The Theory of Resolvent Degree,

after Hamilton, Sylvester, Hilbert, Segre and Brauer.

Jesse Wolfson
University of California, Irvine

No Boundaries - Groups in Algebra, Geometry and Topology
In honor of Benson Farb
October 27, 2017

Ongoing joint work with Benson Farb

Ongoing joint work with Benson Farb

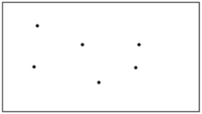
Ongoing joint work with Benson Farb

Given X, how hard is it to find Y?

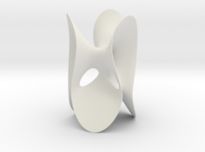
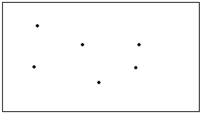
Given X, how hard is it to find Y?

general sextic $x^6 + a_1x^5 + \cdots + a_6$		

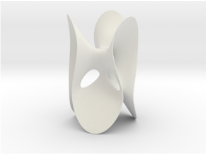
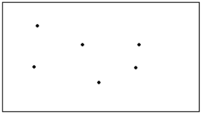
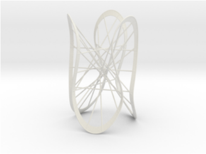
Given X, how hard is it to find Y?

general sextic $x^6 + a_1x^5 + \cdots + a_6$		
a root 		

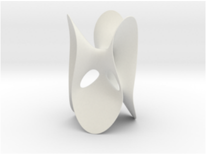
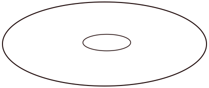
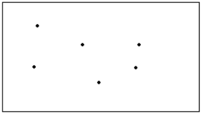
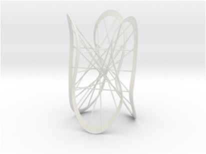
Given X, how hard is it to find Y?

general sextic	cubic surface	
$x^6 + a_1x^5 + \cdots + a_6$		
a root 		

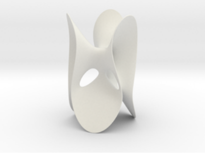
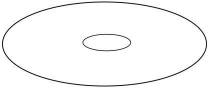
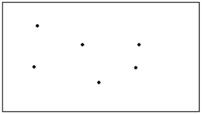
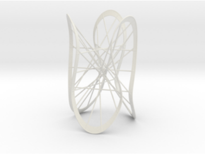
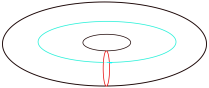
Given X, how hard is it to find Y?

general sextic	cubic surface	
$x^6 + a_1 x^5 + \cdots + a_6$		
a root	a line	
		

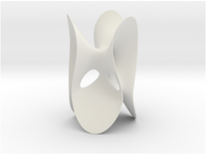
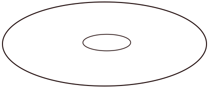
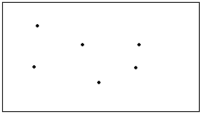
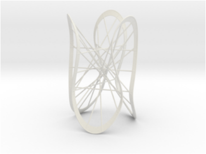
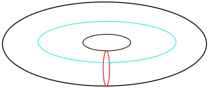
Given X, how hard is it to find Y?

general sextic $x^6 + a_1 x^5 + \cdots + a_6$	cubic surface 	Jacobian 
a root 	a line 	

Given X, how hard is it to find Y?

general sextic	cubic surface	Jacobian
$x^6 + a_1x^5 + \cdots + a_6$		
a root	a line	a level structure
		

Given X, how hard is it to find Y?

general sextic	cubic surface	Jacobian
$x^6 + a_1x^5 + \cdots + a_6$		
a root	a line	a level structure
		

Goal: Explain how these three questions can be precisely related.

