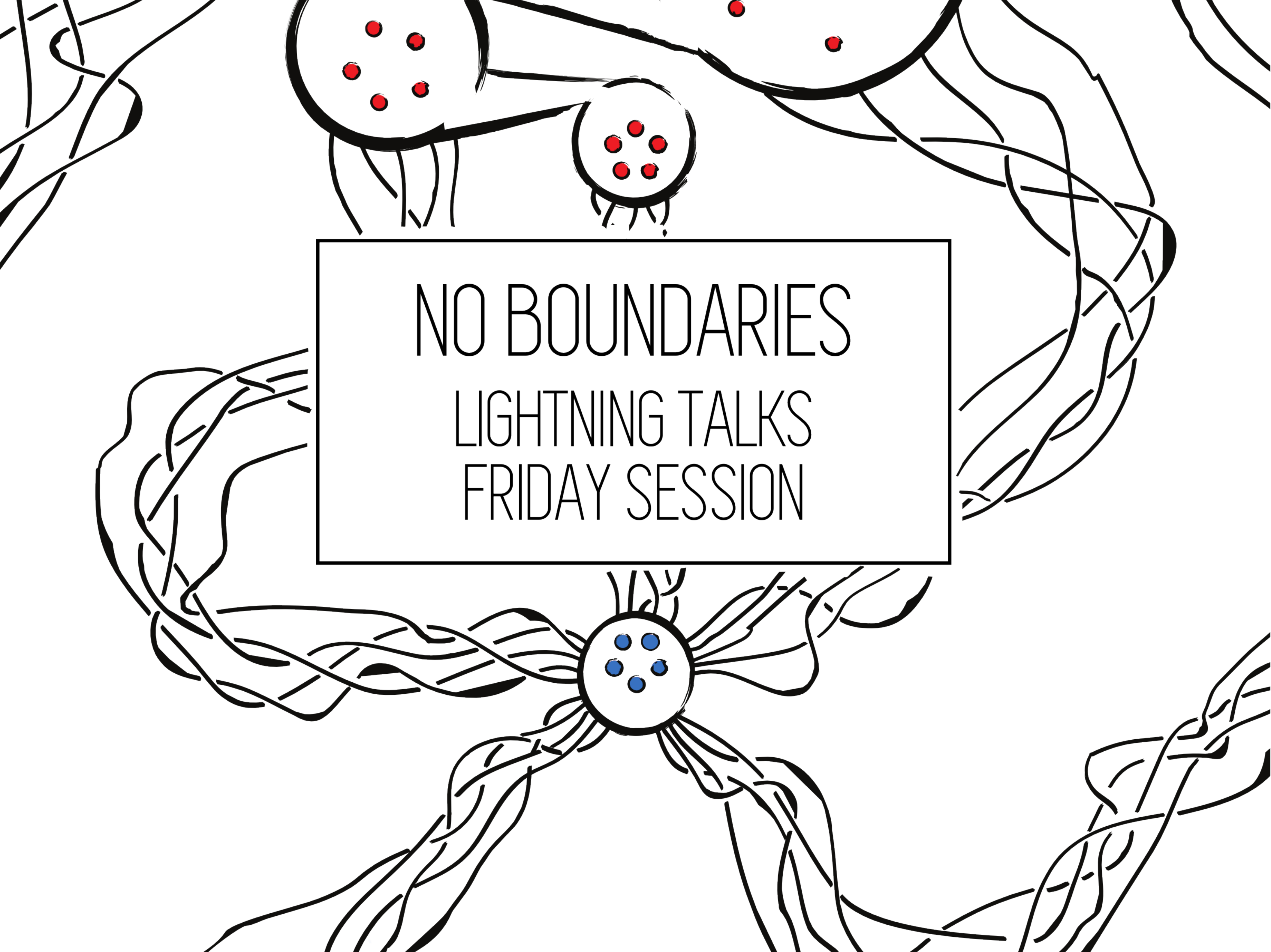
NO BOUNDARIES

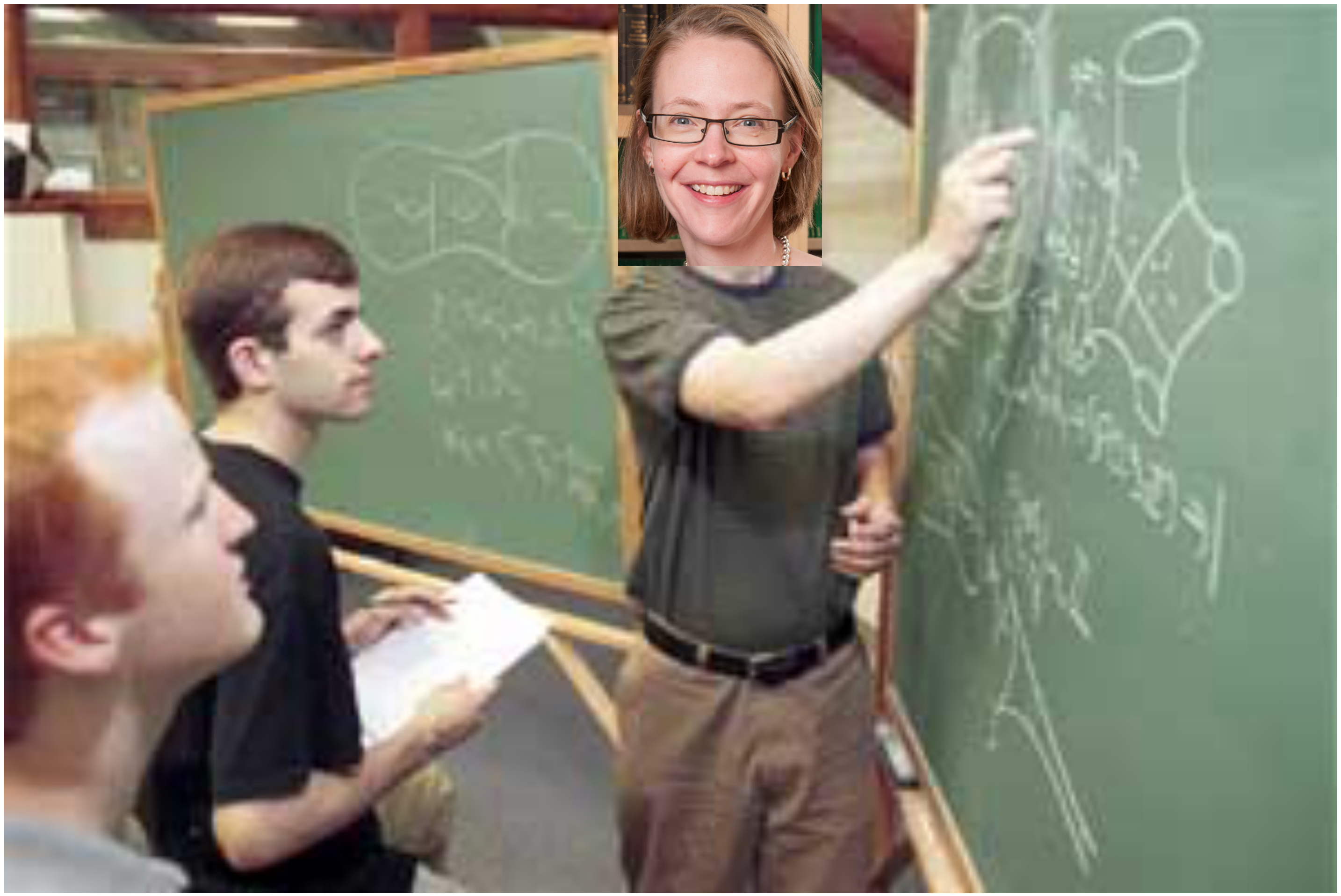
LIGHTNING TALKS
FRIDAY SESSION



Ivanov's Metaconjecture

Tara Brendle
Dan Margalit

No Boundaries: Groups in Algebra, Geometry, and Topology
University of Chicago
October 27, 2017



Automorphisms of the Curve Complex

Theorem (Ivanov). Aut $\mathcal{C}(S_g)$ = MCG(S_g)

Application. Aut MCG(S_g) = MCG(S_g)

Rigidity for Complexes

- Systole Complex — Schmutz-Schaller
- Pants Complex — Margalit
- Nonseparating Curve Complex — Irmak
- Cut System Complex — Irmak-Korkmaz
- Complex of Separating Curves — Brendle-Margalit
- Torelli Geometry — Farb-Ivanov
- Complex of Domains — McCarthy-Papadopoulos
- Arc Complex — Irmak
- Arc and Curve Complex — Irmak-Korkmaz
- Asymptotic Pants Complex — Fossas-Nguyen
- Ideal Triangulation Graph — Korkmaz
- Strongly sep. curve complex — Bowditch
- Hole-bounding Curves and Pairs Complex — Irmak-Ivanov-McCarthy
- Complex of Shirts and Straightjackets — Bridson-Pettet-Souto

Rigidity for Groups

Mapping Class Group
Ivanov

Torelli Group
Farb-Ivanov

Johnson Kernel
Brendle-Margalit

Terms of Johnson Filtration
Bridson-Pettet-Souto

Ivanov's Metaconjecture

Any object naturally associated to a surface *S* and having a sufficiently rich structure has MCG(*S*) as its group of automorphisms.

Rigidity for Groups

Mapping Class Group
Ivanov

Torelli Group
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Johnson Kernel
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Terms of Johnson Filtration
Bridson-Pettet-Souto

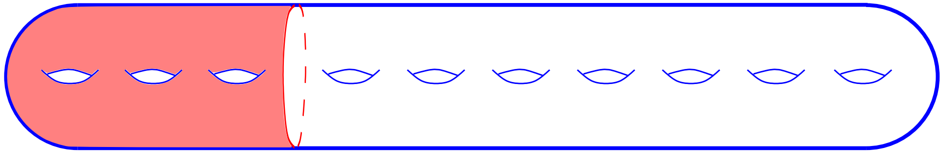
Other Normal Subgroups?

