures easier to understand. It also allows use to use the chunks in a modular way, that is, we can combine the pieces together in new and interesting ways to achieve different purposes. We encourage the reader to think of other multistep processes that can be written as compositions of functions!

Composition and Continuity. What can we say about the continuity of a composition of functions? The next theorem says that we have the best answer possible.

Theorem 3.3. *Suppose that $f(x)$ and $g(x)$ are continuous functions. Then the composition $(f \circ g)(x)$ is a continuous function.*

Just as with Theorem 3.1, there is a more refined version of the theorem, but it is again more subtle. It goes like this:

> *Suppose that $g(x)$ is continuous at $x = a$ and that $f(x)$ is continuous at $g(a)$, then $(f \circ g)(x)$ is continuous at $x = a$.*

In order to prove that Theorem 3.3 is true, we can prove the refined version. In the theorem, $f(x)$ and $g(x)$ are continuous at all x, so if the refined version is true then Theorem 3.3 is true.

So let's aim to understand the refined version. We'll be briefer than in the last section, since all of these proofs are very similar, and we do not want to bore the reader too much.

As in the refined statement, we have a real number a, we are assuming that $g(x)$ is continuous at a, and we are assuming that $f(x)$ is continuous at $g(a)$. We want to show that $(f \circ g)(x)$ is continuous at $x = a$. We have a lot of practice at this! We choose a collapsing collection of open intervals around a. Here is our favorite:

$$(a-1, a+1) \quad (a-1/2, a+1/2) \quad (a-1/3, a+1/3) \quad (a-1/4, a+1/4) \quad \cdots$$

We want to show that the wrappers of the $(f \circ g)$-images of these intervals form a collapsing collection of closed intervals for $(f \circ g)(a) = f(g(a))$. The idea of the argument is very simple. We start with a collapsing collection of open intervals for a. We apply g, and since g is continuous, we obtain a collapsing collection of intervals for $g(a)$. Then when we apply f to the latter collapsing collection, the continuity of f tells us that the f-images of those form a collapsing collection for $(f \circ g)(a) = f(g(a))$. And this is exactly what it means for $f \circ g$ to be continuous.

Fine print. We now take the intuition from the previous paragraph and spell out more of the details. First of all, since $g(x)$ is continuous at a we know that the wrappers of

$$g(a-1, a+1) \quad g(a-1/2, a+1/2) \quad g(a-1/3, a+1/3) \quad g(a-1/4, a+1/4) \quad \cdots$$

form a collapsing collection for $g(a)$. Say that these wrappers are

$$[c_1, d_1] \quad [c_2, d_2] \quad [c_3, d_3] \quad [c_4, d_4] \quad \cdots$$

Since these intervals form a collapsing collection for $g(a)$, the intervals

$$(c_1, d_1) \quad (c_2, d_2) \quad (c_3, d_3) \quad (c_4, d_4) \quad \cdots$$

form a collapsing collection of closed intervals for $g(a)$ (this is not quite true in every case; why not?). Since $f(x)$ is continuous at $g(a)$, it follows that the wrappers of

$$f(c_1, d_1) \quad f(c_2, d_2) \quad f(c_3, d_3) \quad f(c_4, d_4) \quad \cdots$$

form a collapsing collection for $(f \circ g)(a) = f(g(a))$. Now the argument finishes similarly to the last section. The wrappers of the $(f \circ g)$-images of

$$(a-1, a+1) \quad (a-1/2, a+1/2) \quad (a-1/3, a+1/3) \quad (a-1/4, a+1/4) \quad \cdots$$

must be contained inside the wrappers of the

$$f(c_1, d_1) \quad f(c_2, d_2) \quad f(c_3, d_3) \quad f(c_4, d_4) \quad \cdots$$

(why?). In other words, the wrappers of the $(f \circ g)$-images of the $(a-1/n, a+1/n)$ are even smaller than (or no bigger than) the intervals. So it must be that these $(f \circ g)$-images form a collapsing collection for $(f \circ g)(a) = f(g(a))$, which is exactly what it means for $(f \circ g)(x)$ to be continuous at $x = a$. We are done!

And the limit version. We now arrive at the limit version of Theorem 3.3. Try to guess the statement before you read it.

Theorem 3.4. *Suppose that $f(x)$ and $g(x)$ are functions and let a be a real number. Suppose that*

$$\lim_{x \to a} g(x) = L \quad \text{and} \quad \lim_{x \to L} f(x) = M.$$

Then

$$\lim_{x \to a} (f \circ g)(x) = M.$$

Again, the proof of Theorem 3.4 is similar to the proof of Theorem 3.3. And so we leave it to the reader to fill in the details.

Bonus version: limits and continuity together. Theorems 3.3 and 3.4 explain what happens to continuity and limits when we compose two functions. The next theorem gives a version that combines limits, continuity, and compositions all in one.

