Ergodic Theory of Interval Exchange Transformations

Joint with Jon Chaika

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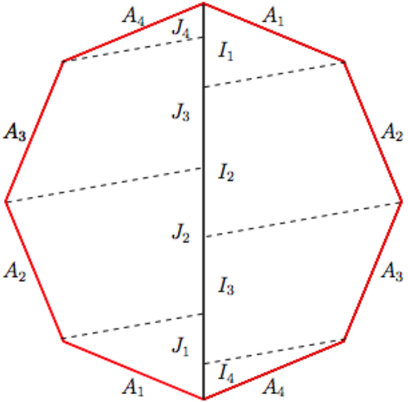
An interval exchange transformation on d intervals is a bijection $T : [0, 1) \to [0, 1)$ given by cutting up $[0, 1)$ into d subintervals and then rearranging the intervals by a translation on each interval.

Historically IET arose in the study of billiard flows in polygons with angles rational multiple of π. and more generally *translation surfaces*.



Vary θ you get 1- parameter family of IET.

The endpoints of intervals are discontinuities of T.

The classical example is when $d = 2$ and we have a number $0 < \lambda < 1$. The map is

$$T(x) = x + \lambda \ (mod 1)$$

so $T[0, 1 - \lambda) = [\lambda, 1)$ and $T[1 - \lambda, 1) = [0, \lambda)$.

Identify $[0, 1)$ with unit circle via $x \to e^{2\pi i x}$. Then T is rotation of circle by angle $2\pi\lambda$.

In the classical case of a torus the first return is IET with $d = 2$.
Kroneker-Weyl Theorem says that

- If $\lambda = \frac{p}{q}$ then for every point x, the orbit is periodic; $T^{(q)}(x) = x$.
- if λ irrational then every orbit $T^{(n)}(x)$ is dense (minimality) and equidistributed on $[0, 1)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^{(j)}(x)) = \int_0^1 f(t)dt$$

for *every* x and f continuous.

This says *all* orbits behave the same way

This last notion is called *unique ergodicity*

Ergodicity says that orbit of *generic* point satisfies this equation.

For example: geodesic flow on closed surface constant negative curvature is ergodic. There are equidistributed orbits, closed orbits, and lots of different behavior.

The flow is not uniquely ergodic

Unique ergodicity is equivalent to T having a unique invariant probability measure.

In case of IET the measure is Lebesgue. .

Question: Does minimality imply unique ergodicity?

Answer: No

In the context of IET and billiards first counterexample due to Veech (1970)

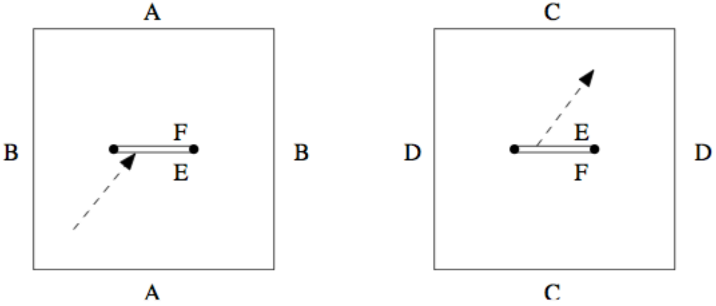


Figure: Slit torus

For some lengths of side E and directions θ of the flow, the flow in direction θ is minimal and not uniquely ergodic.

Construction of IET that are not uniquely ergodic I will describe today essentially originates from Keane(1973)

Other examples were constructed by Satayev (1973)

More recently, examples are constructed using topological methods, limits of simple closed curves and Thurston's sphere of projective measured foliations. (Gabai, Leininger,Modami, Lenzhen, Rafi, Brock).

HOW COMMON ARE NON UNIQUELY ERGODIC IET?

Fix an irreducible permutation π on d letters. The set of IET with permutation π is parametrized by the standard simplex

$$\Delta_d = \{\vec{\lambda} : \sum_{i=1}^{d} \lambda_i = 1\}.$$

.

If the orbit of a point of discontinuity does not hit another discontinuity then, every orbit is dense.(minimality)

This holds except only on a set of positive codimension so we ignore the set of IET which are not minimal.

Keane conjectured that in Δ_d the set of IET that are minimal but not uniquely ergodic should have 0 Lebesgue measure. This was proved by myself and independently, Veech (1982).

Fix a rational billiard table or more generally a translation surface. S.Kerckhoff, M, J. Smillie (1986) showed that the set of directions $\theta \in [0, 2\pi)$ such that the billiard flow in direction θ (or corresponding IET) is not uniquely ergodic has Lebesgue measure 0.

For some billiard tables (translation surfaces) if a direction is minimal it must be uniquely ergodic. Examples are lattice translation surfaces.