Overview

Set-up

k - ground field (for this talk, assume $\mathrm{char}(k) = 0$)

Set-up

k - ground field (for this talk, assume $\mathrm{char}(k) = 0$)
$\mathcal{M}$ - variety over k

Set-up

k - ground field (for this talk, assume $\text{char}(k) = 0$)
$\mathcal{M}$ - variety over k
$\mathcal{M}' \longrightarrow \mathcal{M}$ - finite dominant map (i.e. restricts to a cover of a Zariski open)

Set-up

k - ground field (for this talk, assume $\mathrm{char}(k) = 0$)
$\mathcal{M}$ - variety over k
$\mathcal{M}' \longrightarrow \mathcal{M}$ - finite dominant map (i.e. restricts to a cover of a Zariski open)
Example:

Set-up

k - ground field (for this talk, assume $\mathrm{char}(k) = 0$)
$\mathcal{M}$ - variety over k
$\mathcal{M}' \longrightarrow \mathcal{M}$ - finite dominant map (i.e. restricts to a cover of a Zariski open)
Example:

$\mathcal{P}_n$ - space of monic degree n polynomials

Set-up

k - ground field (for this talk, assume $\mathrm{char}(k) = 0$)
$\mathcal{M}$ - variety over k
$\mathcal{M}' \longrightarrow \mathcal{M}$ - finite dominant map (i.e. restricts to a cover of a Zariski open)
Example:

$\mathcal{P}_n$ - space of monic degree n polynomials
$\widetilde{\mathcal{P}}_n := \{(P, z) \in \mathcal{P}_n \times \bar{k} \mid P(z) = 0\}$
- space of polynomials with a choice of root

Set-up

k - ground field (for this talk, assume $\mathrm{char}(k) = 0$)
$\mathcal{M}$ - variety over k
$\mathcal{M}' \longrightarrow \mathcal{M}$ - finite dominant map (i.e. restricts to a cover of a Zariski open)

Example:

$\mathcal{P}_n$ - space of monic degree n polynomials
$\widetilde{\mathcal{P}}_n := \{(P, z) \in \mathcal{P}_n \times \bar{k} \mid P(z) = 0\}$
- space of polynomials with a choice of root

Forgetting the root gives a branched cover

$$\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$$
$$(P, z) \mapsto P$$

Set-up

Examples of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
  (polynomials with or w/o a root)

Set-up

Examples of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
  (polynomials with or w/o a root)
- $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$
  (cubic surfaces with or w/o a line)

Set-up

Examples of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
  (polynomials with or w/o a root)
- $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$
  (cubic surfaces with or w/o a line)
- $\mathcal{A}_g[L] \longrightarrow \mathcal{A}_g$
  (PPAVs with or w/o a level structure)

Set-up

Examples of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
  (polynomials with or w/o a root)
- $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$
  (cubic surfaces with or w/o a line)
- $\mathcal{A}_g[L] \longrightarrow \mathcal{A}_g$
  (PPAVs with or w/o a level structure)

Want: A common invariant that captures the complexity of specifying a point in the cover given a point in the base.

Warm-up: Essential Dimension

Definition

The essential dimension $\mathrm{ed}_k(\mathcal{M}' \longrightarrow \mathcal{M})$ *is the minimum d for which there exists a Zariski open $U \subset \mathcal{M}$ and a pullback square*

$$\begin{array}{ccc} \mathcal{M}'|_U & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y \end{array}$$

with $\dim_k(Y) = d$*.*

Resolvent Degree

Definition

The resolvent degree $\mathrm{RD}_k(\mathcal{M}' \longrightarrow \mathcal{M})$ *is the minimum d such that there exists a tower of finite dominant maps*

$$E_r \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 = \mathcal{M}$$

with

$$\begin{array}{ccc} E_r & \longrightarrow & \mathcal{M}' \\ & \searrow \quad \swarrow & \\ & \mathcal{M} & \end{array}$$

and with $\mathrm{ed}_k(E_i \longrightarrow E_{i-1}) \leq d$ *for all* i*.*

Remarks

Resolvent degree quantifies "how hard" it is to specify a point in a cover, given a point in the base.

Remarks

Resolvent degree quantifies "how hard" it is to specify a point in a cover, given a point in the base.