Theorem 3.5. *Suppose that $f(x)$ and $g(x)$ are functions and let a be a real number. Suppose further that $f(x)$ is continuous and that*

$$\lim_{x \to a} g(x)$$

exists. Then

$$\lim_{x \to a} (f \circ g)(x) = f\left(\lim_{x \to a} g(x)\right).$$

If $g(x)$ is continuous, then we know the equality in the theorem must be true. Indeed, by the continuity of $g(x)$, the right hand side becomes $f(g(a)) = (f \circ g)(a)$ (go back to our limit definition of continuity!). On the left-hand side, Theorem 3.3 tells us that $f \circ g$ is continuous, and so by the limit definition of continuity, this left-hand side becomes $(f \circ g)(a)$ again.

The point of the last paragraph is that Theorem 3.5 does not tell us anything new when $g(x)$ is continuous. So to understand the theorem, we should think about examples where $g(x)$ is not continuous. As per the theorem, we still need $g(x)$ to have a limit at whatever a we choose. Take for example the removable discontinuity function $r(x)$ above. Let's also take the linear function $\ell(x)$ from above. We already said that

$$\lim_{x \to 3} r(x) = 6$$

Using Theorem 3.5, we can find the limit of $(\ell \circ r)(x)$:

$$\begin{aligned} \lim_{x \to 3} (\ell \circ r)(x) &= \ell\left(\lim_{x \to 3} r(x)\right) \\ &= \ell(6) \\ &= 12 \end{aligned}$$

We could have instead found a formula for the composition $(\ell \circ r)(x)$ and then studied the limit of that directly. But that is the advantage of these theorems—we do not have to get our hands dirty as much.

Exercises

1. Give your own examples of compositions of functions from everyday life.
2. Let $f(x) = 5x - 160/9$ be the temperature conversion function from Fahrenheit to Celsius. First, find a conversion function $g(x)$ from Celsius to Fahrenheit. What are $(f \circ g)(x)$ and $(g \circ f)(x)$?
3. Can the composition of discontinuous functions be continuous?
4. Write down a formula for the function $(\ell \circ r)(x)$. Prove from the definition that $\lim_{x\to 3}(\ell \circ r)(x) = 12$.
5. Find examples of functions $f(x)$ and $g(x)$ so that
$$\lim_{x\to a}(f\circ g)(x) \neq f\left(\lim_{x\to a} g(x)\right).$$

3.3 The squeeze theorem

We have one more basic fact about limits and continuity to squeeze in (I can't help it!). It is called the squeeze theorem. Basically, it gives us a way to say that a function is continuous at a certain value of x by squeezing it between two other continuous functions. We will use this to show that our wiggle function $w(x)$ is continuous at $x = 0$. Can you already guess which continuous functions it is squeezed between?

Theorem 3.6 (Squeeze Theorem for Continuity)**.** *Suppose that $f(x)$ and $h(x)$ are continuous functions, and that $g(x)$ is another function with*

$$f(x) \le g(x) \le h(x)$$

for all values of x and with

$$f(a) = h(a)$$

Then $g(x)$ is continuous at a.

How do we use this for the function $w(x)$? Well if $f(x) = -|x|$ and $h(x) = |x|$, then we have $f(x) \le g(x) \le h(x)$ (why?). So if we believe that $|x|$ is continuous (why is that true?) then the Squeeze theorem tells us that $w(x)$ is continuous. Very neat!

The idea of the squeeze theorem is very simple. We assumed that $f(x)$ and $h(x)$ are continuous. By the limit definition of continuity and the assumption that $f(a) = h(a)$ we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = f(a)(= h(a)).$$

But then since $f(x) \le g(x) \le h(x)$ it must be that $f(a) = g(a) = h(a)$, and that

$$\lim_{x\to a} g(x) = h(a)(= f(a) = g(a)).$$

Applying the limit definition of continuity again, it follows that g is continuous at a. This is the right idea, but we are secretly using the squeeze theorem for limits, which we haven't actually proved yet. See Theorem 3.7 below!

Fine print. Let us give a proof that does not use the Squeeze Theorem for Limits. We want to show that $g(x)$ is continuous at a. As per the definition, we start with a collapsing collection of open intervals for a, such as our favorite one:

$$(a-1, a+1) \quad (a-1/2, a+1/2) \quad (a-1/3, a+1/3) \quad (a-1/4, a+1/4) \quad \cdots$$

We want to check that the wrappers of the g-images of these intervals form a collapsing collection for $g(a)$.

We should right away note that the conditions $f(x) \le g(x) \le h(x)$ and $f(a) = h(a)$ combine to tell us that $g(a)$ is the same as $f(a)$ and $h(a)$. From here the idea of the argument is straightforward.

By the continuity of $f(x)$ and $h(x)$ we have that the wrappers of the f- and g-images of the collapsing collection for a give two different collapsing collections for $f(a) = g(a) = h(a)$.