Dahmani-Guirardel-Osin examples

Main Theorem

If $N \lhd \mathrm{MCG}(S_g)$ has an element with small support then:

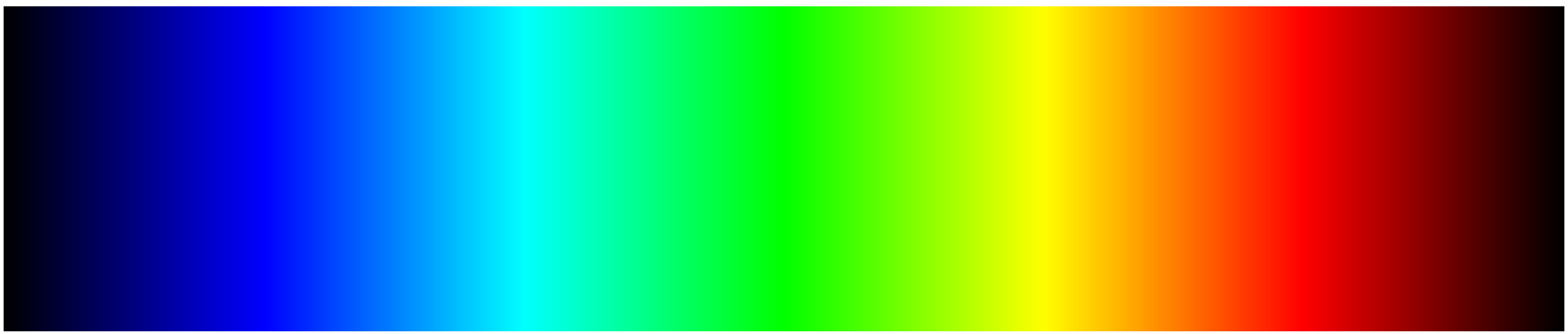
$$\mathrm{Aut}\, N = \mathrm{MCG}(S_g).$$



Normal Subgroups of MCG

Aut >> MCG

Aut = MCG



Infinitely generated RAAGs

Terms of Johnson filtration, Magnus filtration, etc.

Normal Subgroups of MCG

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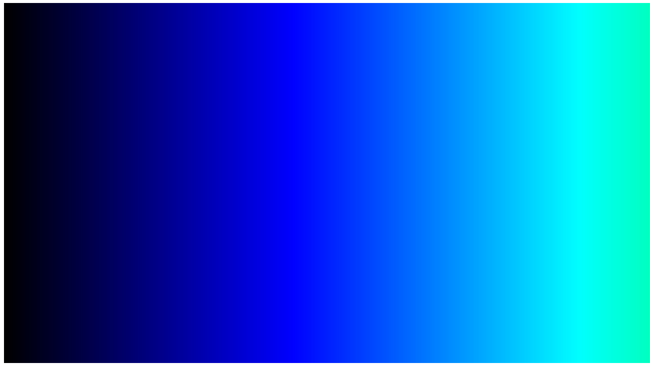
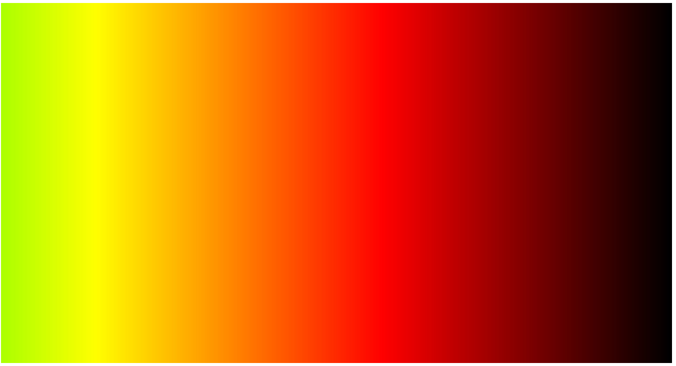


?



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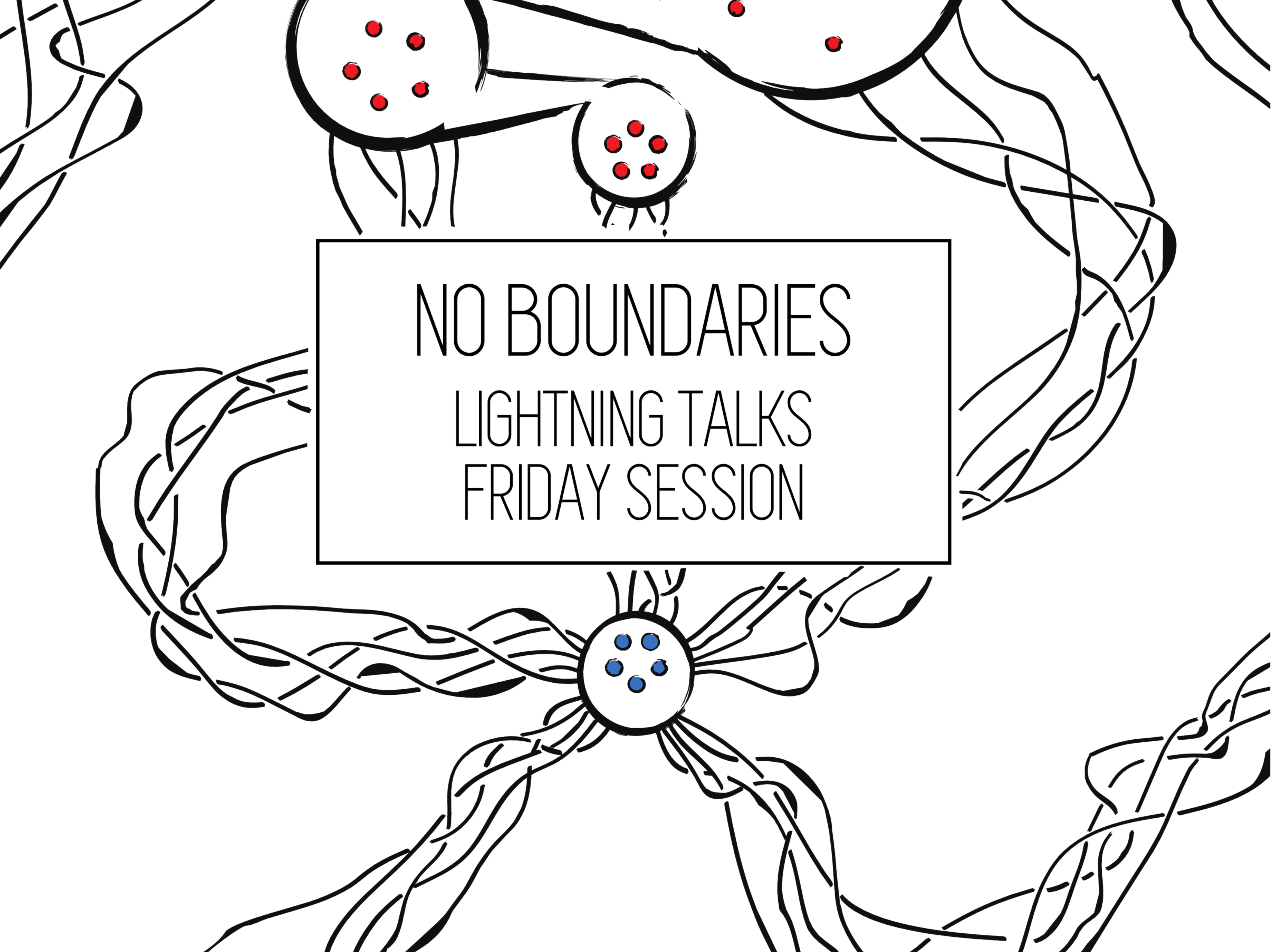
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The Primitive Torsion Problem

Khalid Bou-Rabee

Joint with Patrick W. Hooper

The City College of New York

The Primitive Torsion Problem

- Let F_r be the free group of rank r. A *primitive element* is an element that is part of a basis for F_r.
- Let P_k be the group generated by kth powers of all primitive elements in F_r.

- **The Primitive Torsion Problem**: When is F_r/P_k finite? Finitely presented? Solvable? Nilpotent?

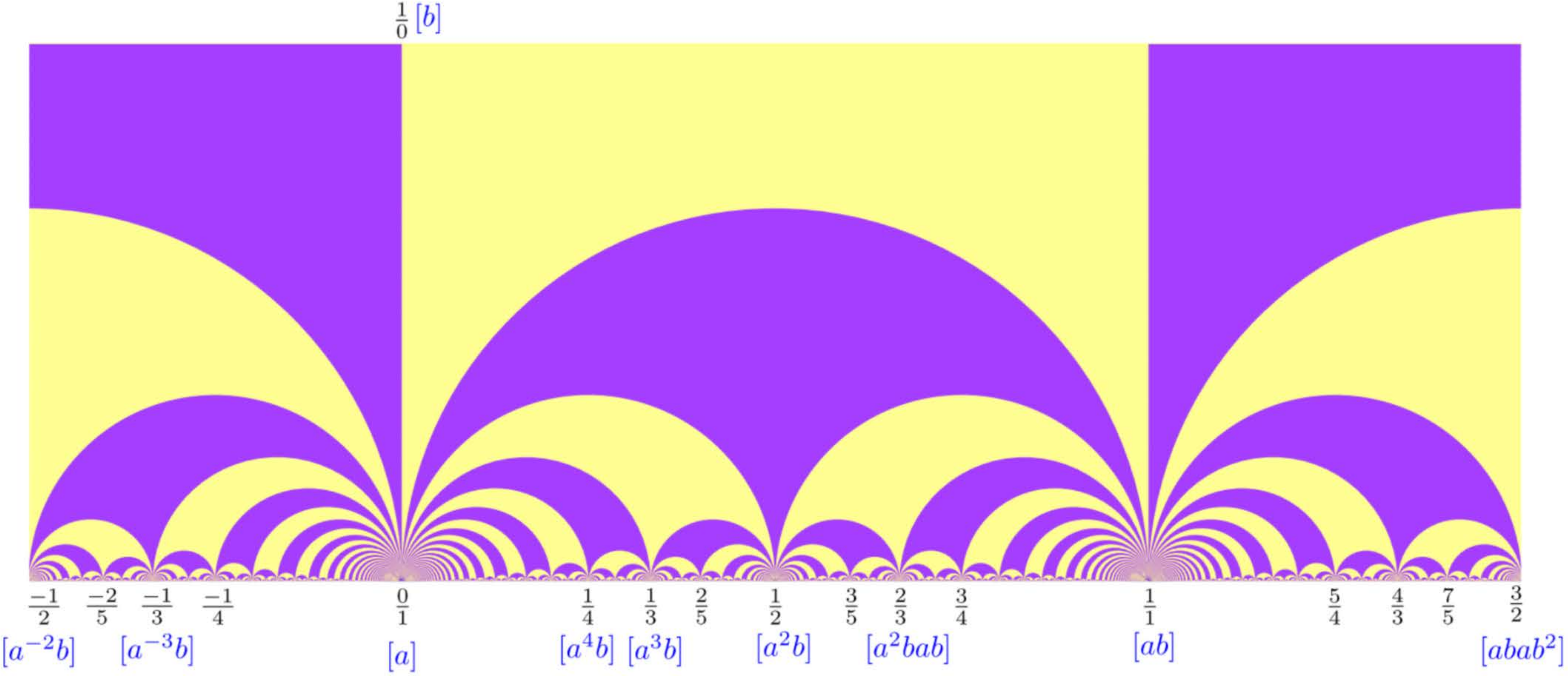
- Similar questions for other groups may be asked…

