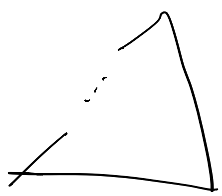
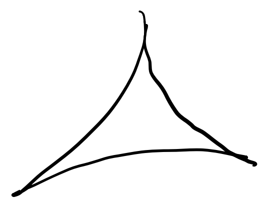
Jordan Ellenberg : "Farblandia"

no... FAR - from - BLAND

def: (geodesic) metric space δ-hyperbolic : one side lies within δ of the other two



unexample

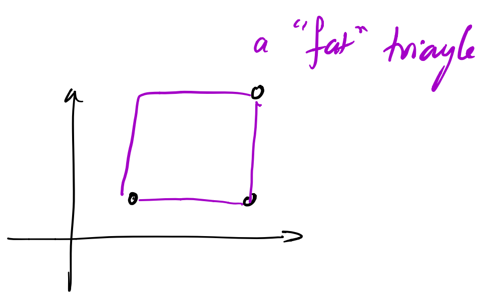


example

↪ statement about geometry and Groups?

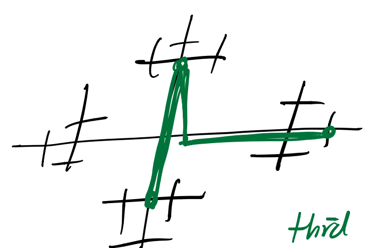
def: a gp. is word hyperbolic (w.r.t. a genset S) if its Cayley graph is δ-hyperbolic.

$\mathbb{Z}^2$



unexample

F_2



third side is contained in others.

upshot: being hyperbolic is a very strong property
"easy to navigate your group"
eg. solvable word problem.

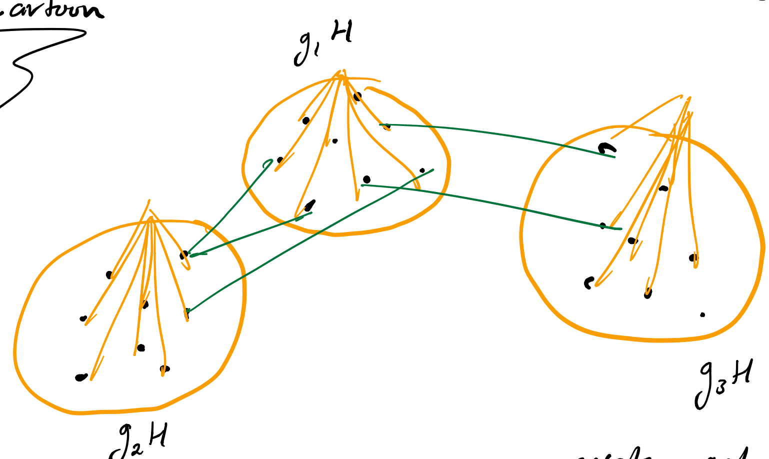
Rk: groups in nature are often NOT hyperbolic.
eg $\pi_1 M$ (for M: noncompact w/ bounded curvature) need NOT be hyperbolic.

$\rightsquigarrow$ Farb's Thesis: notion of relative hyperbolicity.

idea: G: group w/ subgp $H < G$

"enemy" (poison/problem) subgroup.

Cartoon



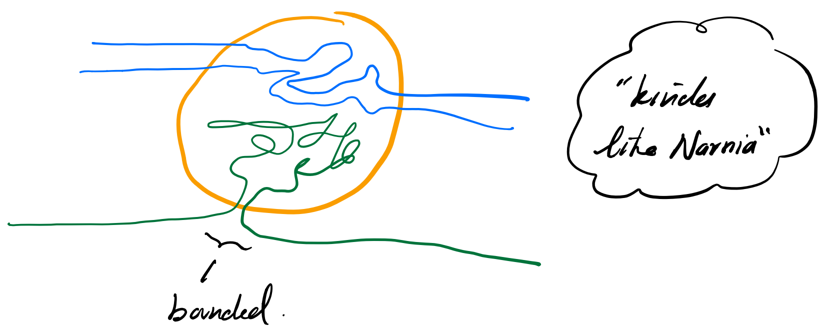
cosets. get coned
crushed is ok, but not as good.

def: G is <u>hyperbolic relative</u> to H

if (1) the coned Cayley graph $\widehat{\Gamma}$ is δ-hyperbolic.

(2) "Bounded coset penetration". (BCP).

- passing in and out of cosets in bounded nbhd.
- pairs of geodesics "fellow travel"-ish



Thm (Farb): rel. hyp $\Rightarrow$ solvable word problem.

