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<u>Mapping Class Groups</u>

$\mathrm{Mod}_g \approx \mathrm{Aut}(\pi_1 \Sigma_g)$

$\mathrm{Aut}(\mathbb{Z}^n)$
$\shortparallel$
$GL_n(\mathbb{Z})$

$\mathrm{Aut}(F_n)$

congruence subgroups

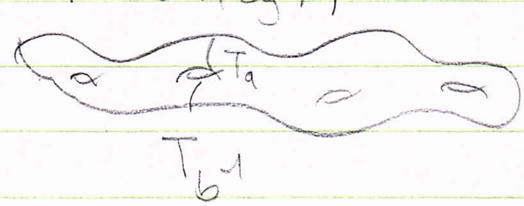
level n. $GL_n\mathbb{Z}(\ell) = \ker \left(\begin{array}{c} \mathrm{Aut}(\mathbb{Z}^n) \\ \downarrow \\ \mathrm{Aut}(\mathbb{Z}/\ell\mathbb{Z}^n) \end{array} \right)$

Q What are the analogues for level n congruence subgroups for $\mathrm{Mod}(S)$?

A $\mathrm{Torelli}_g(k) = \ker\left(\mathrm{Aut}(\pi_1 \Sigma_g) \longrightarrow \mathrm{Aut}\begin{pmatrix}\text{k-step nilpotent} \\ \text{quotient of } \pi_1 \Sigma_g\end{pmatrix}\right)$

$\mathrm{Torelli}_g(0) = \mathrm{Mod}_g$
$\cup$
$\mathrm{Torelli}_g(1) = \ker\left(\mathrm{Aut}(\pi_1 \Sigma_g) \longrightarrow \mathrm{Aut}(H_1 \Sigma_g)\right)$

= subgp gen'd by



$\cup$

$\mathrm{Torelli}(2) = K_g$ " Johnson kernel

subgroup gen'd by separating twists

$\cup$

$\mathrm{Torelli}(3)$

Q: Are these congruent subgroups finitely gen'd?

level 0 : Is $\mathrm{Mod}\,\Sigma_g$ f.g.?

Dehn: yes.

level 1: Birman asked in 1971 if the Torelli gp is f.g.

Kirby's problems from 1970s

solved by Denis Johnson in 1983

A: Yes.

level 2: McCullough-Miller asked in 1986:

Is K_g f.g.

Birman asked in 80's

Morita's problem lists in 90s.

Biss-Farb 2006: no.

Biss-Farb 2008: never mind "jk"

level 3: nothing known. Expectation was no?

<u>Thm (CEP, EH)</u>: Yes, $\forall$ levels. (EH: level 2 for $g \geq 5$)

Thm (CEP): For $g \geq 2k$ any subgroup of $\mathrm{Torelli}_g$ that contains the k^{th} term of the lower central series of Torelli (in particular ~~[illegible]~~ covers $\mathrm{Torelli}_g(k)$.) is f.g.

What is exciting is strength of method to prove this
method gives new way to prove G/N nilpotent, G f.g.
$\Longrightarrow N$ f.g.

Can it be applied to other groups that we wish we knew
were f.g.

Simple case of methods (sufficient for $k=2$)

<u>Thm (CEP)</u> Let $G =$ f.g. group.
Suppose a group Γ acts on G s.t.
1) G should be gen'd by a single Γ orbit
2) The "commuting graph" is connected $CG(C)$
$CG(C)$: vertices = elts of C.
edge $x \frown y \Leftrightarrow x$ and y commute
3) The image of $\Gamma \longrightarrow GL(\mathrm{Hom}(G,\mathbb{R}))$
is irreducible topological space in Zariski topology.

Then $[G,G]$ is f.g.

Apply to $G = \mathrm{Torelli}_g$
$\Gamma = \mathrm{Mod}_g$ acting by conjugation
1) Take $C = \{$all genus 1 BPs$\}$
classification of surfaces $\Rightarrow$ single conjugacy class
2) CG(genus 1 BPs) is connected for $g > 4$
3) Johnson $\mathrm{Hom}(\mathrm{Torelli}_g \mathbb{R}) = \Lambda^3 H_1(\Sigma_g ; \mathbb{R})$
as Mod_g rep
Image of Mod_g = image of $Sp_{2g}\mathbb{Z}$
(Zariski closure: $Sp_{2g}\mathbb{R}$, connected $\Rightarrow$ irred $\to Sp_{2g}\mathbb{Z}$ irred

So $[\mathrm{Torelli}_g, \mathrm{Torelli}_g] \approx K_g$ is f.g.

What goes into this?

Bieri - Neumann - Strebel : BNS invariant

G f.g.

G/N abelian

this tells you when N f.g.

Thm (BNS '87 + Brown '87)

Let G = f.g.

Every abelian action of G on an $\mathbb{R}$-tree is trivial

$\Leftrightarrow [G, G]$ f.g.

I'm allowed to think of this as just "tree"

abelian action : $\exists$ nonzero homomorphism $\rho: G \to \mathbb{R}$

s.t. $\ell(g) = |\rho(g)|$ $\forall g$

trivial : $\exists$ globally invariant line

"Benson, you did think of an example of this and those examples are called trivial"

First time the full strength of this thm used.

"It's like we've had a nuclear bomb in the garage for 30 years"

To prove $K_g \approx [T_g, T_g]$ is f.g.

need to prove every abelian action of Torelli$_g$ on $\mathbb{R}$-tree is trivial

Step 1:

Choose f.g. set

$X = \{x_1, \dots, x_n\}$ of genus 1 BPs

s.t. $CG(X)$ is connected

Step 2: "Some homomorphisms don't give us trouble"

If $\forall i\ \rho(x_i) \neq 0$ then ρ is good

(any ρ-abelian action is trivial)

pf: $\ell(x_i) = |\rho(x_i)| > 0$

all x_i act by hyperbolic isometry on $\mathbb{R}$-tree.

$\Longrightarrow$ each x_i preserves a unique line associated to it $A(x_i)$

But if 2 hyperbolic elts commute

$\Longrightarrow A(x) = A(y)$

$A(x_1) = A(x_2) = A(x_3) = A(x_4) = A(x_5)$

these generate Torelli$_g$. So all of Torelli$_g$ preserves this action

Step 3 Suppose (for a contradiction) λ is bad

$\lambda \neq 0$ but $\exists$ nontrivial λ-abelian action

$\Longrightarrow \lambda^g$ is bad $\forall g \in \mathrm{Mod}_g$

Step 2 $\Rightarrow \underbrace{\{\lambda^g \mid \forall g \in \mathrm{Mod}_g\}}_{\text{single } \mathrm{Mod}_g \text{ orbit} \Rightarrow \text{irreducible}} \subset \underbrace{\{\rho \mid \exists i\ \rho(x_i) = 0\}}_{\text{union of hyperplanes}}$

irreducible $\Rightarrow \exists j$

$\{\lambda^g\} \subset \{\rho \mid \rho(x_j) = 0\}$

$\lambda^g(x_j) = 0\ \forall g \rightarrow \lambda(g x_j g^{-1}) = 0\ \forall g \Rightarrow \lambda\left(\begin{smallmatrix}\text{all genus 1} \\ \text{BPs}\end{smallmatrix}\right) = 0$

$\lambda = 0$ $\boxtimes$