Cyclic covers are much simpler than general covers, for example:

Remarks

Resolvent degree quantifies "how hard" it is to specify a point in a cover, given a point in the base.

Cyclic covers are much simpler than general covers, for example:

- Newton's method (i.e. $z \mapsto z - \frac{P(z)}{P'(z)}$) gives a reliable iterative algorithm for extracting radicals.

Remarks

Resolvent degree quantifies "how hard" it is to specify a point in a cover, given a point in the base.

Cyclic covers are much simpler than general covers, for example:

- Newton's method (i.e. $z \mapsto z - \frac{P(z)}{P'(z)}$) gives a reliable iterative algorithm for extracting radicals.

Theorem (McMullen, 1988)

There is no generally convergent iterative algorithm for finding the roots of a general polynomial of degree ≥ 4.

Remarks

Resolvent degree quantifies "how hard" it is to specify a point in a cover, given a point in the base.

Cyclic covers are much simpler than general covers, for example:

- Newton's method (i.e. $z \mapsto z - \frac{P(z)}{P'(z)}$) gives a reliable iterative algorithm for extracting radicals.

Theorem (McMullen, 1988)

There is no generally convergent iterative algorithm for finding the roots of a general polynomial of degree ≥ 4.

$RD = 1$ reflects this.

Remarks, cont.

Consider $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$ (moduli of polynomials with and w/o a root)

Remarks, cont.

Consider $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$ (moduli of polynomials with and w/o a root)

Theorem (Babylonians)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_2 \longrightarrow \mathcal{P}_2) = 1.$

Remarks, cont.

Consider $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$ (moduli of polynomials with and w/o a root)

Theorem (Babylonians)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_2 \longrightarrow \mathcal{P}_2) = 1.$

Theorem (Tartaglia, Cardano, Ferrari)

For $n \leq 4$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) = 1.$

Remarks, cont.

Consider $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$ (moduli of polynomials with and w/o a root)

Theorem (Babylonians)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_2 \longrightarrow \mathcal{P}_2) = 1.$

Theorem (Tartaglia, Cardano, Ferrari)

For $n \leq 4$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) = 1.$

Theorem (Bring, 1786)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_5 \longrightarrow \mathcal{P}_5) = 1.$

Remarks, cont.

Conjecture (Hilbert)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \geq 2$ *for* $n \geq 6$.

Remarks, cont.

Conjecture (Hilbert)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \geq 2$ *for* $n \geq 6$.

Theorem (Doyle–McMullen, 1989)

There exists a tower of iterative algorithms for extracting the roots of a general polynomial of degree at most 5. For $n > 5$, *no such tower exists.*

Remarks, cont.

Conjecture (Hilbert)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \geq 2$ *for* $n \geq 6$.

Theorem (Doyle–McMullen, 1989)

There exists a tower of iterative algorithms for extracting the roots of a general polynomial of degree at most 5. For $n > 5$*, no such tower exists.*

$\therefore$ If you believe Hilbert's conjecture, Doyle–McMullen is another example of how $\mathrm{RD} > 1$ captures intuitive notions of complexity of a problem.

Remarks, cont.

Essential dimension - introduced in 1998 by Buhler and Reichstein.

Remarks, cont.

Essential dimension - introduced in 1998 by Buhler and Reichstein.
From the definition, $\mathrm{ed}_k \geq \mathrm{RD}_k$.

Remarks, cont.

Essential dimension - introduced in 1998 by Buhler and Reichstein.
From the definition, $\mathrm{ed}_k \geq \mathrm{RD}_k$.
In general, the difference can be arbitrarily large:

Remarks, cont.

Essential dimension - introduced in 1998 by Buhler and Reichstein.
From the definition, $\mathrm{ed}_k \geq \mathrm{RD}_k$.
In general, the difference can be arbitrarily large:

$$\mathcal{M} := \{p(z) \in \mathcal{P}_{2n} \mid p(z) = (z^2 - a_1) \cdots (z^2 - a_n)\}$$
$$\mathcal{M}' := \widetilde{\mathcal{P}}_{2n}|_{\mathcal{M}}.$$

Then for $n > 1$

$$\mathrm{ed}_k(\mathcal{M}' \longrightarrow \mathcal{M}) = n >> \mathrm{RD}_k(\mathcal{M}' \longrightarrow \mathcal{M}) = 1$$

Remarks, cont.