Known results

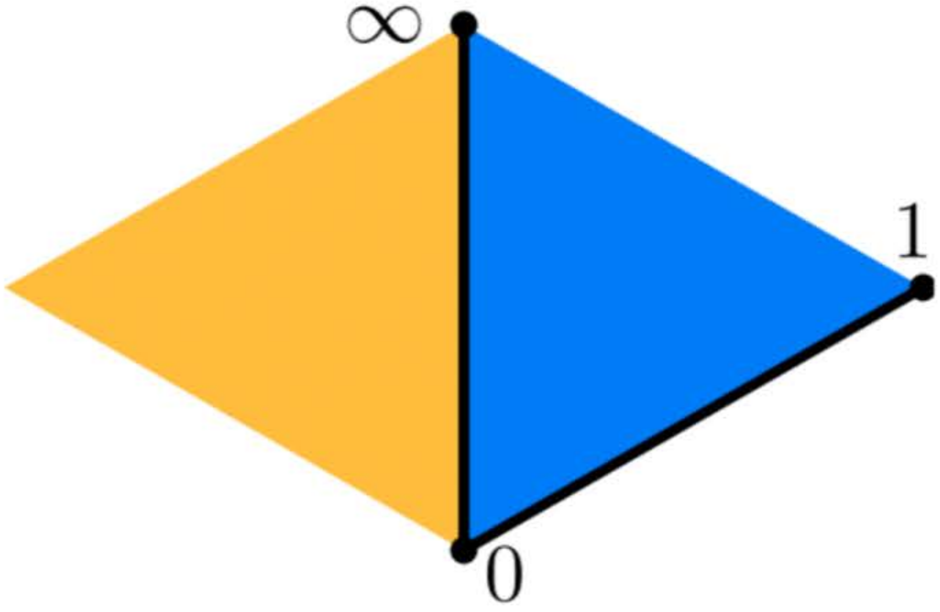
- Theorem (Thomas Koberda and Ramanujan Santharoubane, 2015) For some $k \geq 10$, the group F_r/P_k is infinite.
- Theorem (Andrew Putman and Justin Malestein, 2017) Same result. Different proof.

- Theorem (Patrick W. Hooper and Bou-Rabee, 2017) The group F_2/P_k is finite if and only if $k = 1,2,3$. **Moreover,** F_2/P_4 is virtually nilpotent (we construct an explicit integral representation), and F_2/P_k is finitely presented for $k = 1,2,3,4,5$.

The Farey triangulation:

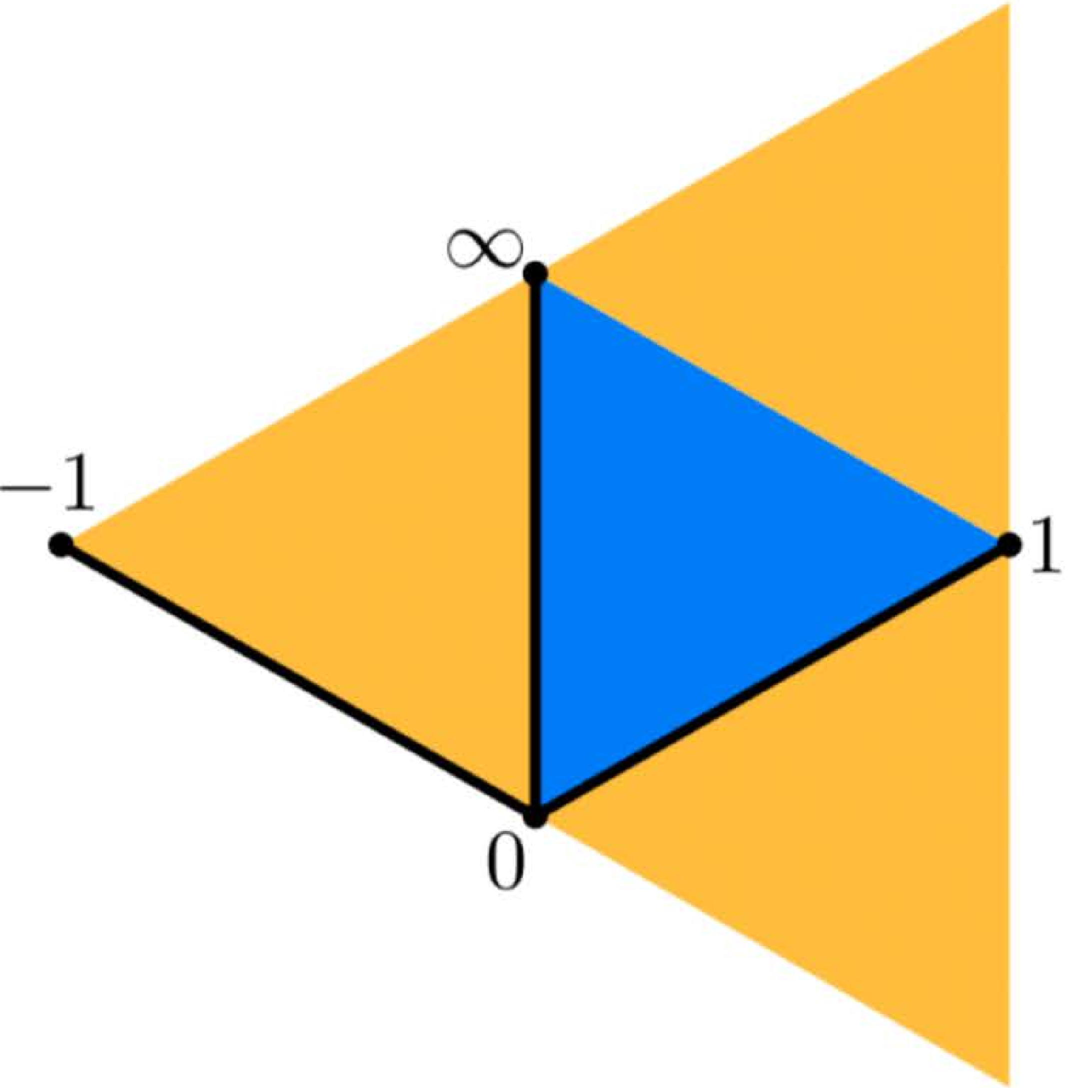


Normal generators for F_2/P_2



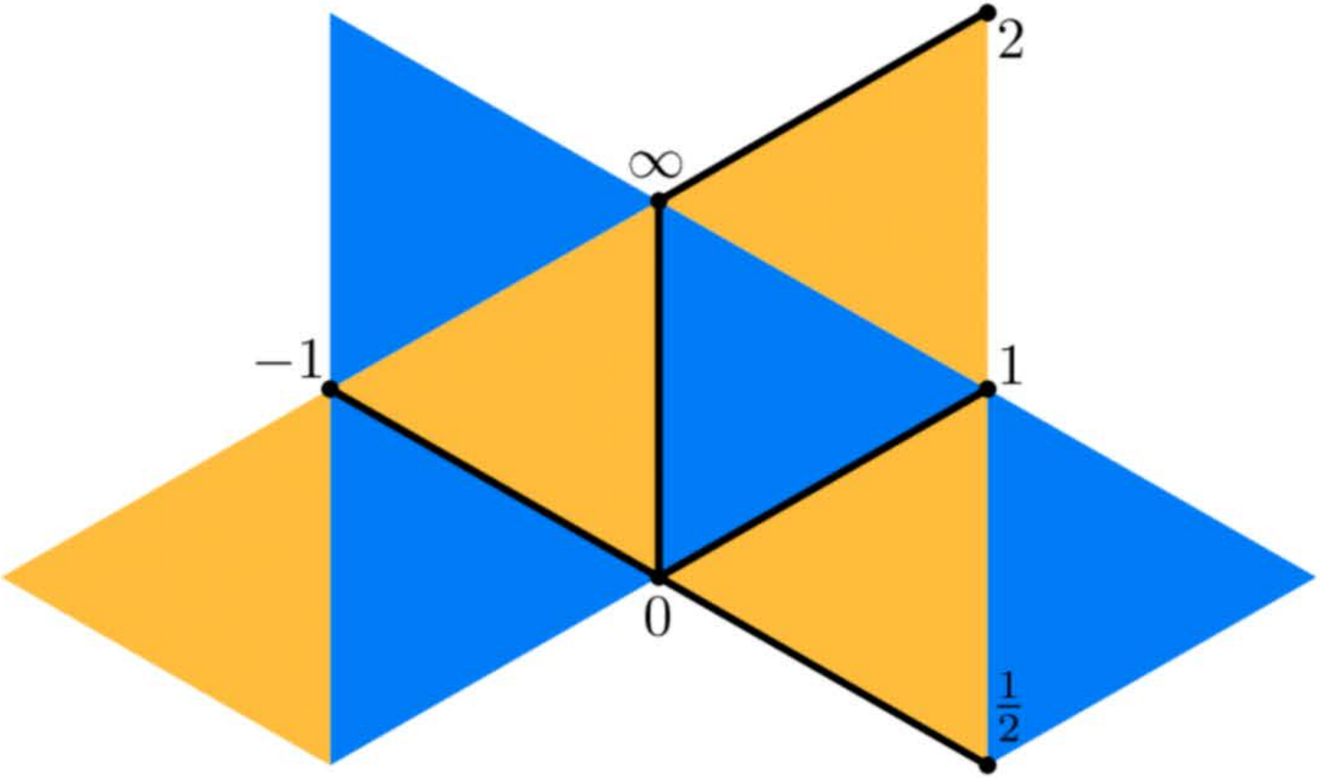
Vertex	Generator of P_2
∞	a^2
0	b^2
1	$(ab)^2$