Here is the trick. We have the wrappers of the f- and h-images of the original collapsing collection, say there are

$$[c_1, d_1] \qquad [c_2, d_2] \qquad [c_3, d_3] \qquad \dots$$

and

$$[c'_1, d'_1] \qquad [c'_2, d'_2] \qquad [c'_3, d'_3] \qquad \dots$$

From these two collections we can make a collection of larger intervals like this:

$$[c_1, d'_1] \qquad [c_2, d'_2] \qquad [c_3, d'_3] \qquad \dots$$

These really are larger: because of the inequality $f(x) \le h(x)$, it must be that $[c_1, d'_1]$ contains both $[c_1, d_1]$ and $[c'_1, d'_1]$ (why?). But there is more: even though we made the intervals larger, this collection of larger closed intervals forms a collapsing collection for $f(a) = h(a)$ (this requires thought!). And now the kicker: because of the inequalities $f(x) \le g(x) \le h(x)$, it must be that the wrappers for the g-images of the original collapsing collection must be contained inside the collapsing collection of larger closed intervals above. Since those form a collapsing collection for $f(a) = h(a)$, it must be that the wrappers of the g-images (being smaller) also form a collapsing collection for $f(a) = g(a) = h(a)$. That's it!

And the limit version. Don't think we would let you leave without doing the limit version!

Theorem 3.7 (Squeeze Theorem for Limits)**.** *Suppose that $f(x)$ and $h(x)$ are functions with*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Suppose that $g(x)$ is another function with

$$f(x) \le g(x) \le h(x)$$

for all values of x. Then

$$\lim_{x\to a} g(x) = L.$$

As per usual, the argument uses the same idea as the continuity version. We encourage the reader to provide all of the details. As hinted above, if we prove Theorem 3.7 first, we can then use the limit definition of continuity to give a quick proof of Theorem 3.6

4 The Intermediate Value and Extreme Value Theorems

In this section we explain two important theorems in Calculus: the Intermediate Value Theorem and the Extreme Value Theorem. These two theorems are the raison d'être for the notion of continuity: if these theorems did not exist, we would not have bothered with continuity. In other words, these theorems are kind of a big deal.

4.1 The Intermediate Value Theorem

We begin with the Intermediate Value Theorem.

Theorem 4.1 (Intermediate Value Theorem)**.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, that $f(a) < 0$, and that $f(b) > 0$. Then there is a number c in (a, b) with $f(c) = 0$.*

Intuitively, the intermediate value theorem is obvious: if you are drawing the graph of a continuous function—and hence drawing without lifting your pencil off the paper—and you start below the x-axis and you end up above the x-axis, then somewhere in between the graph will hit the x-axis. In more plain terms: if you want to get from one side of an (infinite) river to the other, you need to cross the river.

The intuition is right on the money. But we would not call this a mathematical proof (or, formal argument) because we are not using the definition of continuity, just the intuition. So this argument does not adequately address the case of a function that fails to fall under the intuitive notion of continuity (such as the wiggly function above).

Let us prove—or give the argument for—the Intermediate Value Theorem, known affectionately as IVT. First, we define a set of real numbers A by the following rule: A is the collection of all numbers x in $[a, b]$ so that $f(x) < 0$. We know at least one number in this set, namely a. Because A has at least one number, it makes sense to consider its wrapper.

The wrapper of A is a closed interval. The left endpoint of the interval must be a. We can't make it greater than a since a is in A. And we can't make it less than a because A does not have any numbers less than a. The right endpoint of the wrapper must be less than or equal to b, since A does not have any numbers greater than b. In other words, the wrapper of A is of the form $[a, c]$ with $c \leq b$.

Our goal now is to show that $f(c) = 0$ (so not only do we know that there is an x with $f(x) = 0$, we are also giving a description of where x is!). Let's take a collapsing collection of closed intervals for c:

$$(c-1, c+1) \qquad (c-1/2, c+1/2) \qquad (c-1/3, c+1/3) \qquad \cdots$$

Because f is continuous, the wrappers of the f-images

$$f(c-1, c+1) \qquad f(c-1/2, c+1/2) \qquad f(c-1/3, c+1/3) \qquad \cdots$$

form a collapsing set of closed intervals for $f(c)$. We now see that $f(c)$ must be zero. If $f(c)$ were nonzero, say $f(c) = 1$, then by the definition of continuity the wrapper of $f(c - 1/1,000,000, c + 1/1,000,000)$—or maybe the f-image of an interval further down the list—would be a small interval around $f(c) = 1$. More specifically, this wrapper would not contain the number 0. But that means that the f-image of every number in $f(c-1/1,000,000, c+1/1,000,000)$ is positive. But *that* would mean that the wrapper for A would be $[a, c']$ with c' some number less than c. In other words, this would mean we chose the wrong c at the start. That's it!

(It is worth emphasizing that the argument is exactly the same if $f(c)$ is any other number, such as $f(c) = .01$ or $f(c) = -.5$. Whatever $f(c)$ is, we just wait until the wrappers of the f-images of the collapsing collection are small enough so that they do not contain 0.)

This is a sophisticated mathematical argument. Do your best to stay with it and understand it. You might not need to understand this to get an A in your Calculus course, but your brain and your future self will both thank you.

Exercises

1. Was there a time in your life when your height in inches was equal to your weight in pounds (h/t Thomas Banchoff)?

2. Is is true that if a basketball team is losing at halftime and then wins the game, then there must be a point during the second half where the game is tied?

