Rigidity implies geometricity for surface group representations

Kathryn Mann
Brown University
&
Maxime Wolff
Inst. Math. Jussieu

Rigidity

Γ discrete group (e.g. $\pi_1(\Sigma_g) = \Gamma_g$), G topological group
Study representations $\rho : \Gamma \to G$.
think: G linear (rep. theory) or $G = \mathrm{Homeo}(M)$, $\mathrm{Diff}(M)$ (dynamics)

Definition: $\rho : \Gamma \to G$ is *rigid* if "only trivial deformations"
$\rho \in \mathrm{Hom}(\Gamma, G)/G$ is an isolated point.

Problem: quotient space typically not Hausdorff
e.g. $\mathrm{Hom}(\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))/\,\mathrm{SL}(2,\mathbb{C}) \leftrightarrow$ trace *except* $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution: Define "character space"
$X(\Gamma, G) :=$ largest Hausdorff quotient of $\mathrm{Hom}(\Gamma, G)/G$
for $\mathrm{SL}(n,\mathbb{C})$ this *is* characters; for G complex, reductive Lie group, it is GIT quotient

Change definition: Rigid means isolated point in $X(\Gamma, G)$.

Rigidity from geometry

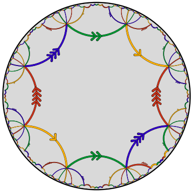
Mostow rigidity (Calabi): $\Gamma = \pi_1(M^n)$ hyperbolic manifold
$\Gamma \to SO(n,1)$ embedding as cocompact lattice is rigid in $X(\Gamma, SO(n,1))$

Analog in non-linear setting ?

Definition: $\rho : \Gamma \to \text{Homeo}(M)$ is *geometric* if factors through

$$\underset{\text{cocompact lattice}}{\Gamma} \hookrightarrow \underset{\text{transitive Lie group}}{G} \hookrightarrow \text{Homeo}(M)$$

Example 1. $\pi_1(\Sigma_g) \to PSL(2,\mathbb{R}) \to \text{Homeo}(S^1)$



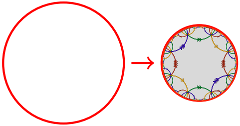
Theorem (Matsumoto '87)
The example above is *rigid* in $X(\pi_1(\Sigma_g), \text{Homeo}(S^1))$.

Geometric reps to Homeo(S^1)

Fact: Connected, transitive Lie groups in Homeo(S^1) are

- SO(2)
- finite cyclic extensions of PSL$(2, \mathbb{R})$

$\mathbb{Z}/k\mathbb{Z} \to G \to \mathrm{PSL}(2, \mathbb{R})$



Cor.: can describe all geometric actions of $\pi_1(\Sigma_g) = \Gamma_g$ on S^1.

(lifts of Fuchsian actions)

Theorem (Mann, 2014)

If $\rho : \Gamma_g \to \mathrm{Homeo}(S^1)$ is geometric, then it is rigid.

Theorem (Mann–Wolff, 2017)

Converse: if $\rho \in X(\Gamma_g, \mathrm{Homeo}_+(S^1))$ is rigid, then it is *geometric.*

Plan:

1. What is $X(\Gamma_g, \mathrm{Homeo}_+(S^1))$?
2. Idea of proof for rigid $\Rightarrow$ geometric.

What is $X(\Gamma_g, \text{Homeo}_+(S^1))$?

- Space of *flat (foliated), topological S^1 bundles over Σ_g*

- Points are *semi-conjugacy classes* of actions

- Parametrized by *rotation numbers* of elements.

analog of trace coordinates for $X(\Gamma, \text{SL}(2, \mathbb{R}))$

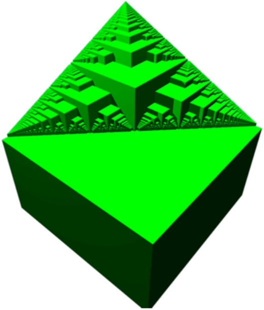
- Topologically... complicated

Not known:

- Finitely many connected components?