Normal generators for F_2/P_3



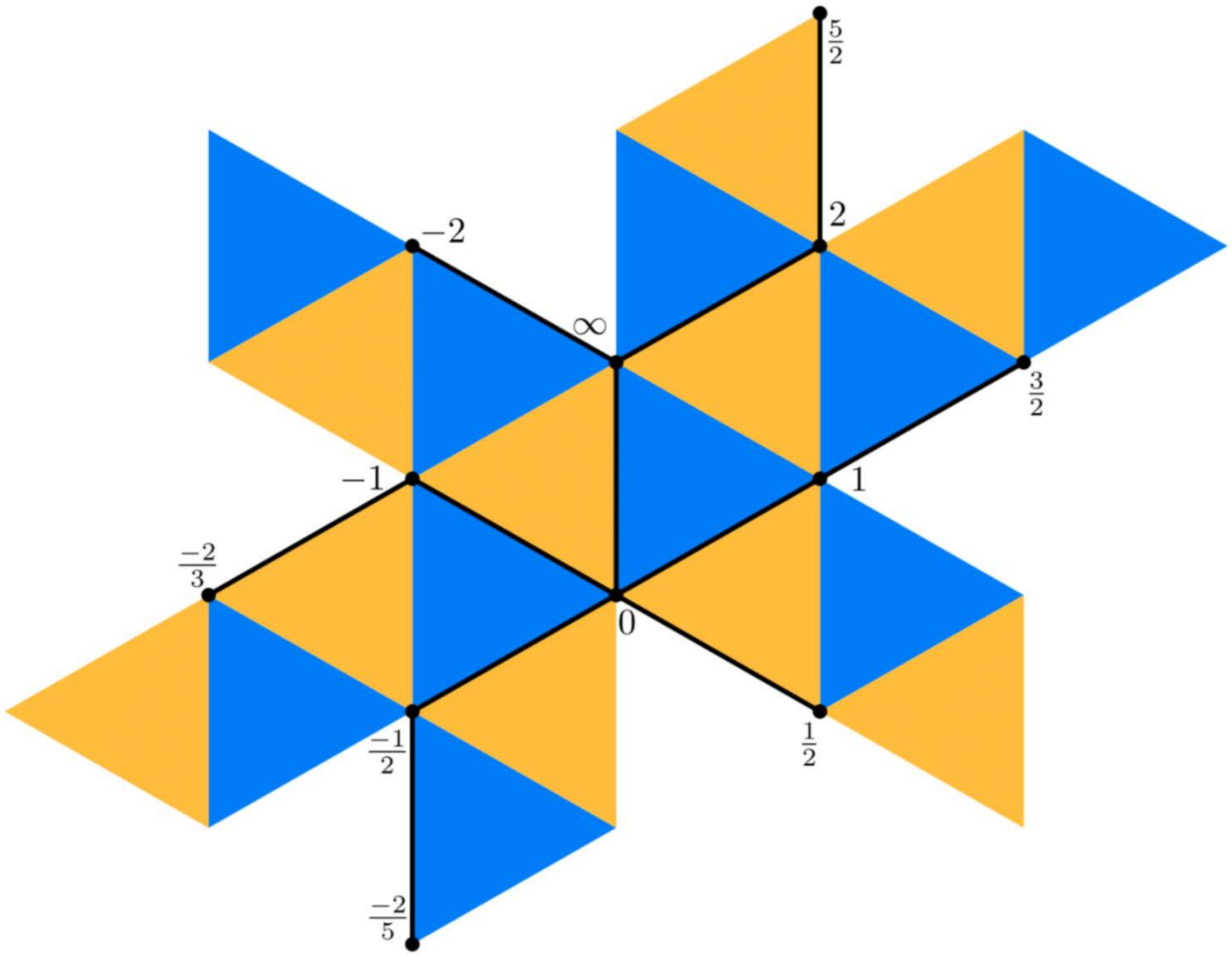
Vertex	Generator of P_3
∞	a^3
0	b^3
1	$(ab)^3$
-1	$(ab^{-1})^3$

Normal generators for F_2/P_4



Vertex	Generator of P_4
∞	a^4
0	b^4
1	$(ab)^4$
-1	$(ab^{-1})^4$
2	$(a^2b)^4$
$\frac{1}{2}$	$(ab^2)^4$

Normal generators for F_2/P_5



Vertex	Generator of P_5
∞	a^5
0	b^5
1	$(ab)^5$
-1	$(ab^{-1})^5$
2	$(a^2b)^5$
$\frac{1}{2}$	$(ab^2)^5$
-2	$(a^2b^{-1})^5$
$\frac{-1}{2}$	$(ab^{-2})^5$
$\frac{3}{2}$	$(a^2bab)^5$
$\frac{-2}{3}$	$(ab^{-1}ab^{-2})^5$
$\frac{5}{2}$	$(a^3ba^2b)^5$
$\frac{-2}{5}$	$(ab^{-2}ab^{-3})^5$

New notion

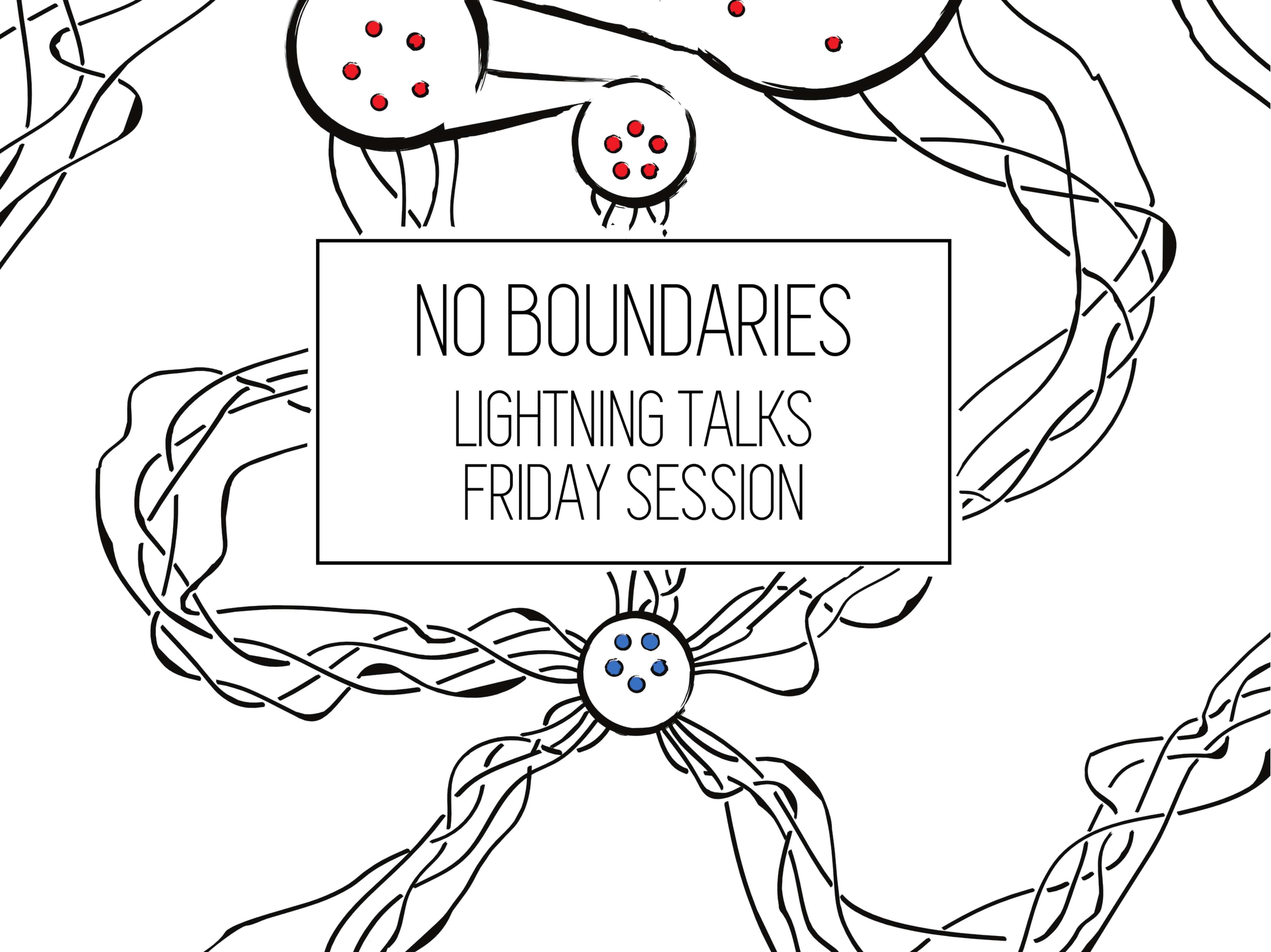
- A representation of F_2 is *characteristic* if for any automorphism ψ of F_2, there is an automorphism Ψ of $GL(n,\mathbb{C})$ so that $\Psi \circ \rho \circ \psi^{-1}(g) = \rho(g)$ for all $g \in F_2$.

