Large-scale geometry of right-angled Coxeter groups

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No boundaries – Groups in algebra, geometry and topology

In honor of Benson Farb

October 28, 2017

Γ= defining graph

Then W_Γ = RACG based on Γ is the group with the presentation:

- one generator for each vertex
- each generator has order 2
- one commuting relation for each edge

Examples

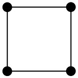
$\Gamma =$ 

$W_\Gamma = D_\infty := \mathbb{Z}_2 * \mathbb{Z}_2$

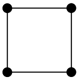
$\Gamma =$ 

$W_\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$

Examples of RACGs

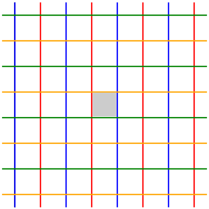


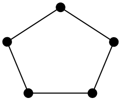
$$W_\Gamma = D_\infty \times D_\infty$$



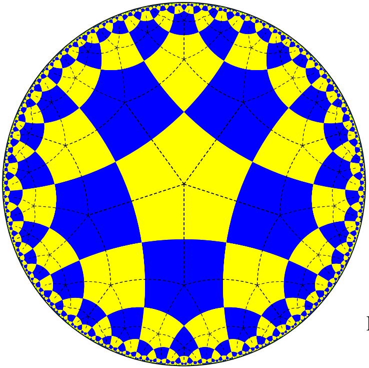
$$W_\Gamma = D_\infty \times D_\infty$$

$=$ group generated by reflections in the sides of a square in $\mathbb{R}^2$



$\Gamma =$ 

W_Γ = group generated by reflections in sides of a right-angled pentagon in $\mathbb{H}^2$



Picture by Jon McCammond

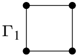
More generally, W_Γ acts properly and cocompactly on its *Davis complex* Σ_Γ.

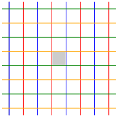
- Σ_Γ is a CAT(0) cube complex.
- The generators of W_Γ act by reflections.

When Γ is triangle-free, Σ_Γ is 2-dim'l. (Fill in squares in Cayley graph)

Many properties of RACGs can be seen in the graph. For example:

1. W_Γ hyperbolic $\iff$ Γ has no induced squares (Moussong)





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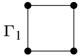
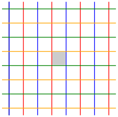
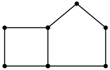




2. W_Γ splits over a finite group, i.e. $W_\Gamma = A *_C B$ with C finite $\iff$ Γ has a separating clique (Mihalik–Tschantz)



$W_\Gamma = W_{\Gamma_1} *_{\mathbb{Z}_2 \times \mathbb{Z}_2} W_{\Gamma_2}$

General question (Gromov):
Classify finitely generated groups up to quasi-isometry (QI)

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- lattices in semi-simple Lie groups (Farb and friends)
- Baumslag–Solitar, abelian-by-cyclic groups (Farb–Mosher)

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Classify finitely generated groups up to quasi-isometry (QI)

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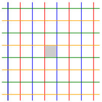
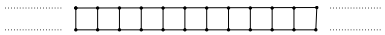
Goal: Classify RACGs up to quasi-isometry in terms of Γ.

QI invariant: ends

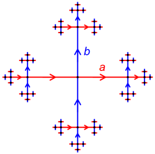
0 ends



1 end



2 ends



∞ ends

QI invariant: ends

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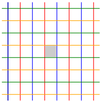
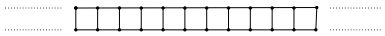
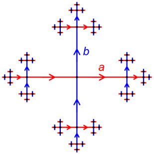


1 end



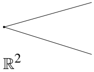
2 ends

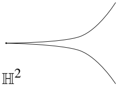


∞ ends

- A f.g. group has 0, 1 2 or ∞ ends. (Hopf)
- G has ∞ ends $\iff$ G splits over a finite group. (Stallings)
- (Papasoglu–Whyte) ∞ ends case $\rightsquigarrow$ 1 end case.
- Focus on 1-ended RACGs.