- How different from $X(\Gamma_g, \text{Diff}_+(S^1))$?

  (see work of J. Bowden)

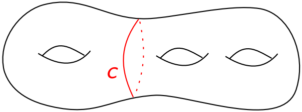


Calegari–Walker
Ziggurats and Rotation numbers

Proof ideas for “Rigid $\Rightarrow$ Geometric”

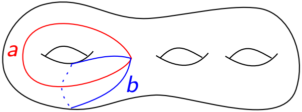
Dynamical lemma: ρ rigid $\Rightarrow \rho(\gamma)$ has rational rotation number for every simple closed curve γ.

Key tool: *Bending deformations* works in $\mathrm{Hom}(\Gamma_g, G)$ for any G



$\Gamma_g = A *_{\langle c \rangle} B$

Bending ρ along c:
take c_t commuting with $\rho(c)$.
Define $\rho_t = c_t \rho c_t^{-1}$ on B,
$\rho_t = \rho$ on A.



$\Gamma_g = F *_{\langle a \rangle}$

Bending ρ along a: similar, define $\rho_t(b) = a_t \rho(b)$.

if $a_1 = \rho(a)$, like Dehn twist

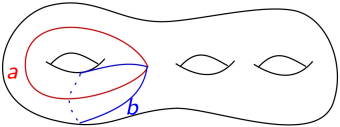
Headaches: • based curves. • centralizers. • 1-parameter subgroups.

Proof ideas for "Rigid $\Rightarrow$ Geometric"

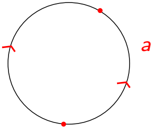
Main idea: $\rho(\gamma)$ has periodic points (lemma), so
take bending ρ_t and study movement of periodic points of $\rho_t(\gamma)$.

ρ rigid $\Rightarrow$ combinatorial structure of $\mathrm{Per}(\rho_t(a))$, $\mathrm{Per}(\rho_t(b))$
"won't change" e.g. having common point, cyclic order of points
From this, "reconstruct" the structure of geom. rep.

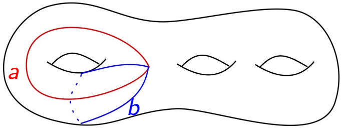
Baby version of main idea



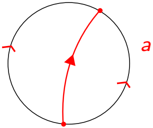
Suppose $\rho(a)$ and $\rho(b)$ have hyperbolic dynamics:



Baby version of main idea

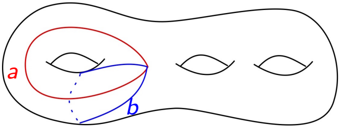


Suppose $\rho(a)$ and $\rho(b)$ have hyperbolic dynamics:

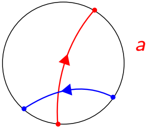


Claim: ρ rigid $\Rightarrow$ axes cross.

Baby version of main idea

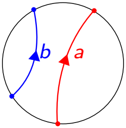


Suppose $\rho(a)$ and $\rho(b)$ have hyperbolic dynamics:



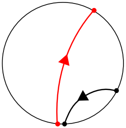
Claim: ρ rigid $\Rightarrow$ axes cross. *"reconstruct topology of Σ_g"*

Proof: Suppose

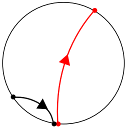


Bending: $\rho_t(b) = a_t\rho(b)$
$\rho_t(a) = a$

Picture: axis of $a^{-N}\rho(b)$ for $N >> 0$:



or



repelling point near $\rho(b)^{-1}(a_+)$

deformation gives non-conjugate picture, contradiction □

In real life...

This line of argument “works” if $|\mathrm{Per}(\rho(a))| < \infty$.

- “axes” of SCC’s “intersect” only when (based) curves do.
- w/ combinatorial technique of Matsumoto (2015), get geometricity.

Much work to arrive at deformation so that $|\mathrm{Per}(\rho(a))| < \infty$,
build machinery to modify and track combinatorics of periodic sets.

Many open questions remain...

Thanks!