- We say $\rho: F_2 \to GL(n,\mathbb{C})$ is an *oriented characteristic representation* if:
  - For each $\psi \in Aut_+(F_2)$ there is an $M \in GL(n,\mathbb{C})$ so that $M\, \rho \circ \psi^{-1}(g) M^{-1} = \rho(g)$ for all $g \in F_2$.
  - For each $\psi \in Aut_-(F_2)$ there is an $M \in GL(n,\mathbb{C})$ so that $M \cdot \overline{\rho \circ \psi^{-1}(g)} \cdot M^{-1} = \rho(g)$ for all $g \in F_2$.

Improvement scheme

- Assume $\rho: F_2 \to GL(n, \mathbb{C})$ is an oriented characteristic representation factoring through G_k. We produce an oriented characteristic representation $\tilde{\rho}: F_2 \to GL(n+m, \mathbb{C})$ factoring through G_k (hopefully with $m > 0$) so that there is a short exact sequence of the form $1 \to \mathbb{Z}^d \to \tilde{\rho}(F_2) \to \rho(F_2) \to 1$ where $d \geq 0$ is the rank of the abelian image $\tilde{\rho}(\ker \rho)$ (hopefully $d > 0$).

- Using this scheme we get an explicit faithful representation for F_2/P_4 and infinite representations for F_2/P_k for $k \geq 4$.
- What will this scheme give us for F_2/P_5? We are working on it.



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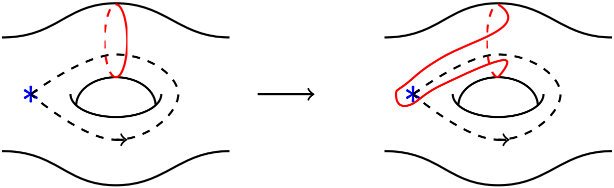
Algebraic Characterizations in the Mapping Class Group

Victoria Akin

An Example



The Point-Pushing Subgroup

$$1 \to P(S_g) \to \mathrm{Mod}(S_{g,*}) \to \mathrm{Mod}(S_g) \to 1$$

An Example

Algebraic Characterization

- Abstractly isomorphic to $\pi_1(S_g)$
- Normal in $\mathrm{Mod}(S_g)$

An Example

(Ivanov-McCarthy) $\mathrm{Out}(\mathrm{Mod}^{\pm}(S_{g,*}) \cong 1$

An Example

$$\text{(Ivanov-McCarthy) } \mathrm{Out}(\mathrm{Mod}^{\pm}(S_{g,*}) \cong 1$$

- Burnside:
  If a centerless group G is charcteristic in $\mathrm{Aut}(G)$, then $\mathrm{Aut}(\mathrm{Aut}(G)) \cong \mathrm{Aut}(G)$. That is, $\mathrm{Out}(Aut(G)) \cong 1$.

An Example

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- Dehn-Nielsen-Baer:
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- Dehn-Nielsen-Baer:
  $\mathrm{Aut}(\pi_1(S_g)) \cong \mathrm{Mod}^{\pm}(S_{g,*})$.

- Uniqueness of Point-Pushing:
  $\mathrm{Out}(\mathrm{Aut}(\pi_1(S_g))) \cong \mathrm{Out}(\mathrm{Mod}^{\pm}(S_{g,*})) \cong 1$.

In General

For $H < G$ geometrically/topologically defined, can we find a purely algebraic characterization?

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- Braid group?

$$1 \to \pi_1(\mathrm{Conf}_n(S_g)) \to \mathrm{Mod}(S_{g,n}) \to \mathrm{Mod}(\Sigma_g) \to 1$$

- Disk pushing?
- Handle pushing?

In General

What other normal/non-normal subgroups are unique?

In General

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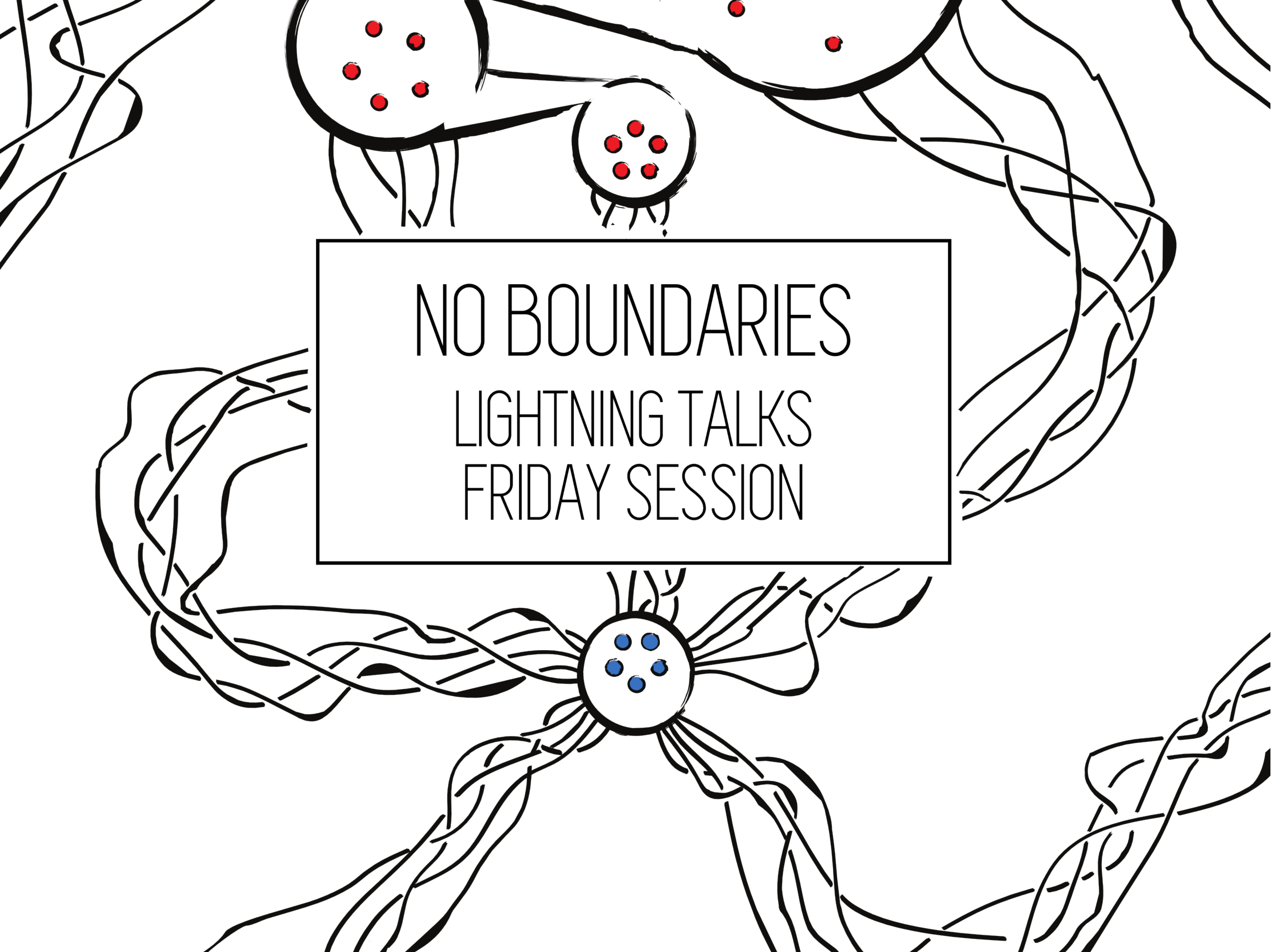
- (with D. Margalit) Torelli? Johnson Kernel? Higher terms in the Johnson Series?

Thank you



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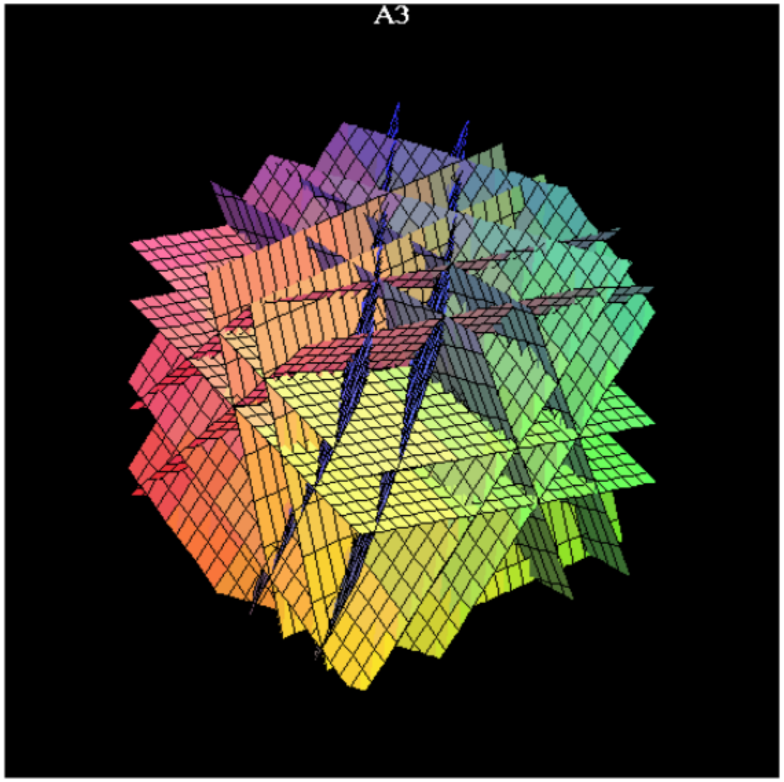
Representation stability for Finitely Generate Arrangements



No Boundaries
Oct 2017

Nir Gadish

Linear subspace arrangements

A collection $\bigcup_{i=1}^{n} L_i \subset \mathbb{C}^d$

linear subspaces

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Determines $M_{\mathcal{A}} = \mathbb{C}^d \setminus \bigcup_{i=1}^{n} L_i$

Linear subspace arrangements

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Determines $M_{\mathcal{A}} = \mathbb{C}^d \setminus \bigcup_{i=1}^{n} L_i$

Fundamental problem: compute $\mathrm{H}^*(M_{\mathcal{A}})$.

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Arno'ld, Orlik-Solomon, Goresky-MacPherson...