3. If we know that f is a continuous function with $f(a) \leq 0$ and $f(b) \geq 0$, do we know that there is a number c in (a, b) with $f(c) = 0$?

4. Use the IVT to prove that if f is a continuous function with $f(a) \leq L$ and $f(b) \geq L$, then there is a c in $[a, b]$ with $f(c) = L$.

5. Use the IVT to prove that if f is a continuous function with $f(a) = y$ and $f(b) = z$, then $f([a, b])$ contains $[y, z]$. In other words, every number between y and z is an output of f.

6. Find a function with $f(1) < -1$, $f(2) > 1$

and so that there is no number c with $f(c) = 0$.

7. What part of the proof of IVT fails if f is not continuous?

4.2 The Extreme Value Theorem

And now the Extreme Value Theorem. An extreme value for a function $f(x)$ is an output, say $f(c)$ that is smaller than, or bigger than, every other output. These extreme values are also called a minimum and maximum. This sounds like a natural thing, but there are lots of functions that do not have extreme values. For example $f(x) = x^2$ does not have a maximum, because if you plug in bigger and bigger inputs, you get bigger and bigger outputs. It does have a minimum (what is it?). The function $f(x) = x^3$ has no maximum or minimum.

What if we change the domain to be an interval? For example, let's consider the function

$$f : [0, 1] \to \mathbb{R}$$

given by the formula $f(x) = x^2$. The notation means that the inputs, instead of being allowed to be any real numbers, are forced to be in the interval $[0, 1]$. Now we have what we want: $f(0) = 0$ is the minimum and $f(1) = 1$ is the maximum.

So that seems like the fix: if we look at functions with domain being a (finite) interval, then we always have maxima and minima. Not so fast! What if we again consider $f(x) = x^2$ but with the domain $(0, 1)$. Then there is no maximum or minimum (why?). What if we use the domain $[0, 1)$?

The Extreme Value Theorem, also known as EVT, says that we will definitely have extreme values, as long as the domain is a closed interval and—wait for it—the function is continuous. In the statement, the notation $f : [a, b] \to \mathbb{R}$ means that the domain (or set of inputs) is the closed interval $[a, b]$ and the potential outputs (or co-domain) is the set of real numbers.

Theorem 4.2 (Extreme Value Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Then f achieves an absolute maximum and an absolute minimum. Specifically, there are numbers c and d in $[a, b]$ so that*

$$f(c) \leq f(x) \leq f(d)$$

for all numbers x in $[a, b]$.

The proof of this theorem is the most challenging one in these notes. You absolutely can understand it, although it may take some time. Mastering this proof will be satisfying, and bring you one step closer to mathematical nirvana.

In order to prove the Extreme Value Theorem, we will begin by proving the following boundedness statement. In the statement, when we say that $f(x)$ is bounded on $[a, b]$ we mean that there are numbers n and N so that $n \leq f(x) \leq N$ for every x in $[a, b]$. More informally, this means that $f(x)$ does not go off to infinity as x varies in $[a, b]$. Here is the boundedness statement:

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Then $f(x)$ is bounded on $[a, b]$.

The argument for this boundedness statement has a similar flavor to our argument for the Intermediate Value Theorem. We define a set of numbers A to be the numbers c in $[a, b]$ so that $f(x)$ is bounded on $[a, c]$. We have the following four steps:

1. a is in A
2. A is an interval with left endpoint a
3. A is a closed interval
4. $A = [a, b]$.

The last step is exactly what we want for the boundedness statement. Indeed, this last step tells us that b is in A. By the way we defined A, this means that $f(x)$ is bounded on $[a, b]$. Since that is the entire domain of $f(x)$, this means that $f(x)$ is bounded.

Let's get to the four steps, starting with the first. To say that a is in A is to say that $f(x)$ is bounded on $[a, a]$. Remember that this strange looking interval is just the single number a. It is true that $f(x)$ is bounded on $[a, a]$, since the only output is $f(a)$. So there is certainly no going off to infinity here. Step 1 complete!

For the second step, we first convince ourselves that if c is in A then all the numbers between a and c are in A. This is true because if $f(x)$ is bounded on an interval, it should also be bounded on any smaller interval. Now think about this: if we have a set of numbers in $[a, b]$, and a is in the set, and every time we have a number c in the set, all the numbers between a and c are in the set. Then it must be that this set is an interval with left endpoint a. Convince yourself of this! Done? That finishes Step 2.

As we turn to Step 3. If A is not a closed interval, then it must be of the form $[a, e)$ for some number e in $[a, b]$. We want to show that this is impossible. Indeed, if $f(x)$ is bounded on $[a, e)$ then it is also bounded on $[a, e]$. Informally, if $f(x)$ does not go to infinity on $[a, e)$, then adding one more number e cannot make $f(x)$ go to infinity.

Finally, we have Step 4, which is where we finally use the assumption that $f(x)$ is continuous. We already know that A is a closed interval $[a, e]$. We need to show that this mystery number e is actually b. The basic idea is that if we know that $f(x)$ is bounded on the interval $[a, e]$ with $e < b$, then we can show that $f(x)$ is bounded on a slightly larger interval. This should make some sense, if you imagine drawing the graph of $f(x)$ and you use the informal idea of continuity as meaning that we can draw the graph of $f(x)$ without lifting our pencil off of the paper. But, the whole point of these notes is to get away from relying only one the intuition. Time for some fine print.

