Tullia Dymarz — Quasi-Isometric Rigidity and Solvable Groups

~~GT w/~~

Recall (Farb - Mosher):

Studies large-scale geometry of solvable gps

- $BS(1,m) = \langle a, t \mid tat^{-1} = a^m \rangle \cong \mathbb{Z} *_m$
- $\Gamma_M = \mathbb{Z}^n *_M$, $\det(M) > 1$.

$= \mathbb{Z}^n \rtimes_M \mathbb{Z}$

Note: • $BS(1,m) \curvearrowright X_m := \mathbb{H}^2 \times_h T_{m+1}$

horocyclic product

• $\Gamma_M \curvearrowright X_M := G_M \times_h T_{\det(M)+1}$

where $G_M = \mathbb{R}^n \rtimes_M \mathbb{R}$.

[note that $BS(1,m)$ is just the case M is scalar

Today: Eigenvalues have modulus > 1

Thm (Farb–Mosher) If $\Lambda \simeq_{QI} \Gamma_M$, then $\Lambda \curvearrowright X_N$, (in fact M, N have rational pow. w/ identical Jordan form)

Pf Sketch $QI(X_M) \cong \text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{Q}_m)$

(idea: $\rightarrow$ is by action on boundary)

and $QI(X_M) \cong \text{Bilip}(\partial_\infty G_M \cancel{\setminus} *) \times \text{Bilip}(\partial_\infty T_{\det M+1} \setminus *)$

We have

$$\begin{array}{c} \Lambda \longrightarrow QI(X_M) \\ \cup \\ \text{Isom}(X_M) \end{array}$$

$\boxed{\text{Goal}}$

Note that $\text{Isom}(X_M) \subseteq \text{Sim}(\partial_\infty G_M) \times \text{Sim}(\partial_\infty T_{\det M}$

Strategy: Try to conjugate Λ into similarities

[This is not Farb–Mosher's strategy, bu

Mosher–Sageev–Whyte: "Uniform" subgps of $\text{Bilip}(T_{\det(M)+1})$ into $\text{Sim}(\partial_\infty T_{\det M+1}$

"Tullia's Actual Thesis" {Dymarz – Xie : Conjugate subgps of $\mathrm{Bilip}(\partial_\infty G_M)$ to $\mathrm{Sim}(\partial_\infty G_N)$

~~More~~ Further results:

- [Reiter-Peng, Dymarz-Xie]: $\Gamma_\varphi = N \rtimes_\varphi$ (N nilpotent)

  $\Gamma_\varphi \curvearrowright G_\varphi \times_h T_\varphi$

- [Eskin-Fisher-Whyte, Peng].

  Program on QI Rigidity of polycyclic gps.

Precise statement of thm:

Thm (Dymarz – Xie). Let $\Gamma \subseteq \mathrm{Bilip}($

M diagonaliz

Assume Γ is uniform, amenable,

$\Gamma \curvearrowright$ ~~Bilip~~ $\partial_\infty G_M \times \partial_\infty G_M \setminus$

cocmptly

Then Γ is conjugate into $\mathrm{Sim}(\partial_\infty G_M)$

Pf In the case $M = e^A$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\partial_\infty G_M = (\mathbb{R}^2, d_M)$$

w/ $d_M(p,q) := \max\{\Delta x, (\Delta y)^{1/2}\}$

Bilipschitz maps are of the form:

$$f(x,y) = (f_1(x,y), f_2(y))$$

Tullia's thesis: Up to conjugacy, these are

$$\gamma(x,y) = \theta \circ (x + h(y), y).$$

similarity | h is $\frac{1}{2}$-Hölder, $h(0) = 0$

Set $B = \left\{ h : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{l} h(0)=0 \\ h \text{ is } \frac{1}{2}\text{-Hölder} \end{array} \right\}$.

Banach space

Note: $\Gamma \curvearrowright B$ affinely and isometrically

w/ norm $\|h\| = \sup\limits_{y \neq z} \dfrac{|h(y) - h(z)|}{|y-z|^{1/2}}$

Now we use:

<u>Thm</u> (Day) If Γ is amenable, and acts by affine isometric transf on a compt convex subset of a Banach space, then it has a fixed pt

To finish the proof, conjugate so the fixed pt is at 0.