Example of lattice surface are billiards in regular *n*-gon

(Veech 1990) For a lattice surface; for any direction θ, the flow in direction θ is either completely periodic or minimal and uniquely ergodic.

HAUSDORFF DIMENSION

Here again we are working with the $d-1$ dimensional simplex Δ_d which parametrizes all IET with d intervals. Denote NUE as non uniquely ergodic IET.

Smillie and M(1991) showed that if $d \geq 4$ then for each π there is a constant $0 < \delta < 1$ such that $HDim(NUE) = d - 1 - \delta$.

Question: What is value of δ?

Theorem

(joint with Jon Chaika)
Suppose $d \geq 4$. Let π be a permutation in the Rauzy class of the symmetric permutation $(d, d-1, \ldots, 1)$. Then $HDim(NUE) = d - 1 - \frac{1}{2}$.

For $d = 4$ proved previously by Athreya-Chaika. (There is only one Rauzy class)

I want to focus on LOWER bound that the Hausdorff dimension is at least $d - 1 - \frac{1}{2}$. This means constructing large sets of non uniquely ergodic IET. The method we used is called Rauzy induction. (Upper bound proved previously myself using Teichmüller dynamics)

Rauzy induction $\mathcal{R}$ is map from Δ_d with a permutation to Δ_d with a possibly different permutation. It is a renormalization procedure which when iterated tells you about the dynamics of the given IET. (It is a different iteration than iterating T by forming $T^{(n)}$). Rauzy induction is one of the standard tools in the subject of IET, Teichmüller dynamics.

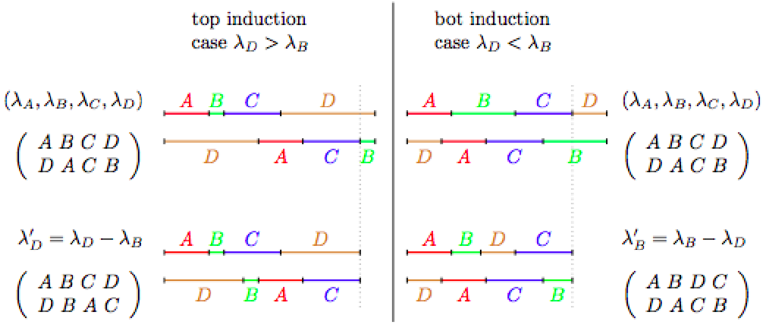


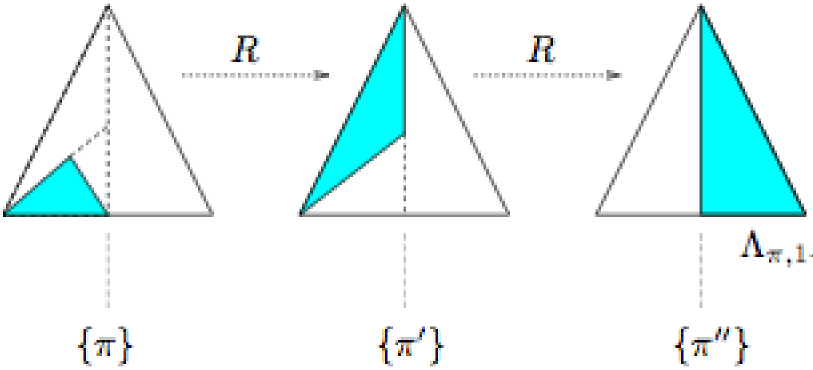
FIGURE 8. The two cases of the Rauzy induction for interval exchange transformation.

Figure: Rauzy induction

The permutations are in same Rauzy class

In case $d = 2$ where only two intervals I_1 and I_2 compete, the number of times one interval say I_1 beats I_2 in a row is determined by continued fraction expansion of λ.

Repeating Rauzy induction puts more and more restrictions on IET.



It is a standard fact that an IET is uniquely ergodic iff the nested sequence of simplices it determines converges down just to that point (and not a positive dimension simplex).

GOAL: To prove our theorem: make choices of winners and losers in Rauzy induction to build lots of sequences of nested simplices whose infinite intersections are larger than a point. By lots I mean so we get big Hausdorff dimension.

Associated to $\mathcal{R}$ is a $d \times d$ elementary (projective) matrix

$$A : \Delta_d \to \Delta_d.$$

It has 1 on diagonal. In first case ($\lambda_m < \lambda_j$) it has 1 in $[m, j]$ place and 0 elsewhere.