Fine print. Let us delve into the details of Step 4. We want to argue that if $f(x)$ is bounded on $[a, e]$ with $e < b$, then $f(x)$ is bounded on $[a, e']$ with $e' < e \leq b$. This would mean that the right endpoint of the closed interval A cannot be less than b. Since the left endpoint is a, this means that $A = [a, b]$, which is what we want.

We are assuming that $f(x)$ is bounded on $[a, e]$, which means that there are numbers n and N so that

$$n \leq f(x) \leq N$$

for every x in $[a, e]$. Since $f(x)$ is continuous, it is continuous at e. This means that if we take the collapsing collection of open intervals

$$(e-1, e+1) \qquad (e-1/2, e+1/2) \qquad (e-1/3, e+1/3) \qquad \cdots$$

and take the wrappers of the f-images of these intervals, we obtain a collapsing collection of closed intervals for $f(e)$. But this implies that, if we go far enough down the list in the collapsing collection, maybe $(e - 1/100, e + 1/100)$, then the wrapper of the f-image is as small as we want, say, smaller than $[f(e) - 1, f(e) + 1]$. But this means that the f-image of $(e - 1/100, e + 1/100)$ is contained in $[f(e) - 1, f(e) + 1]$. But this means that $f(x)$ is bounded on $[a, e + 1/100)$ (why?). But this means that A is larger than $[a, e]$, which is exactly what we wanted to show.

In summary, we showed that if $f(x)$ is bounded on $[a, e]$ with $e < b$, then it is bounded on a larger interval. And so the only available possibility left is that $f(x)$ is bounded on $[a, b]$, as desired. This is the end of the boundedness statement!

Back to EVT. Now, how do we go from the boundedness statement to the Extreme Value Theorem? We will use a technique called proof by contradiction. Here is how this works. We assume that EVT is false. Then we use logical reasoning to show that some false statement is true. We conclude from that that our assumption "EVT is false" was false. Which means EVT is true.

This might seem a little convoluted, but sometimes this is the best way to prove something. You probably also use proof by contradiction in every day life. Suppose you are trying to prove that you did not commit a crime. You might say: Suppose that I did do the crime. There is evidence that I was 100 miles away from the crime scene a mere 5 minutes after the crime. If I traveled 100 miles in 5 minutes, I must have traveled at an average speed of 1200 miles per hour. This cannot be, since the fasted known mode of transportation is an airplane, which travels at less than 600 miles per hour. As a result, I conclude that the original assumption is false, meaning that I did not commit the crime.

So let us prove EVT. Assume that it is false. This means that there is a continuous function $f : [a, b] \to \mathbb{R}$ and that $f(x)$ does not have a maximum value (or minimum, but we'll just stick with maximum). The boundedness statement tells us that the wrapper of $f([a, b])$ is a closed interval $[n, N]$. Since $f(x)$ does not have a maximum, there is no number c in $[a, b]$ with $f(c) = N$. Next, we define a new function $g : [a, b] \to \mathbb{R}$ given by the formula

$$g(x) = \frac{1}{N - f(x)}.$$

This function makes sense because, as $f(x)$ is never equal to N, we are never dividing by 0. Also, $g(x)$ is continuous by Theorem 3.1 (this requires thought!). Therefore $g(x)$ is bounded on $[a, b]$. Say $g(x) \leq M$ for all x in $[a, b]$. Inverting both sides of the last inequality gives

$$N - f(x) \geq \frac{1}{M}$$

for all x in $[a, b]$, or $f(x) \leq N - 1/M$. But that cannot be right: the wrapper of $f([a, b])$ was supposed to be $[n, N]$. Now we are saying that $f([a, b])$ is contained in $[n, N - 1/M]$, meaning that our supposed wrapper is too large. That is a contradiction. So it must be that EVT is true!

You might be asking yourself: how would I ever think of writing down that function $g(x)$? I certainly wouldn't expect that you would come up with this yourself, unless you spent a very long time thinking about it. At the same time, you can still appreciate the argument, just as you can appreciate a painting without being able to paint.

So how can you ever discover something new in mathematics, if we are saying that we do not expect you to have figured out this argument? The answer is that, if you learn enough of these arguments, you start to get a feel for how they work. It is like learning a language (or any other skill, really). Eventually you become fluent enough that you can start making arguments like this on your own. It takes time, but you can do it, and the satisfaction one obtains from doing this is special. Go for it!

5 ϵ–δ

As mentioned in the introduction to these notes, there is a standard way to define limits that appears in almost every textbook on Calculus. We would be remiss not to discuss this, and explain why the definition given above is equivalent. If you go on to take more advanced courses in mathematics, you will see this standard definition and will be expected to be familiar with it.

Again, as per the standard definition, we say

$$\lim_{x \to a} f(x) = L$$

(or, the limit of $f(x)$ as x approaches a equals L) if

for all $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$ it is true that $|f(x) - L| < \epsilon$.