The divergence Div of G measures the "spread" of geodesics in G



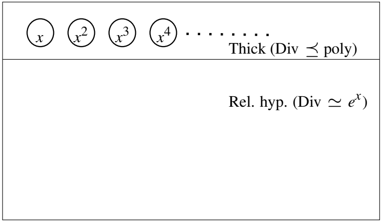


QI invariant: divergence

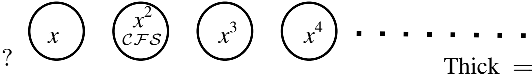
The divergence Div of G measures the “spread” of geodesics in G

For one-ended RACGs:

- (D.-Thomas)
  $\forall d \in \mathbb{N}, \exists W_\Gamma$ with Div $\simeq x^d$
- (Behrstock-Hagen-Sisto)
  W_Γ is
  - rel. hyp. ($\implies$ Div $\simeq e^x$) or
  - thick ($\implies$ Div $\preceq x^d$, some d)
- (D.-Thomas, Levcovitz)
  Div $\simeq x^2 \iff$
  Γ is $\mathcal{CFS}$ and not a join



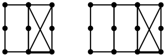
Further information: divergence in 1-ended RACGs



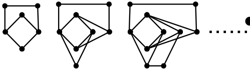
Thick $\Longrightarrow$ Div $\preceq$ polynomial

Some families can be distinguished using

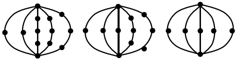
Rel. hyp $\Longrightarrow$ Div $\simeq e^x$



• contracting boundaries
(Charney–Sultan)
(Behrstock)



• divergence spectra
(Tran)



• work of Drutu–Sapir
(Hruska–Stark–Tran)

? x x^2 $\mathcal{CFS}$ x^3 x^4 . . . .

Thick $\Longrightarrow$ Div $\preceq$ polynomial

Some families can be distinguished using

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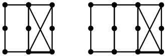
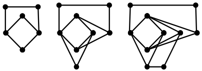
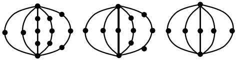


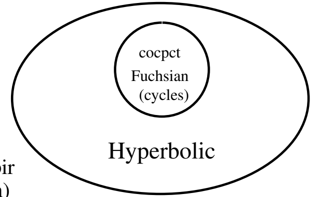
• contracting boundaries (Charney–Sultan) (Behrstock)



...... • divergence spectra (Tran)



• work of Drutu–Sapir (Hruska–Stark–Tran)



Suppose G is

- one-ended hyperbolic
- not cocompact Fuchsian
- admits splittings over 2-ended subgroups

  $\rightsquigarrow$ Bowditch's JSJ tree $\mathcal{T}_G$

- $\mathcal{T}_G$ is defined using the local cut point structure of the boundary ∂G.
- quasi-isometry invariant.

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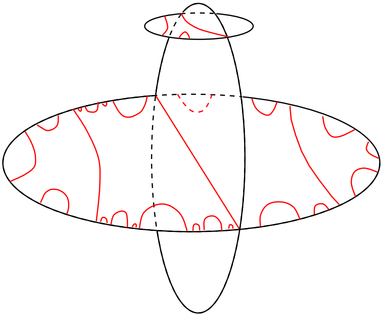
Theorem (D.–Thomas)

For W_Γ as above, with Γ triangle free, the JSJ tree $\mathcal{T}_\Gamma$ can be constructed visually. The vertices and edges of $\mathcal{T}_\Gamma$ are defined in terms of subsets of vertices of Γ.

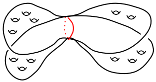
–Finite valence vertices:
(valence $k \geq 3$) $\leftrightarrow$ $\{x, y\} \subset \partial G$ s.t. $\partial G \setminus \{x, y\}$ has k cpts.

–*Quadratically hanging vertices* (valence ∞)$\leftrightarrow$ maximal $S \subset \partial G$ s.t. $\partial G \setminus \{x, y\}$ has 2 components. $\forall x, y \in S$

–Edges: inclusion of sets.