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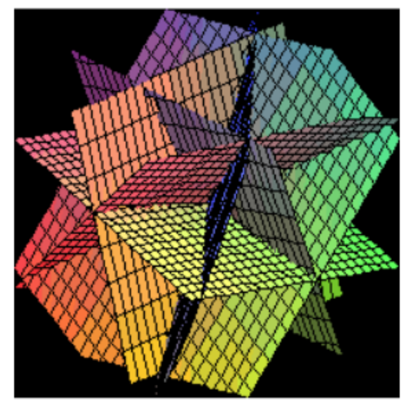
Arno'ld, Orlik-Solomon, Goresky-MacPherson...

(and Farb!)

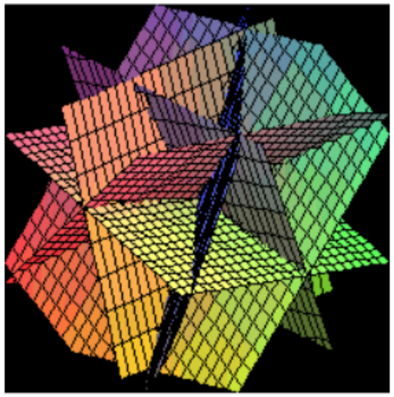
Examples



Examples

1) **Configurations:** $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{z_i = z_j\}$

"the braid arrangement".



Examples

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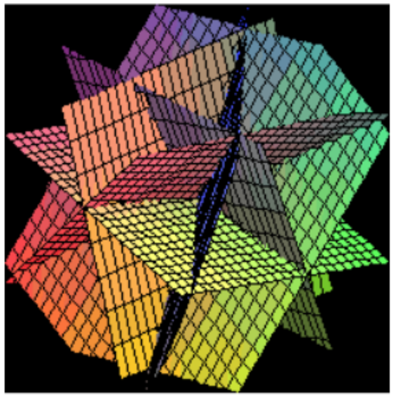
"the braid arrangement".

2) **Rational maps:** $$\underset{z_i \in}{\mathbb{C}^n} \times \underset{p_i \in}{\mathbb{C}^n} \setminus \bigcup_{i,j} \{z_i = p_j\}$$



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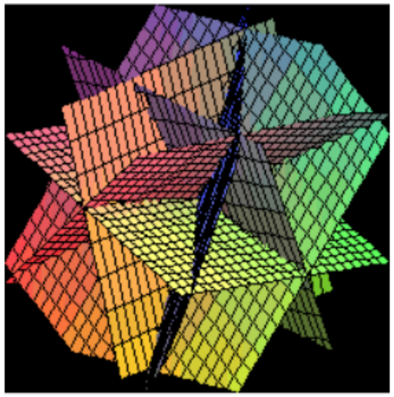
3) **Type B:** $\mathbb{C}^n \setminus \bigcup \{z_i = \pm z_j\}$



Examples

1) **Configurations:** $\mathbb{C}^n \setminus \bigcup_{i\neq j} \{z_i = z_j\} \circlearrowright S_n$

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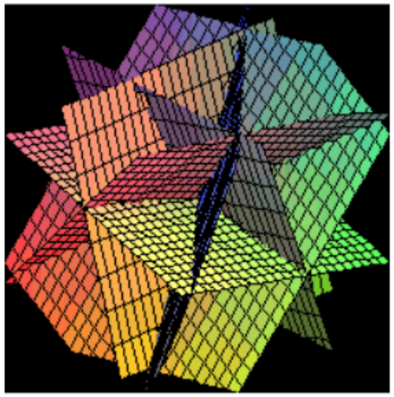
2) **Rational maps:** $\underset{z_i}{\mathbb{C}^n} \times \underset{p_i}{\mathbb{C}^n} \setminus \bigcup_{i,j} \{z_i = p_j\} \circlearrowright S_n \times S_n$

3) **Type B:** $\mathbb{C}^n \setminus \bigcup \{z_i = \pm z_j\} \circlearrowright S_n \ltimes \mathbb{Z}/2\mathbb{Z}^n$



Notice: (a) group actions!

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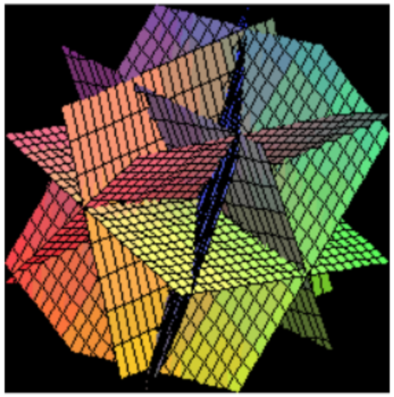

Notice: (a) group actions!

(b) come in families!

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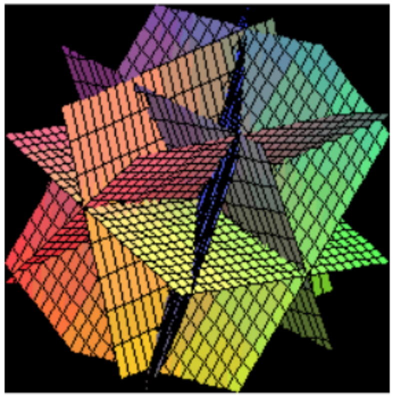

Goal: Understand $\mathrm{H}^*(M_{\mathcal{A}})$ in this context.

Mechanism: C-subspace arrangements

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Family = functor!

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Family = functor!

e.g. **FI** = **F**inite set and **I**njective functions.

$$\begin{array}{ccccccccccc} \{1\} & \to & \{1,2\} & \to & \{1,2,3\} & \to & \dots & \to & \{1,\dots,n\} & \to & \dots \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & & & \circlearrowleft & & \\ S_1 & & S_2 & & S_3 & & & & S_n & & \end{array}$$

Mechanism: C-subspace arrangements

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S_1 & & S_2 & & S_3 & & \downarrow & & S_n & & \\
 & & & & & & & & & & \\
\mathcal{A}_1 & \to & \mathcal{A}_2 & \to & \mathcal{A}_3 & \to & \dots & \to & \mathcal{A}_n & \to & \dots \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & & & \circlearrowleft & & \\
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$$

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One object! e.g. braid arrangements.

$$\mathbb{C}^{\bullet} \setminus \bigcup_{i \neq j} \{z_i = z_j\}$$

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$$\mathbb{C}^{\bullet} \setminus \bigcup_{i \neq j} \{z_i = z_j\}$$

only "one equation" (?)

Representation stability

Representation stability

Applying cohomology:

$$
\begin{array}{ccc}
 & & \mathbf{Arr} \\
 & \overset{\mathcal{A}_\bullet}{\nearrow} & \downarrow \mathrm{H}^i \\
\mathbf{FI} & \longrightarrow & \mathbf{Vect}
\end{array}
$$

Representation stability

Applying cohomology:

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get an **FI**-module - $[n] \mapsto \mathrm{H}^i(M_{\mathcal{A}_n})$.

Representation stability

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Theorem [G]: the ***C***-module $\mathrm{H}^*(M_{\mathcal{A}})$ of a finitely generated ***C***-arrangement exhibits representation stability.

Representation stability

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(a) Polynomial dimensions.

Representation stability

Applying cohomology:

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get an **FI**-module - $[n] \mapsto \mathrm{H}^i(M_{\mathcal{A}_n})$.

Theorem [G]: the ***C***-module $\mathrm{H}^*(M_{\mathcal{A}})$ of a finitely generated ***C***-arrangement exhibits representation stability.

(a) Polynomial dimensions.

(b) Polynomial characters.

Representation stability

Applying cohomology:

$$\begin{array}{ccc} & & \mathbf{Arr} \\ & \overset{\mathcal{A}_\bullet}{\nearrow} & \downarrow \mathrm{H}^i \\ \mathbf{FI} & \longrightarrow & \mathbf{Vect} \end{array}$$

get an **FI**-module - $[n] \mapsto \mathrm{H}^i(M_{\mathcal{A}_n})$.

Theorem [G]: the ***C***-module $\mathrm{H}^*(M_{\mathcal{A}})$ of a finitely generated ***C***-arrangement exhibits representation stability.

(a) Polynomial dimensions.

(b) Polynomial characters.

(c) Inductive description.

Concrete consequences

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1. Configuration space

$$\chi_{\mathrm{H}^2(\mathrm{PConf}^n(\mathbb{C}))} = 3\binom{X_1}{1} + \binom{X_1}{2}X_2 - \binom{X_2}{2} - X_4 + 2\binom{X_1}{3} - X_3$$

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$$\dim \mathrm{H}^3(\mathrm{PRat}^n(\mathbb{C})) = 12\binom{n}{2}\binom{n}{3} + 2n\binom{n}{3} + 3\binom{n}{2}\binom{n}{2}$$

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Applications

- SET-free sets [Harman].

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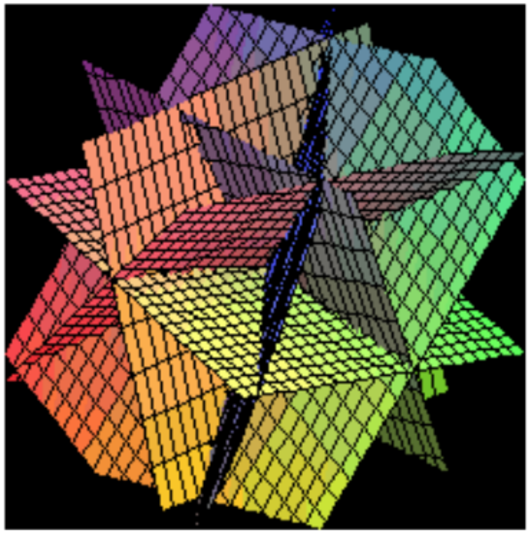
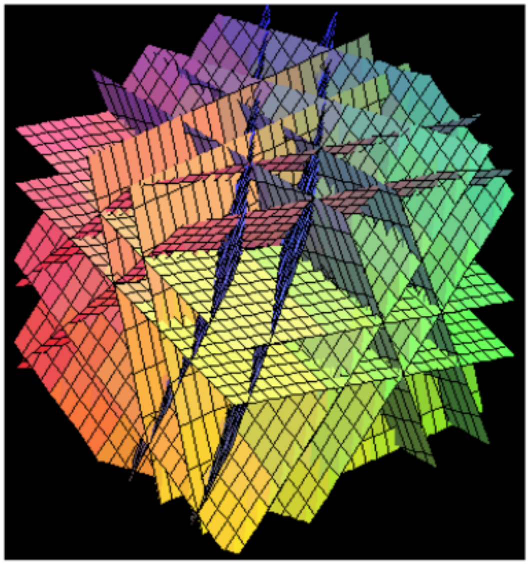
Applications

- SET-free sets [Harman].
- Arithmetic statistics of rational maps.