Essential dimension - introduced in 1998 by Buhler and Reichstein.
From the definition, $\mathrm{ed}_k \geq \mathrm{RD}_k$.
In general, the difference can be arbitrarily large:

$$\mathcal{M} := \{p(z) \in \mathcal{P}_{2n} \mid p(z) = (z^2 - a_1)\cdots(z^2 - a_n)\}$$
$$\mathcal{M}' := \widetilde{\mathcal{P}}_{2n}|_{\mathcal{M}}.$$

Then for $n > 1$

$$\mathrm{ed}_k(\mathcal{M}' \longrightarrow \mathcal{M}) = n >> \mathrm{RD}_k(\mathcal{M}' \longrightarrow \mathcal{M}) = 1$$

$\therefore$ ed_k very sensitive to arithmetic of the fields k and $k(\mathcal{M})$.
RD_k captures traditional notions of complexity of a problem.

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)
  - $\mathcal{M}_{4,2}(1) \longrightarrow \mathcal{M}_{4,2}$ (quartic curves with or w/o a bitangent)

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)
  - $\mathcal{M}_{4,2}(1) \longrightarrow \mathcal{M}_{4,2}$ (quartic curves with or w/o a bitangent)

  $\vdots$

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)
  - $\mathcal{M}_{4,2}(1) \longrightarrow \mathcal{M}_{4,2}$ (quartic curves with or w/o a bitangent)

    $\vdots$
- Congruence subgroups, i.e.

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)
  - $\mathcal{M}_{4,2}(1) \longrightarrow \mathcal{M}_{4,2}$ (quartic curves with or w/o a bitangent)

    $\vdots$
- Congruence subgroups, i.e.

  $X(\Gamma) = \Gamma \setminus G/K$ arithmetic, locally symmetric space.

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)
  - $\mathcal{M}_{4,2}(1) \longrightarrow \mathcal{M}_{4,2}$ (quartic curves with or w/o a bitangent)

    $\vdots$
- Congruence subgroups, i.e.

  $X(\Gamma) = \Gamma \setminus G/K$ arithmetic, locally symmetric space.
  $\Gamma' \subset \Gamma$ finite index subgroup

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)
  - $\mathcal{M}_{4,2}(1) \longrightarrow \mathcal{M}_{4,2}$ (quartic curves with or w/o a bitangent)

    $\vdots$
- Congruence subgroups, i.e.

  $X(\Gamma) = \Gamma \setminus G/K$ arithmetic, locally symmetric space.
  $\Gamma' \subset \Gamma$ finite index subgroup
  - $X(\Gamma') \longrightarrow X(\Gamma)$.

Remarks, cont.

Three major sources of $\mathcal{M}' \longrightarrow \mathcal{M}$:

- Roots of polynomials, e.g. $\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n$
- Classical enumerative problems, e.g.
  - $\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$ (cubic surfaces with or w/o a line)
  - $\mathcal{M}_{4,2}(1) \longrightarrow \mathcal{M}_{4,2}$ (quartic curves with or w/o a bitangent)

    $\vdots$
- Congruence subgroups, i.e.

    $X(\Gamma) = \Gamma \setminus G/K$ arithmetic, locally symmetric space.

    $\Gamma' \subset \Gamma$ finite index subgroup
  - $X(\Gamma') \longrightarrow X(\Gamma)$.

Resolvent degree gives a common, natural invariant of each.

Remarks, cont.

Many (most) examples come with towers of covers, e.g.

Remarks, cont.

Many (most) examples come with towers of covers, e.g.

- $\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$

Remarks, cont.

Many (most) examples come with towers of covers, e.g.

