The Grothendieck p-curvature conjecture – variations

No Boundaries

But: this talk will be all about boundar[ies]

Let $X/\mathbb{C}$ smooth, alg variety

(V, ∇) a vector bundle + integrable (flat) connection $\nabla : V \to V \otimes \Omega^1_{X/\mathbb{C}}$

$\mathrm{Ker}(\nabla)$ on $X(\mathbb{C}) \rightsquigarrow$ a local system $\to \int(V,\nabla) : \pi_1(x) \to GL_n(\mathbb{C})$

Eg. $X = \mathbb{G}_m = \mathbb{C}^* = \mathbb{C} - \{0\}$

$V = \mathcal{O}_X \ni z$ $\quad \nabla(f) = df - a\frac{dz}{z}$, $a \in \mathbb{C}$ $\quad \ker(\nabla) = z^a \cdot \mathbb{C}$

- $\int(V,\nabla) : \pi_1(\mathbb{G}_m) \to \mathbb{C}^*$

$\pi_1(\mathbb{G}_m) = \mathbb{Z} \ni \gamma \longmapsto e^{2\pi i a}$

- Suppose X, V, ∇ defined over a number field K

$\rightsquigarrow$ reduce mod p $\quad X_p, V_p, \nabla_p$

Eg. $a \in K$ $\quad \nabla(f) = df - a\frac{dz}{z}$ mod p

OK $\forall\forall p$

The p-curvature ψ_p $\quad \psi_p \equiv 0 \Leftrightarrow (V, \Delta)_p$ has a full set of algebraic solutions

If $D \in T_{X_p}$ $\quad \nabla(D) : V \to V \otimes \Omega^1_{X/\mathbb{C}} \xrightarrow{\langle\, , D \rangle} V$

$$\begin{array}{c} \| \\ D^{p}_{ia}(\mathcal{O}_X, \mathcal{O}_X) \end{array}$$

$D^p \in T_{X_p}$ $\quad \psi_p(D) := \nabla(D^p) - \nabla(D)^p \in \mathrm{End}_{\mathcal{O}_{X_p}} V_p$

Conj (Grothendieck): If $\psi_p \equiv 0$ $\forall\forall p \Rightarrow \rho_{(V,\nabla)}$ has finite image

$a \in K$ – number field

$\psi_p(z \frac{d}{dz})(1) = \bar{a} - \bar{a}^p \pmod p$

$\psi_p \equiv 0 \Leftrightarrow \bar{a} - \bar{a}^p = 0 \Leftrightarrow \bar{a} \in \mathbb{F}_p \Leftrightarrow$ In $\mathbb{Q}(a)$ almost all primes p split completely

$\Leftrightarrow a \in \mathbb{Q}$

$\Rightarrow e^{2\pi i a}$ – a root of 1

Thm (Katz) If $\psi_p \equiv 0$ $\forall\forall p$ then $\rho(V, \nabla)$ has finite local monodromy

In general, $X \hookrightarrow \bar{X}$ = compact st. $\bar{X} \setminus X$ is a normal crossing

The proof is similar to $\mathbb{G}_m$ case one shows (V, ∇) has regular singular points.

Cor ~~Thm~~ (Farb-K) Let $\mathcal{A}_g$ the moduli space of principally polarized abelian var's. If $g \geq 2$ the conjecture holds for $\mathcal{A}_g$.

More general: Suppose $X = \Gamma \backslash {}^{G(\mathbb{R})}/K_\infty$ locally symmetric with $G(\mathbb{R})/K_\infty$ Hermitian symmetric, $\Gamma \subseteq G(\mathbb{R})$ arithmeti

Thm (Farb-K) Suppose that G is simple.
If Either i) G has $\mathbb{R}$-rank ≥ 2 and is classical (A, B, C, D)
or ii) ______ and $\mathbb{Q}$-rank ≥ 1
Then the conjecture holds for X.

Pf of (ii): • X has toroidal compactifications $X \hookrightarrow \overline{X}$
$\overline{X} \backslash X \underset{\text{incl}}{\subseteq} \overline{X}$

• $rk_{\mathbb{Q}}\, G \geq 1 \Rightarrow \overline{X} \backslash X$ nonempty

• a loop around a boundary component is given by a unipotent $1 \neq \gamma \in \Gamma = \pi_1(X)$

$\rho(v, \Delta): \pi_1(X) \longrightarrow GL_n(\mathbb{C})$

By Katz, $\rho(v, \nabla)(\gamma)$ has finite order
$\Rightarrow \rho(v, \nabla)(\gamma^i) = 1$ for some $i > 0$

$\Rightarrow \ker \rho(v, \Delta)$ is infinite $\Rightarrow$ ker has finite index $\Rightarrow$ conj

Change setup: Suppose $X/\mathbb{C}$ is a closed complex curve
which is generic

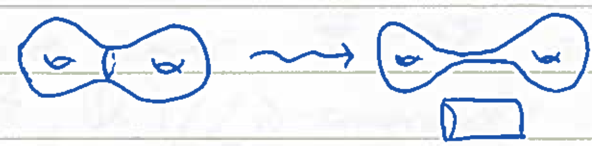
- $\operatorname{Spec} \mathbb{C} \longrightarrow \mathcal{M}_g/\mathbb{Q}$
  has Zariski dense image

or,

- Field of definition K of $\mathbb{C}$ satisfies
  $\operatorname{tr\,deg} K/\mathbb{Q} \geq \dim \mathcal{M}_g$

Thm (Ananth Shankar): Suppose (V,∇) or $X/\mathbb{C}$ satisfies $\psi_p \equiv 0$ $\forall\forall p$. If $\gamma \subseteq X$ a simple closed curve then $\rho_{(V,\nabla)}(\gamma)$ has finite order.

Idea: $\gamma \subseteq X$



Q: Does this imply $\operatorname{Im} \rho_{(V,\nabla)}$ is finite?

Thm (Koberda-Santharoubane) No.