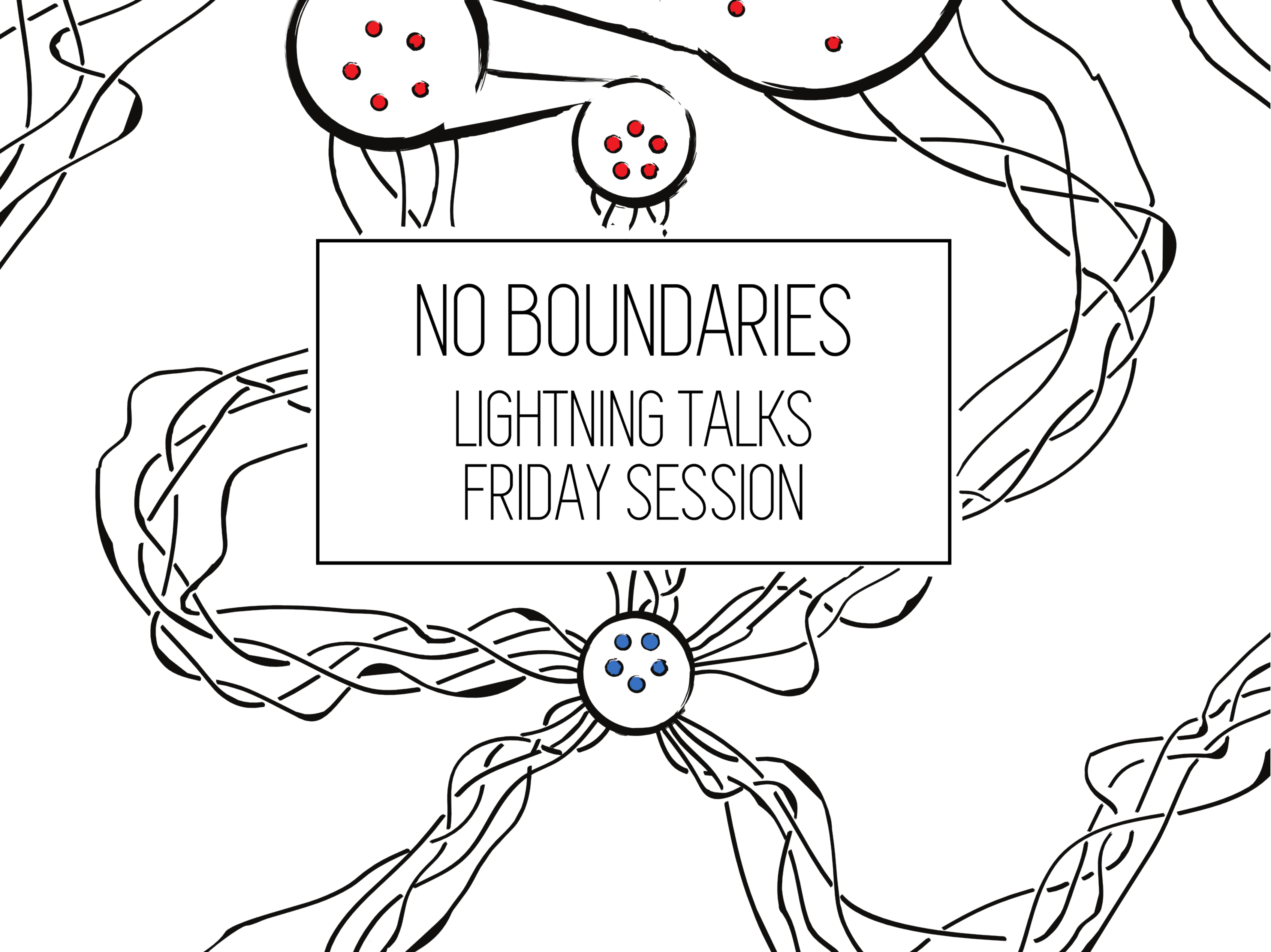
via Étale cohomology.

Thank you!





Any questions?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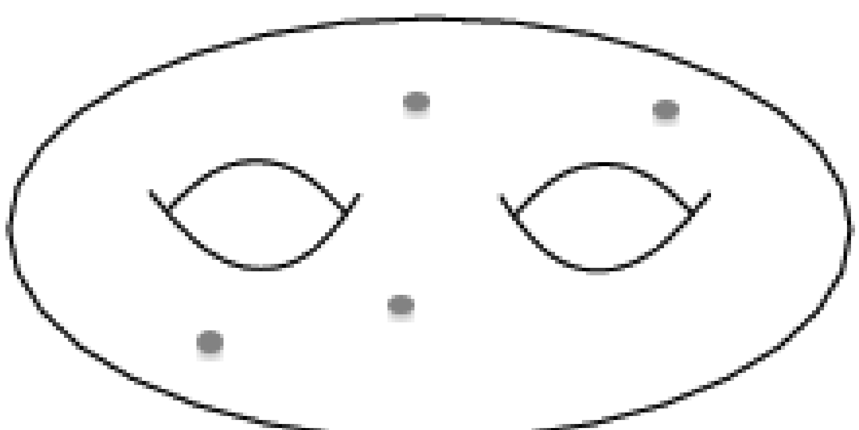
FRIDAY SESSION

Bounding the cohomology of conf guration spaces and rationality of Poincaré series

Kevin Casto

Configuration spaces

- $\mathrm{PConf}_n(M) = \{(m_i) \in M^n \mid m_i \neq m_j\}$
- $\mathrm{Conf}_n(M) = \mathrm{PConf}_n(M)/S_n$
- So $H^i(\mathrm{PConf}_n(M); \mathbb{Q})$ is an S_n-representation, and $H^i(\mathrm{PConf}_n(M))^{S_n} = H^i(\mathrm{Conf}_n(M))$



$\in \mathrm{Conf}_4(\Sigma_2)$

Representation stability

- Recall that irreps of S_n are parameterized by partitions: $\{S^\lambda \mid \lambda \vdash n\}$

- If $m \geq n + \lambda_1$, can extend to $\lambda[m] = (m-n, \lambda_1, \ldots, \lambda_k) \vdash m$

- Given $\{V_n\}$ with V_n an S_n-rep, satisfies *representation stability* [CF] if $\langle V_n, S^{\lambda[n]} \rangle_{S_n}$ is eventually constant

- Church [Ch] proved $H^i(\mathrm{PConf}_n(M))$ satisfies repr. stability for a "nice" manifold M.

- Taking the trivial rep, this means $H^i(\mathrm{Conf}_n(M))$ satisfies homological stability

What about varying *i* ?

- In applications, need to bound $\langle H^i(\mathrm{PConf}_n(X)), S^{\lambda[n]}\rangle$ as i varies

- *A priori*, rep stability doesn't help, since that's only about each fixed i

- **Theorem** ([Ca]). *For M "nice",*

$$|\langle H^i(\mathrm{PConf}_n(M)), S^{\lambda[n]}\rangle| \leq P(i)$$

where $P(i)$ is a polynomial independent of n

Poincaré series rationality

- Put

$$F_{M,\lambda}(x) = \sum_{i \geq 0} \langle H^i(\mathrm{PConf}(M)), S^{\lambda[n]} \rangle \, t^i$$

- Basic fact: if a power series is rational and has poles at roots of unity, its coefficients are a *quasipolynomial*
- Means there are poly's $p_0, \ldots, p_{d-1}$ s.t. $a_i = p_{i \bmod d}(i)$, so a_i bounded by a polynomial
- **Question:** Is $F_{M,\lambda}(x)$ always rational with poles roots of unity (for M nice)?

Partial results

- Question inspired by W. Chen [Che] – using work of [KL], showed answer is "yes" for $M = \mathbb{C}$ (explicit formula)
- Farb-Wolfson-Wood [FWW] prove answer is yes for the trivial rep ($\lambda = \emptyset$) if M is a conn. open submanifold of $\mathbb{R}^{2r}$
- In this case ($\lambda = \emptyset$) we are just looking at power series of stable Betti numbers of $\mathrm{Conf}_n(X)$
- Orlik-Solomon [OS] says that

$$H^*(\mathrm{PConf}_n(\mathbb{C})) = \Lambda^*\langle e_{ij}\rangle/(e_{ij}e_{jk} + e_{jk}e_{ik} + e_{ik}e_{ij})$$

If we don't quotient by ideal, calculations suggest analogous question for exterior algebra *fails*!

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[CF13] T. Church and B. Farb. Representation theory and homological stability. *Advances in Mathematics*, 245:250–314, 2013.

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[Chu12] T. Church. Homological stability for configuration spaces of manifolds. *Invent. Math.*, 188(2):465–504, 2012.

