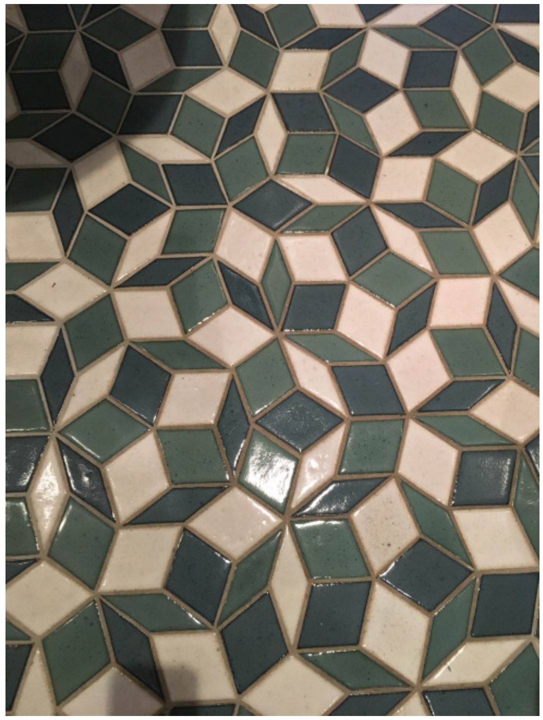
COLORING MAPS ON SURFACES



This image of a tiled bathroom shows a Penrose tiling of the plane with 3 differently colored parallelograms so that no adjacent tiles have the same color.

Theorem 1 (Sibley-Wagon)**.** Any tidy tessellation of the plane by parallelograms is 3-colorable.

Definition 2. In a *tidy* tessellation, parallelograms are stacked edge-to-edge. They do not look like bricks.

More generally, we'll consider quadrangulations of the plane and build the dual graph, which is 4-valent.

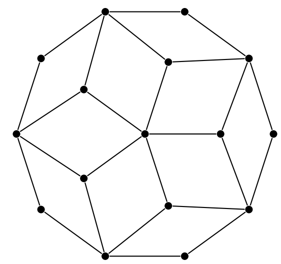
Question 3. Is every 4-valent planar graph 3-colorable?

Answer: No.

Theorem 4 (Stockmeyer, 1973)**.** 3-colorability of finite graphs is NP-complete. This is also true for planar graphs, and (Daley, 1980) for 4-regular planar graphs.

Theorem 5 (Brooks)**.** 4-colorability of 4-valent planar graphs is easy.

If we take a regular decagon, we can tessellate it by rhombuses (see the following page). The goal will be to consider finite subsets of a tiling of the entire plane and 3-color them. (If we can 3-color all finite subsets, then we can find a global 3-coloring.)

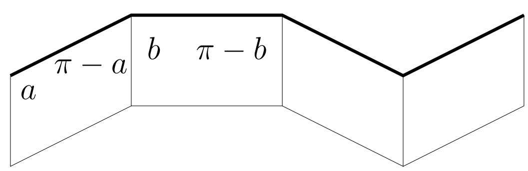


Definition 6. Given a finite region tiled by parallelograms, an *elbow* is a parallelogram with at most two neighbors.

As long as every finite region of the tiling has an elbow, we can 3-color the region by induction.

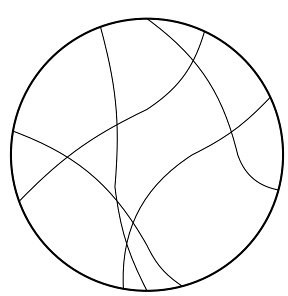
Lemma 7 (3-elbow lemma)**.** A finite tidy tiling by parallelograms has at least 3 elbows.

Proof. Suppose there are no elbows. Draw such a finite tiling and consider the path along the outside of the region. Each edge in this path must belong to a different parallelogram (since there are no elbows). Each parallelogram the path passes contributes $a + (\pi - a) = \pi$ interior angle. If there are n edges, the total interior angle of the path is $n\pi$. On the other hand, the outside of the region forms some polygon with n sides, so the interior angles must sum to $(n-2)\pi$, a contradiction. Therefore there must be elbows.

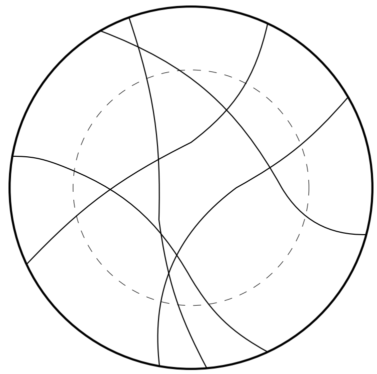


As the path passes across an elbow, the elbow can reduce the interior angle sum by less than π, so there must be at least three elbows. □

We can now discuss Wise's Greendlinger's Lemma in cubical small cancellation theory: look at quadrangulations of disks coming from a CAT(0) cube complex by extending midcube arcs (hyperplanes). As an initial step, we can clean up the picture by pushing off bigons and thus assume that the arcs pairwise intersect in at most one point. There exist at least 3 "corn squares" (elbows).



In the figure, two of the arcs don't cross at all. In the decagon tiling above, you can check that the midcube arcs would pairwise intersect. In our case, we can extend the region to a larger circle, adding intersections as needed so that all of the arcs pairwise intersect:



Then we can draw the dual quadrangulation to get a tessellation. Tessellations of this type are projections of hypercubes.

Theorem 8. Quadrangulations of the plane with embedded hyperplanes pairwise crossing at most once are 3-colorable.

Tilings by parallelograms satisfy this criterion.

Question 9. When can such quadrangulations be realized by parallelograms?

Take a map in the plane where every country has at least 6 neighbors. You get a CAT(0) dual triangulation. By connecting up the graph and its dual, you get a quadrangulation. The fact that the quadrangulation is 3-colorable implies that the graph itself is 4-colorable.