$\downarrow$



surface amalgam

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(valence ∞)$\leftrightarrow$ maximal
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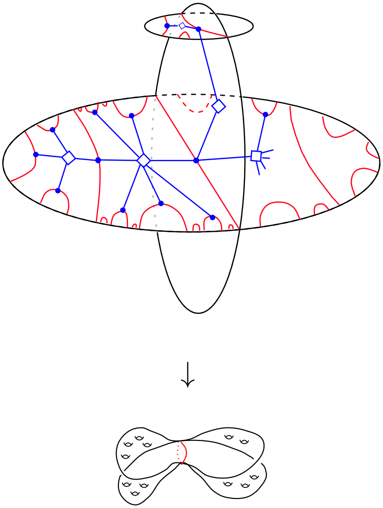
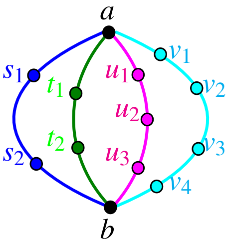
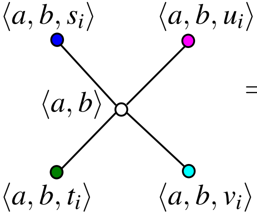
–*Rigid vertices* (valence ∞).
These "fill in the gaps"



surface amalgam

$\Gamma =$



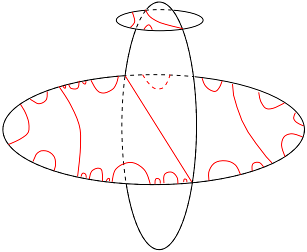


$= \mathcal{T}_\Gamma / W_\Gamma$

The Davis complex for W_G looks like space from prev. example

– orbit of branching geodesics $\leftrightarrow$ $\{a, b\}$ (separating)

– orbits of complementary regions $\leftrightarrow$ branches of Γ



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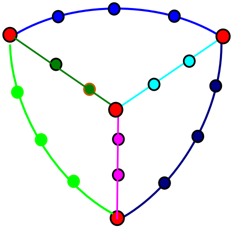
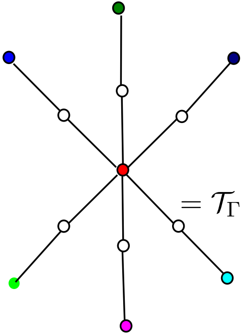




$= \mathcal{T}_\Gamma / W_\Gamma$

Red= rigid vertex of $\mathcal{T}_\Gamma$

Theorem (D.–Thomas)

The JSJ tree $\mathcal{T}_\Gamma$ has no rigid vertices $\iff$ Γ *has no K_4 minors.*

Theorem (D.–Thomas)

The JSJ tree $\mathcal{T}_\Gamma$ *has no rigid vertices* $\iff$ Γ *has no* K_4 *minors.*

Combining this with work of Malone:

Corollary

If Γ, Γ' *have no* K_4 *minors, and* W_Γ, $W_{\Gamma'}$ *as before, then the following are equivalent*

1. W_Γ *and* $W_{\Gamma'}$ *are quasi-isometric*
2. *There is a type-preserving isomorphism* $\mathcal{T}_\Gamma \to \mathcal{T}_{\Gamma'}$

Theorem (Cashen–D.–Thomas)

For $n \geq 4$, let $\hat{K}_n$ be a sufficiently subdivided copy of the complete graph on n vertices. Then

- *For all m, n, we have $\mathcal{T}_{\hat{K}_n} \cong \mathcal{T}_{\hat{K}_m}$.*
- *$W_{\hat{K}_n}$ q.i. $W_{\hat{K}_m}$ if and only if $m = n$.*

Two groups G_1 and G_2 are *(abstractly) commensurable* if there exist finite-index subgroups $H_1 < G_1$ and $H_2 < G_2$ such that $H_1 \cong H_2$.

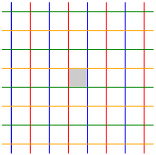
Commensurable $\implies$ quasi-isometric.

Question: To what extent is the converse true?