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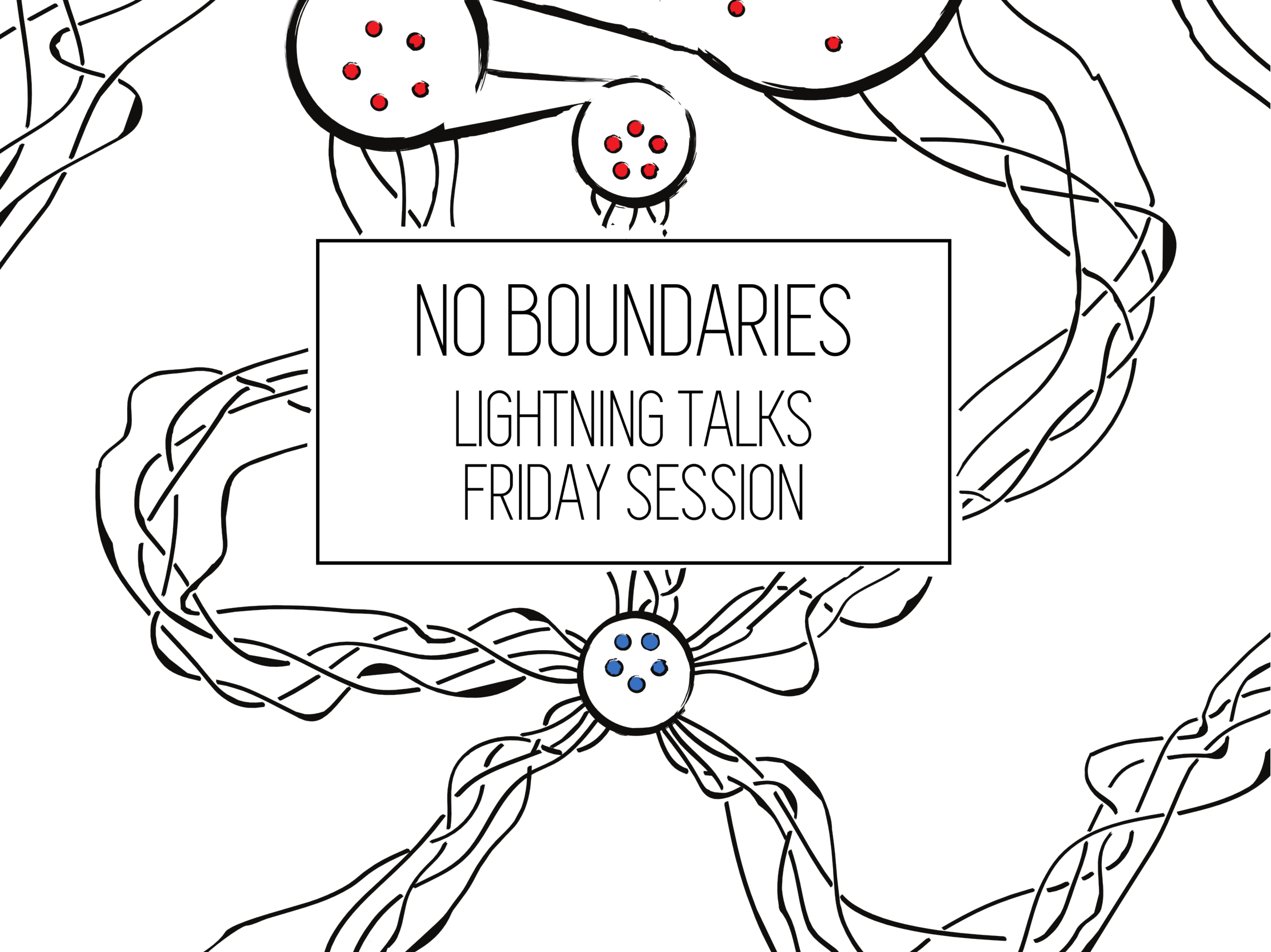
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FRIDAY SESSION

Coarse geometry of expanders from homogeneous spaces

Wouter van Limbeek

University of Michigan

27 Oct 2017

Joint work with D. Fisher and T. Nguyen

Discretizing group actions (Vigolo, '16)

- Γ f.g. group
- M closed Riem. manifold
- $\Gamma \curvearrowright M$ (bi-Lipschitz)

$\rightsquigarrow$

Family of graphs
$(X_t)_{t>0}$

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Action $\Gamma \curvearrowright M$

Mesh $< t^{-1}$

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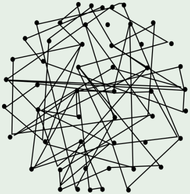
Graphs X_t



Vertices: Regions R_i

Edges: $sR_i \cap R_j \neq \emptyset$.

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Roe's Warped Cone

Assembles all X_t
$\rightsquigarrow \mathscr{C}(\Gamma \curvearrowright \mathrm{M})$.

Dynamics and coarse geometry

Dynamics of $\Gamma \curvearrowright M$

$\overset{?}{\leftrightsquigarrow}$

Coarse geometry of graphs $(X_t)_t$
Or Warped Cone $\mathscr{C}(\Gamma \curvearrowright \mathrm{M})$

Dynamics and coarse geometry

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Theorem (Vigolo,'16)

Spectral gap for $\Gamma \curvearrowright M$ $\Longrightarrow$ $(X_n)_n$ *expander.*

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$\underset{\Longleftarrow}{\textit{Sawicki}}$

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Subgroups of compact Lie groups $\rightsquigarrow$ Spectral gap

Margulis, Sullivan, Drinfeld, Gamburd–Jakobson–Sarnak, Bourgain–Gamburd ($\times 2$), Benoist-De Saxcé, ...

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From now on:

- $M = G$ compact semisimple Lie
- $\Gamma \subseteq G$ dense, fin. pres.

Theorems

Coarse geometry of cones $\overset{?}{\leftrightsquigarrow}$ Dynamics of $\Gamma \curvearrowright M$

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Coarse geometry of cones $\overset{?}{\longleftrightarrow}$ Dynamics of $\Gamma \curvearrowright M$

Theorem (De Laat–Vigolo, Sawicki '17)

Warped cones are QI $\Longrightarrow$ *Groups are Stably QI*

Theorems

Coarse geometry of cones $\overset{?}{\leftrightsquigarrow}$ Dynamics of $\Gamma \curvearrowright M$

Theorem (De Laat–Vigolo, Sawicki, '17)

Warped cones are QI $\Longrightarrow$ *Groups are Stably QI*

$$\mathscr{C}(\Gamma \curvearrowright \mathrm{M}) \simeq_{QI} \mathscr{C}(\Lambda \curvearrowright \mathrm{N}) \Longrightarrow \Gamma \times \mathbb{R}^{\dim M} \simeq_{QI} \Lambda \times \mathbb{R}^{\dim N}.$$

Theorems

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Does the QI type of the cone capture any of the action?

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Theorem (Fisher–Nguyen–vL, '17)

Warped cones are QI $\implies$ *actions are commensurable*

Theorems

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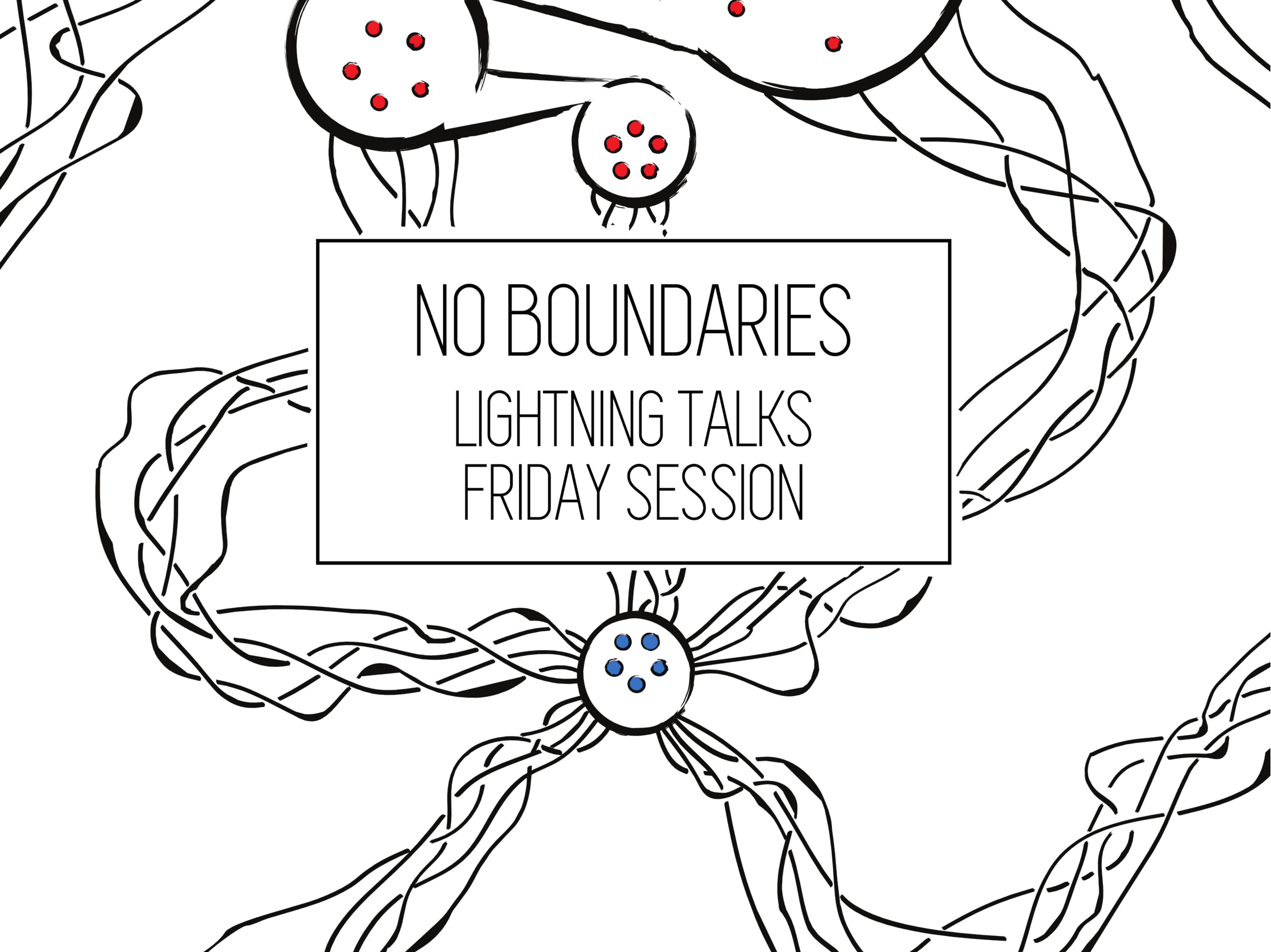
Warped cones are QI $\implies$ *actions are commensurable*

Similar result for graphs $\implies$

Theorem (Fisher–Nguyen–vL, '17)

There exist continua of QI disjoint expanders.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