Similarly, we say that $f(x)$ is continuous at $x = a$ if

for all $\epsilon > 0$ there is a $\delta > 0$ so that whenever $|x - a| < \delta$ it is true that $|f(x) - f(a)| < \epsilon$.

These are called the ϵ–δ definitions for limits and continuity. Since the latter is slightly simpler we will focus on that and leave the equivalence of the limit definitions to the reader.

The ϵ–δ challenge. First, we should gain some basic familiarity with the ϵ–δ definition. We usually think of this in terms of a challenge: First, player 1 choose some number $\epsilon > 0$. This is the challenge.

Then the job of player 2 is to choose (if they can!) a $\delta > 0$ so that whenever $|x-a| < \delta$ it is true that $|f(x) - f(a)| < \epsilon$. It is in player 1's interest to choose ϵ to be very small. That makes the challenge harder. Player 2 should then try to choose a small enough δ so that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. Remembering that absolute values represent distance, this means that $f(x)$ is within ϵ of $f(a)$ whenever x is within δ of x. If player 2 can meet every possible challenge, the function $f(x)$ is continuous at a. (Why is this equivalent to the formal description above?) Hopefully this ϵ–δ challenge resonates with our intuitive notion of continuity, which also stipulates that $f(x)$ is required to be close to $f(a)$ whenever x is close (enough) to x.

From our definition to the standard. There are two directions. We want to show that a function satisfying our definition satisfies the ϵ–δ definition and vice versa. We begin with the first direction. Suppose that $f(x)$ is continuous at $x = a$ under our definition. This means that the wrappers of the f-images of

$$(a-1, a+1) \quad (a-1/2, a+1/2) \quad (a-1/3, a+1/3) \quad (a-1/4, a+1/4) \quad \cdots$$

form a collapsing collection of closed intervals for $f(a)$. We want to show that the ϵ–δ definition is satisfied. Let $\epsilon > 0$ be a positive real number. Since the wrappers of the $f(a - 1/n, a + 1/n)$ form a collapsing collection for $f(a)$, it follows, once we choose a very large N, that the wrapper of $f(a - 1/N, a + 1/N)$ is completely contained in the interval $(f(a) - \epsilon, f(a) + \epsilon)$ (why is this true?). But this means that $f(a - 1/N, a + 1/N)$ (without the wrapper) is completely contained in $(f(a)-\epsilon, f(a)+\epsilon)$. Let $\delta = 1/N$. To say that $|x-a| < \delta$ is to say that x is in $(a-\delta, a+\delta)$, or that x is in $(a-1/N, a+1/N)$. For any such x we have already shown that $f(x)$ lies in $(f(a)-\epsilon, f(a)+\epsilon)$. But this is the same as saying that $|f(x) - f(a)| < \epsilon$. So we have met the challenge! Since ϵ was an arbitrary positive real number, we have shown that we can meet any ϵ-challenge with an appropriate δ. And so $f(x)$ is continuous under the ϵ–δ definition, as desired.

From the standard definition to ours. The argument for the other direction is very similar in spirit. We start by assuming that $f(x)$ satisfies the ϵ–δ definition. This means that for all $\epsilon > 0$ there is a $\delta > 0$ so that whenever $|x - a| < \delta$ it is true that $|f(x) - f(a)| < \epsilon$. We want to show that $f(x)$ satisfies our definition of continuity. To this end, we choose a collapsing collection of open intervals for a. Since we get to choose, we will use the standard one

$$(a-1, a+1) \quad (a-1/2, a+1/2) \quad (a-1/3, a+1/3) \quad (a-1/4, a+1/4) \quad \cdots$$

We want to argue that the wrappers of

$$f(a-1, a+1) \quad f(a-1/2, a+1/2) \quad f(a-1/3, a+1/3) \quad f(a-1/4, a+1/4) \quad \cdots$$

form a collapsing collection for $f(a)$. The first three properties are straightforward. The wrappers are closed intervals by definition. They must contain $f(a)$ since a is in every interval in the original collapsing collection. Each interval is contained in the previous since each interval in the original collapsing collection is contained in the previous. Finally, we must check that $f(a)$ is the only number in all of these wrappers.

Let y be a real number that is different from $f(a)$. We want to find some $f(a - 1/N, a + 1/N)$ so that y is not in the wrapper. As usual, we assume that $y > f(a)$, with the other case being essentially the same. Let $\epsilon = (y - f(a))/2$ and let $\delta > 0$ be the response to this challenge. This means that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. Now choose N so that $N > 1/\delta$, or $1/N < \delta$. By the choice of δ, we have that $f(a-1/N, a+1/N)$ is completely contained in $(f(a)-\epsilon, f(a)+\epsilon)$. By the choice of ϵ, we have that y is not in this interval, nor is y in its wrapper. So y is not in all of the wrappers of all of the $f(a-1/n, a+1/n)$. Since we choose y to be arbitrary, this means that there are no other numbers, besides $f(a)$, in all of the wrappers of all of the $f(a - 1/n, a + 1/n)$, and we are done!