In second case it has 1 in $[j, m]$ place

We can repeat Rauzy induction with new IET T' and get 2 new possible matrices depending on who wins. After doing Rauzy induction n times we get matrices $A_1, \ldots, A_n$, a new IET $\mathcal{R}^n(T)$, new lengths $\vec{\lambda}_n$,

We take product of matrices

$$M_n = A_1 A_2 \cdots A_n : \Delta_d \to \Delta_d.$$

We think of M_n two ways. On one hand it relates lengths of $\mathcal{R}^n(T)$ to original T by

$$\vec{\lambda} = M_n \vec{\lambda}_n$$

The second way is that the j^{th} column of M_n records the number of visits of j^{th} interval of $\mathcal{R}^n(T)$ to the intervals of T.

In going from M_{n-1} to M_n by multiplying on right by A_n, say with I_m beating I_j we add m^{th} column of M_{n-1} to j^{th} column of M_{n-1} to get M_n.

As we have said $M_n\Delta_d \subset \Delta_d$ is a decreasing sequence of simplices and T is uniquely ergodic iff $\cap_{n=1}^{\infty} M_n\Delta$ is a single point

We can see that by saying that if it converges to a point then the columns are projectively almost the same. Since they are given by visitations, this says all points visit the original intervals with the same distribution .

This gives (another) proof of Weyl Theorem holds ($d = 2$). There are only 2 columns. One column is added to the other a number of times and then the second is added to the first.

An essentially trivial fact: you have vectors v_1, v_2 in $\mathbb{R}^d$ and $|v_1| \geq |v_2|$. Then v_1 and $v_1 + v_2$ are closer in angle than v_1 and v_2 by a definite factor. For $d = 2$ it happens infinitely often that a large column is added to a smaller one. It follows that the 1-dimensional simplicies converge to a point.

If you want to build non uniquely ergodic IET for $d \geq 4$ you want at least one pair of columns to be projectively different as $n \to \infty$. In our case the first $d - 2$ columns will converge projectively to one single limit, (the angles between them goes to 0) but $d - 1$ and d columns to a different limit.

Here is a rough idea of how we do this. The major point is that at any step you have a choice of which interval you want to win.

All columns are ultimately added to all others, but as stated in the last slide we cannot add a column C_i; $i \leq d - 2$ to C_{d-1} or C_d column when the first $d - 2$ columns are larger than the last two and vice versa.

We will repeat infinitely often the following steps

1. The first $d-2$ columns start much smaller than the last two. We act as if we have IET on $d-2$ intervals and do any Rauzy induction we please on these $d-2$ intervals, except that we beat the last two intervals when in conflict. This we call freedom.
2. At a moment in this process we need the first $d-2$ columns to start to become large compared to the last two columns. Then for a period of Rauzy induction they are only added to each other which means they never even compete with the last two intervals. This is called restriction. At end of this sequence of restriction, the first $d-2$ columns are now much bigger than the last two
   Morally we have a genus $g-1$ surface glued to a torus and they barely interact.
3. We repeat the previous two steps starting now with the last two columns being much smaller.

Restriction drastically cuts the measure of the collection of IET we build (as it must since at end we have measure 0).
The way our permutations work during restriction is that the first interval I_1 always loses so it cuts down measure, but we have enough control to give Hausdorff dimension.

Where does 1/2 in Theorem come from in our case?

Somewhat analogous statement. The set of reals in their continued fraction expansion $x = [a_0, a_1 \ldots]$ that satisfy $a_n \to \infty$ has dimension 1/2

Here is a snapshot of one thing we prove. We cut Δ_d into a fixed family of parallel planes P and along a subsequence of steps k of Rauzy induction, intersect all our subsimplicies with planes P to give exponentially (in k) many disjoint polygons $\Delta_k^j \subset P$ each with diameter r_k such that

- For most planes P the number n_k of polygons Δ_k^j satisfies for all $\epsilon > 0$

$$\lim_{k \to \infty} n_k r_k^{3/2-\epsilon} = \infty.$$

- If $\ldots \Delta_{k+2}^j \subset \Delta_{k+1}^j \subset \Delta_k^j \ldots \subset \Delta_1^j$ is a nested sequence of polygons then $\cap_{k=1}^{\infty} \Delta_k^j \cap P \subset NUE$

With some other things one needs to prove such as estimates on distance between disjoint polygons, estimates on how r_k decays in k, then one can put a Borel measure μ on a positive measure set of planes P such that

- $\mu(P \cap NUE) > 0$
- $\mu(B(x, r)) < r^{3/2}$.

Use Frostman's Lemma to say

$$HDim(P \cap NUE) \geq 3/2 = 2 - 1/2$$

for a positive measure set of planes. One promotes this to the whole simplex. That is where loss of 1/2 comes from.

THANK YOU

HAPPY BIRTHDAY BENSON!