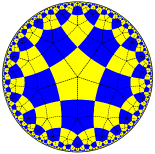
$\Gamma_n = n$-gon, with $n \geq 3$. Commensurability classes in $\{W_{\Gamma_n}\}$:

1. $\{W_{\Gamma_3}\}$ (finite)

2. $\{W_{\Gamma_4}\}$ (infinite, not hyperbolic)

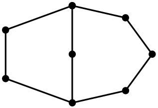


3. $\{W_{\Gamma_n} | n \geq 5\}$ (hyperbolic)



Consider $W_{m,n}$ defined by:

$\Gamma_{m,n}$ = an $(m+3)$-gon + an $(n+3)$-gon, identified along a pair of edges



$\Gamma_{m,n}$

$\chi_1 = -\frac{(m-1)}{4}$

$\chi_2 = -\frac{(n-1)}{4}$

orbicomplex $\mathcal{O}_{m,n}$

Theorem (Crisp–Paoluzzi)

The groups W_{p_1,p_2} and W_{q_1,q_2} are abstractly commensurable if and only if

$$\frac{\chi_1}{\chi_2} = \frac{\chi_1'}{\chi_2'}$$

Crisp–Paoluzzi examples

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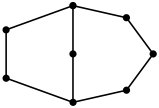

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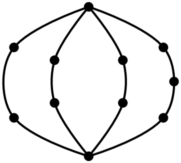
$$\frac{\chi_1}{\chi_2} = \frac{\chi_1'}{\chi_2'}$$

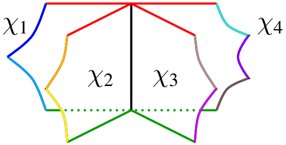
$\iff$ (χ_1, χ_2) *and* (χ_1', χ_2') *are commensurable.*

Vectors v_1, v_2 *commensurable* $\iff$ $\exists\, M, N \in \mathbb{Z}$ such that $Mv_1 = Nv_2$

Generalized theta graphs



$\Theta(n_1, \ldots, n_k)$



$\mathcal{O}(n_1, \ldots, n_k)$

$\chi(\Theta) = (\chi_1, \ldots, \chi_k)$

Theorem (D.–Stark–Thomas)

Let $\Theta = \Theta(n_1, \ldots, n_k)$ and $\Theta' = \Theta(m_1, \ldots, m_{k'})$. Then W_Θ and $W_{\Theta'}$ are commensurable if and only if

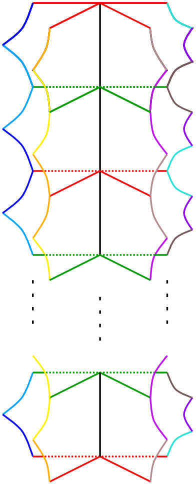
1. $k = k'$, *and*
2. $\chi(\Theta)$ *and* $\chi(\Theta')$ *are commensurable.*

Consider $W(\Theta)$ and $W(\Theta')$ with

$k = k'$, and $q\chi(\Theta) = p\chi(\Theta')$, with $p, q \in \mathbb{Z}$

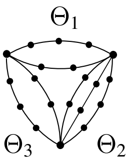
Construct common covers!

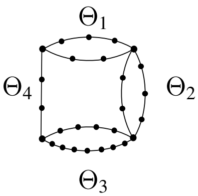
Unfold one p times and the other q times.



So $W(\Theta)$ and $W(\Theta')$ are commensurable.

Examples



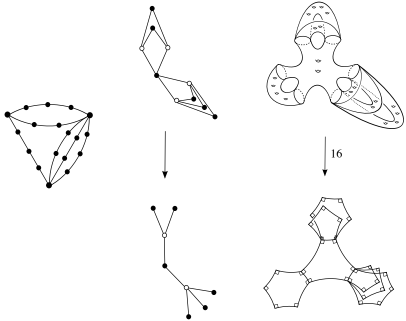


Theorem (D.–Stark–Thomas)

We give a complete commensurability classification of the RACGs defined by cycles of generalized theta graphs.

- conditions for commensurability: equations of Euler characteristic vectors.
- There are infinitely many commensurability classes in each q.i. class

1. Construct degree 16 surface amalgam covers with a cyclic structure:



2. Use the cyclic structure, the conditions on χ vectors and well-known results on covers of surfaces with boundary to construct common covers.