A final word. The point of this section is to convince the reader that our definition of continuity is equivalent to the standard ϵ–δ definition. We do not want to minimize the difficulty in even parsing the sentence that is the ϵ–δ definition. We did not even spend time here discussing the meanings of the terms "for all" and "there exists." These terms are called quanitifers. Their precise meaning is the same as their usual meaning in spoken English (insofar as they are used in conversation); still, it takes some work to understand how to process these terms. We hope that our arguments above give some intuition for how this all works. Again, if you are going to go on in mathematics, you will eventually come to grips with these terms. Good luck!

APPENDIX.

A Sets of real numbers

This appendix is to help the reader get on solid footing with the idea of a set of real numbers and with the various notations related to sets.

In mathematics, the word "set" is the word we use for any collection of objects. We can talk about the set of real numbers, the set of rational numbers, the set of integers, the set of even integers, the set of prime numbers, the set of letters in the English alphabet, the set of students in a class, etc. Here, we will be only concerned with sets of real numbers.

Example 1: an open interval. A basic example of a set of real numbers is an interval, such as

$$(0, 1)$$

This is an example of what is called an open interval. This is the set of numbers between 0 and 1, not including 0 or 1. In other words, these are all the numbers that can be written as a decimal with only 0 to the left of the decimal and that has at least one number to the right of the decimal that is not 0 or 9 (why?). Here are some examples of numbers in the set $(0, 1)$:

$$0.5, \quad 1/7, \quad \pi/4, \quad 0.8675309, \quad \text{and} \quad 0.12345678910111213....$$

There are infinitely many numbers in the interval $(0, 1)$. Between any two numbers in the interval, there are infinitely more numbers! For example, between 0.5 and 0.5000001 we have the number 0.50000005 and 0.50000000000000000017, etc.

Example 2: a closed interval. Another example of a set of real numbers is the closed interval

$$[0, 1]$$

This is the set of all real numbers between 0 and 1, including 0 and 1 themselves. In other words, the interval $[0, 1]$ consists of the open interval $(0, 1)$ together with the numbers 0 and 1. Unlike the open interval $(0, 1)$, the closed interval has a smallest number and a largest number.

Example 3: singletons. There is another important type of set of real numbers, which we call singletons. For example, the set

$$\{7\}$$

has exactly one element: the number 7. The curly braces are read as "the set consisting of." Think of the braces as clothes for the number. We will sometimes write $\{7\}$ as

$$[7, 7]$$

Think about why this makes sense!

We can use curly braces for other types of sets. For instance

$$\{8, 6, 7, 5, 3, 0, 9\}$$

is the set consisting of 0, 3, 5, 6, 7, 8, and 9 (the order does not matter!). We will not use set builder notation in these notes, but if you learn what that is, you will be able to write down much more interesting sets.

Example 4: half-open intervals. Half-open intervals come in two varieties. Two examples are $[0, 1)$ and $(0, 1]$. The first of these is the set $[0, 1]$ without the number 1 and the second is $[0, 1]$ without the number 0.

Example 5: unions of intervals. The union of two or more intervals is the set obtained by combining intervals together. For example, the set

$$(0,1) \cup [2,3]$$

the union of the open interval $(0,1)$ and the closed set $[2,3]$ is the collection of numbers that lie in one of the intervals. For example, the numbers 0.5, 2, and 2.5 are in the set, but the number 1.5 is not. We pronounce the symbol $\cup$ as "union." We think of it similarly to the word "and." The set $(0,1) \cup [2,3]$ is $(0,1)$ and $[2,3]$ together.

Example 6: a number a. For this example, we want to discuss intervals containing an unspecified number a, for example

$$(a-1, a+1)$$

If we decide that a should be 5 then this is the interval $(4,6)$. If we decide that a should be 100, this is $(99,101)$. But there will be times when we want to talk about the number a without saying which number a we are talking about. That's a strange thing if you are not used to it. But it is also powerful, because we get to make statements like:

> *For every number a, the interval $(a-1, a+1)$ is an open interval of length 2 that contains a.*

By plugging in different numbers for a, we obtain infinitely many true statements from one!

More examples. Here is an example we will encounter a few times: the set of numbers of the form $1/n$ for $n = 1, 2, 3, \ldots$. This is the set of numbers

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots$$

There is a vast world of other examples of sets of real numbers. We encourage you to think of your own. That said, the only examples we will see in the discussion below are intervals and unions of intervals.

Exercises

1. Is 0 in the set $(-1,0) \cup (0,1]$?
2. How would you write $[0,1] \cup (1,2]$ as a single interval?
3. Is there a smallest number in $(0,1)$?
4. How many numbers would we have to add to $(0,1)$ in order to obtain a closed interval?
5. For which real numbers a is $2a$ in

$$(a-10, a+10)?$$

B Variants

As with any good mathematical theory, there are different versions that may be needed in different situations. We will briefly discuss several variations here, including:

1. other domains and co-domains,
2. left and right limits,
3. continuity from the left and right,
4. limits at infinity, and
5. limits of sequences.