Thm ("): M noncompact manifold w/ bounded curvature and H = cusp subgp

$\Rightarrow$ $(\pi_1 M, H)$ is rel. hyp. (with BCP).

corollary: $\pi_1 M$ has solvable word problem (in $O(n \log n)$ time)

THESIS: Groups are all different

ANTITHESIS: Groups are all the same

SYNTHESIS: "<u>FARBLANDIA</u>"

§ QI - rigidity.

Hilbert's modular group

$K = \mathbb{Q}(\sqrt{d})$ real quadratic field.

$\Lambda \subset SL(2, \mathcal{O}_K) \hookrightarrow PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.

$\Lambda \curvearrowright \mathbb{H}^2 \times \mathbb{H}^2$ lattice.

Farb, Schwartz: Λ, Λ' two nonuniform irreducible lattices in $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. are quasi isometric

$\Rightarrow$ Λ, Λ' are commensurable.

large scale geometry

"space when viewed from far away"

eg. $SL(2, \mathbb{Z}\sqrt{11})$ and $SL(2, \mathbb{Z}\sqrt{13})$ are commensurable

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concern:

$\Lambda \curvearrowright \mathbb{H}^2 \times \mathbb{H}^2$ not cocompact

eg $\quad \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \quad \mathbb{Z}$

$\longleftrightarrow \quad \mathbb{R} \qquad \mathbb{Z} \simeq \mathbb{R}$.

eg. $\mathbb{Z} \times \mathbb{Z}/2 \simeq \mathbb{Z}$.

fix: BRUTALLY remove horoballs

$\leadsto$ Neutered space $\subset \mathbb{H}^2 \times \mathbb{H}^2$ on which Λ acts cocompactly

Rk: horoballs have Sol geometry $\begin{bmatrix} \varepsilon^k & \mathbb{Z}[1] \\ & \varepsilon^{-k} \end{bmatrix}$

$\rightsquigarrow$ quasi isometric rigidity for lattices
(many others) in semisimple gps of any rank

§ Mapping class groups.

$$\mathrm{Mod}(\Sigma_g) = \pi_0 \,\mathrm{Diff}\, \Sigma_g$$

Q: is $\mathrm{Mod}\,\Sigma_g$ "like a lattice" ?

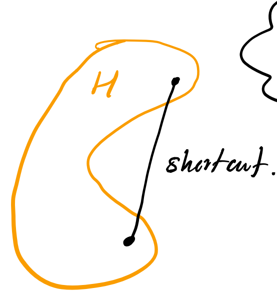
If so, what rank? (rank-1 or higher rank?).

Further, $H < \mathrm{Mod}\,\Sigma_g$

Q: $\exists$? shortcuts in H.

For example $H = \langle \gamma \rangle$.

(quasi)-convexity
distortion



Q: Does word length of γ^n grow like n? (more slowly?).

in $SL(2)$ $\begin{bmatrix} 1 & n \\ & 1 \end{bmatrix}$ requires n letters.

$SL(3)$ $\begin{bmatrix} 1 & 0 & n \\ & 1 & 0 \\ & & 1 \end{bmatrix}$ length $\sim \lg n$.

(commutators).

Farb, Lubotsky, Minsky (2001).

$\mathbb{Z} \hookrightarrow \mathrm{Mod}\,\Sigma_g$ is $\underbrace{\text{quasiconvex}}_{\text{no shortcuts.}}$ / undistorted

Broaddus, Farb, Putman (2009)

Torelli gp : $\mathrm{Tor}\,\Sigma_g = \ker\left(\mathrm{Mod}\,\Sigma_g \longrightarrow Sp(H_1\Sigma_g)\right)$

$\mathrm{Tor}\,\Sigma_g \hookrightarrow \mathrm{Mod}\,\Sigma_g$ is $\underbrace{\text{distorted}}_{\text{many shortcuts.}}$

distances b/w $g, h \in \mathrm{Tor}\,\Sigma_g$
are exponentially smaller in $\mathrm{Mod}\,\Sigma_g$.

§ Representation stability.

Farb, Church, Ellenberg, ...

~ surprisingly? ~ FINITE GROUPS.

consider $P\,\mathrm{Conf}^n X = (x_1, \ldots, x_n) \in X$
$x_i \neq x_j$

$H_i\,(P\mathrm{Conf}^n(X), \mathbb{C})$ admits S_n-action.

$H_1(PConf^n(\mathbb{C}), \mathbb{C})$ has dim $\binom{n}{2}$.

spanned by pairs of elements

$\rightsquigarrow$ same representations, but different groups

ideas

$$\cdots \quad PConf^5 \leftleftarrows PConf^6 \leftleftarrows PConf^7 \quad \cdots$$

$$\circlearrowright S_6$$

Future: $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$?