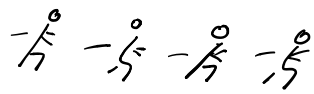
Lee Mosher : Hyperbolic actions of Out F_n and its subgroups

How to proof (via Benson)


. . . .

Proper actions of G on spaces $\rightsquigarrow$ (Gromov) δ-hyperbolic.

Gromov suggested notion of rel. hyperbolic. $\rightsquigarrow$ not developed fully.
"cusped mfld."

Farb (thesis) : Careful treatment of relative hyperbolicity.
(weak hyperbolicity).

2 notions :

(Strong) relative hyperbolicity

weak relative hyperbolicity. $\rightsquigarrow$ Gave rise to "islands" of fruitful areas of research.

"defn?" $\exists\, G \curvearrowright X$ cocompact
where X: conn.
simp. conn.
δ-hyp. w/ simplicial metric

Gps acting on trees
(Bass-Serre theory)

Acylindrically
hyperbolic

Hierarchically
hyperbolic. (more today).

$|\mathrm{stab}(e)| < \infty$
$\Rightarrow$ strong rel. hyp.

. . . first gp. Lee + Benson worked w/.

Soluble Baumslag-Solitar group

B-S tree

$$BS(1,n) = \langle a, t \mid tat^{-1} = a^n \rangle \curvearrowright T_{n+1}$$

$$= \mathbb{Z}[\tfrac{1}{n}] \rtimes \mathbb{Z}$$

matrix rep^{n}

$$\begin{bmatrix} n^k & \mathbb{Z}[\tfrac{1}{n}] \\ & n^{-k} \end{bmatrix}$$

2 natural embeddings

$\xhookrightarrow{①} SL(2, \mathbb{Q}_n) \curvearrowright T_{n+1}$ n-adic rat'l's

$\xhookrightarrow{②} SL(2, \mathbb{R}) \curvearrowright \mathbb{H}^2$.

Both yield highly improper actions.

?

Galois conjugation action.

Galois "galaxy"

\# see NumberTheorists....

However,

$BS(1,n) \curvearrowright T_{n+1} \times \mathbb{H}^2$

is proper.!

further acts cocompactly on a subset

$\Gamma < Isom(\mathbb{H}^2 \times \mathbb{H}^2) \curvearrowright \mathbb{H}^2 \times \mathbb{H}^2$

cocompact lattice $\rightsquigarrow$ induces actions on each $\mathbb{H}^2$-factor

Q (Thurston): is Γ automatic ??

$\rightsquigarrow$ Towards first HHG.

Masur-Minsky (mid 1990's) : S: fin. type

$MCG(S)$ = mapping class group. $\mathcal{F}$: { isotopy classes of essential, connected, cosubsurfaces }

$F \subset \mathcal{F}$ $\qquad CF$ = curve cplx. of of F

Masur-Minsky: $\mathcal{C}(F)$ are all Gromov hyperbolic.

$\mathcal{G} = \coprod \mathcal{C}(F)$ admits action by $MCG(S)$

Masur-Minsky: Action is "proper" (weak hyperbolicity).

also $\exists$ invariant core $\underset{QI}{\sim} MCG(S)$

w/ truncated ℓ^1-metric (cutoff function).

Quasi-isometric rigidity

→ Behrstock, Kleiner, Minsky, Mosher | Hamenstadt

→ H_b^2-alternative: either $G^{fg} < MCG(S)$

is (1) virtually abelian

(2) H_b^2 contains a copy of ℓ^1

(suggestive of Out $\mathbb{F}_n$).

Handel, Mosher: Out $\mathbb{F}_n$ satisfies H_b^2-alternative.

first case: "irreducible" subgroups of Out $\mathbb{F}_n$ satisfy $H_b^2(G;\mathbb{R})$-alternative

Bestvina, Feighn | Hamenstadt

relies on simplification of irreducible by Harker?.

ideas for H_b^2-alternative: use hierarchical framework in Out $\mathbb{F}_n$.

Recall $\mathcal{G} = \sqcup$ (Hyp. spaces).

The Galaxy of Hyperbolic complexes

Hyperbolicity : (Handel, Mosher) (Bestvina, Feighn) (B. Mann)

$$\begin{tikzcd} \mathcal{FS}_n \arrow[r] & \mathcal{FF}_n \arrow[r] \arrow[d] & \mathcal{ZS}_n \text{ : cycl. splitting} \\ & \text{Intersection graph} \approx \text{Cosurface graph} & \end{tikzcd}$$

(Mann). (Dowdall, Taylor)

gp-invariant ?

#Agol

$\longrightarrow$ also inherits cocompact action.

" $\longrightarrow$ " Kapovich, Rafi.

$\exists$ Lipschitz surjection w/ quasiconvex point inverses.

(Not reversible! b/c relies on characterization of dendrites)

Want to use Bestvina, Feighn, Handel "attracting laminations"

(analog of laminations in Thurston classification)

$\varphi \in \mathrm{Out}\, F_n$

$\mathcal{L}(\varphi)$ = fin. set of attracting laminations

each $\Lambda \in \mathcal{L}(\varphi)$ has "free factor support"

$\Lambda \neq \Lambda' \in \mathcal{L}(\varphi) \Rightarrow$ supports are unequal

$\mathcal{L}(\varphi) \longleftrightarrow \mathcal{L}(\varphi')$ BIG SETS???

Restrict to f.i. torsion-free

$\ker(\longrightarrow H_2(\, ; \mathbb{Z}_3))$ $\overline{\mathrm{Out}}(F_n)$

$G^{f.g.} < \overline{Out(\mathbb{F}_n)}$

$A < \mathbb{F}_n$ minimal rank free factor set.

$[A]$ is G-invariant.

$im(G \longrightarrow Out\,A)$ NOT virtually abelian.

Case 1: G has ≥ 2 lamination pairs w/ free factor support A.

$\Rightarrow$ use $G \curvearrowright \mathcal{FS}_n$ to show $\exists\, \ell' \subseteq H^2_b(G)$.

Case 2:

G has ≤ 1 lamination w/ supp. $= A$

$\rightsquigarrow$ convert to $Aut(A)$ - question

in MCG this is virtually abelian case $\Rightarrow$ Done.

Handel-Mosher in $Out\,\mathbb{F}_n$.

Case 2a: G has no lamination pairs w/ support A.

$G \curvearrowright T$ are "A-type"

Case 2b: G has 1 lamination pair w/ support A

$1 \rightarrow B \rightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1$ (analog of cyclic fibre bundle)

"hyperbolic semidirect product"

find $A < B$.

Hamenstädt: higher dimensional Bestvina, Fujiwara condition,

Mahan, Sardar: for improper action.

Ivanov: classification of $G < MCG(S)$

G not nec. f.g

Handel, Mosher: classification of $G^{f.g} < MCG(S)$.

Bestvina, Feighn, Handel: Tits alternative f.g

Horbez: Tits alternative (no f.g.).