B.1 Other domains and co-domains

The domain of a function is the set of inputs. The co-domain is any set of potential outputs. The range is the set of actual outputs. It's a strange thing that we actually get to choose the codomain for our functions. The only rules are that it must be possible to plug every number of the domain into the function, and that the co-domain contains the entire range.

For all of the functions above, the domain and co-domain are both equal to $\mathbb{R}$, the set of real numbers. For a function $f(x)$ with domain and co-domain $\mathbb{R}$, we write

$$f : \mathbb{R} \to \mathbb{R}$$

and read this as: f is a function from $\mathbb{R}$ to $\mathbb{R}$.

For the function $\sin(x)$, we can say that the domain and co-domain are both $\mathbb{R}$. We could also say that the domain is $\mathbb{R}$ and the co-domain is $[-1, 1]$ and write

$$\sin : \mathbb{R} \to [-1, 1]$$

For any function, we can make another function by making the domain smaller. We call this a restriction of the function. For example, we could consider the restriction of sin to, say, the interval $[0, 2\pi]$ and write

$$\sin : [0, 2\pi] \to [-1, 1]$$

Restrictions happen a lot in applications. For instance, if we consider the position of a particle over the course of one second, we might restrict the domain to the interval $[0, 1]$. One potential confusion is that we often change the domain of a function without changing the name of the function. So to completely specify a function, we should give the formula and specify the domain and co-domain.

B.2 Continuity from the left and right

This one is easy. We say that a function $f(x)$ is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Similarly, $f(x)$ is continuous from the right at a if

$$\lim_{x \to a^+} f(x) = f(a)$$

We can see that $f(x)$ is continuous at $x = a$ exactly when $f(x)$ is continuous from the left and from the right at a (assuming the left and right limits are defined and a is in the domain for f).

B.3 Limits at infinity

We somehow made it this far in our discussion without ever mentioning infinity. That is about to change. To say that

$$\lim_{x \to \infty} f(x) = L$$

means that as we make x larger and larger (without bound), the outputs get closer and closer to L. We can make a formal definition for this, just like we did for limits at (non-infinite) numbers.

The only thing we need is an analogue for collapsing collections. We don't need to worry about puncturing in this case, but we do need to use intervals at infinity. Here is a typical collapsing collection of open intervals for ∞:

$$(1, \infty) \qquad (2, \infty) \qquad (3, \infty) \qquad \cdots$$

We then take images and wrappers as usual, and check if the resulting collection of intervals is a collapsing collection. As a first example, we encourage the reader to write out the formal definition of a limit at infinity and check that

$$\lim_{x \to \infty} 1/x = 0$$

and that
$$\lim_{x\to\infty} \sin(x)$$
does not exist. Of course, we can define limits at $-\infty$ in a similar way.

B.4 Infinite limits

We also have situations where the limit of a function is infinity. For instance, we might be willing to believe that
$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$
Here the input should we a punctured collapsing collection of intervals for 0, as usual, say
$$\begin{gathered}(-1,0)\cup(0,1)\\ (-\tfrac{1}{2},0)\cup(0,\tfrac{1}{2})\\ (-\tfrac{1}{3},0)\cup(0,\tfrac{1}{3})\\ (-\tfrac{1}{4},0)\cup(0,\tfrac{1}{4})\\ \vdots\end{gathered}$$
The wrappers of the images of these intervals under $f(x) = 1/x^2$ are
$$[1,\infty) \qquad [4,\infty) \qquad [9,\infty) \qquad [16,\infty) \qquad \cdots$$
We say that a collection of half-closed intervals is a collapsing collection for ∞ if all of the intervals are of the form $[a,\infty)$ and there is no number in all of the intervals. This holds for the above collection, and so we have verified that the limit is ∞.

Again, there is an analogue for
$$\lim_{x\to a} f(x) = \infty$$
We can also start combining ideas. We can have limits like
$$\lim_{x\to a^+} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = -\infty$$
et cetera.

B.5 Limits of sequences

Sequences and limits of sequences also arise in calculus. A sequence is simply a list of numbers that goes on forever. We say that a sequence converges to a number L if the numbers get closer and closer to L as we go further down the list. For instance, the limit of the sequence
$$1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$$
is 0. As with limits of functions, we want to say exactly what we mean by closer and closer... in other words, we want to give a formal definition.

We already have most of the tools we need to give a formal definition of the limit of a sequence. First we define the tails of a sequence to be the sequences obtained by removing the first term, the first two terms, etc. For instance, the first few tails of the above sequence are
$$\begin{gathered}1,\ 1/2,\ 1/3,\ 1/4,\ \ldots\\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ \ldots\\ 1/3,\ 1/4,\ 1/5,\ 1/6,\ \ldots\\ 1/4,\ 1/5,\ 1/6,\ 1/7,\ \ldots\end{gathered}$$
We then find the wrappers of these sequences:
$$[0,1] \qquad [0,1/2] \qquad [0,1/3] \qquad [0,1/4] \qquad \cdots$$
We see that this is a collapsing collection for the number 0, and this is how we verify that the limit of the above sequence is 0.

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