- $\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$
- $\cdots \longrightarrow \mathcal{M}_g[\ell^n] \longrightarrow \mathcal{M}_g[\ell^{n-1}] \longrightarrow \cdots \longrightarrow \mathcal{M}_g$

Remarks, cont.

Many (most) examples come with towers of covers, e.g.

- $\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$
- $\cdots \longrightarrow \mathcal{M}_g[\ell^n] \longrightarrow \mathcal{M}_g[\ell^{n-1}] \longrightarrow \cdots \longrightarrow \mathcal{M}_g$

Much of algebraic geometry pre-1930 concerned with:

Remarks, cont.

Many (most) examples come with towers of covers, e.g.

- $\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$
- $\cdots \longrightarrow \mathcal{M}_g[\ell^n] \longrightarrow \mathcal{M}_g[\ell^{n-1}] \longrightarrow \cdots \longrightarrow \mathcal{M}_g$

Much of algebraic geometry pre-1930 concerned with:

> Given one datum, specify others.

I.e. specify the relationships between intermediate covers.

Remarks, cont.

Many (most) examples come with towers of covers, e.g.

- $\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}$
- $\cdots \longrightarrow \mathcal{M}_g[\ell^n] \longrightarrow \mathcal{M}_g[\ell^{n-1}] \longrightarrow \cdots \longrightarrow \mathcal{M}_g$

Much of algebraic geometry pre-1930 concerned with:

Given one datum, specify others.

I.e. specify the relationships between intermediate covers.
Resolvent degree gives a natural framework for understanding and organizing classical work.

Remarks, cont.

Resolvent degree first defined (as far as we can find) in Brauer, 1975.

Remarks, cont.

Resolvent degree first defined (as far as we can find) in Brauer, 1975.
However, explicit study of $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n)$ is classical:

Remarks, cont.

Resolvent degree first defined (as far as we can find) in Brauer, 1975.

However, explicit study of $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n)$ is classical:

- Tschirnhausen, 1683

Remarks, cont.

Resolvent degree first defined (as far as we can find) in Brauer, 1975.

However, explicit study of $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n)$ is classical:

- Tschirnhausen, 1683
- Bring, 1786

Remarks, cont.

Resolvent degree first defined (as far as we can find) in Brauer, 1975.
However, explicit study of $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n)$ is classical:

- Tschirnhausen, 1683
- Bring, 1786
- Hamilton, 1836

Remarks, cont.

Resolvent degree first defined (as far as we can find) in Brauer, 1975.
However, explicit study of $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n)$ is classical:

- Tschirnhausen, 1683
- Bring, 1786
- Hamilton, 1836
- Sylvester, 1886

$\vdots$ (more below)

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds.

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds. However:

There are no known nontrivial lower bounds on RD_k.

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds. However:

There are no known nontrivial lower bounds on RD_k.

Conjecture (Hilbert)

1. $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) = 2$.
2. *(Hilbert's 13th Problem)* $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_7 \longrightarrow \mathcal{P}_7) = 3$.

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds. However:

There are no known nontrivial lower bounds on RD_k.

Conjecture (Hilbert)

1. $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) = 2.$
2. *(Hilbert's 13th Problem)* $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_7 \longrightarrow \mathcal{P}_7) = 3.$

Question

Is $\mathrm{RD}_{\mathbb{C}}(-) \equiv 1$*?*

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds. However:

There are no known nontrivial lower bounds on RD_k.

Conjecture (Hilbert)

1. $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) = 2$.
2. *(Hilbert's 13th Problem)* $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_7 \longrightarrow \mathcal{P}_7) = 3$.

Question

Is $\mathrm{RD}_{\mathbb{C}}(-) \equiv 1$*?*

Hilbert thought no.

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds. However:

There are no known nontrivial lower bounds on RD_k.

Conjecture (Hilbert)

1. $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) = 2.$
2. *(Hilbert's 13th Problem)* $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_7 \longrightarrow \mathcal{P}_7) = 3.$

Question

Is $\mathrm{RD}_{\mathbb{C}}(-) \equiv 1$*?*

Hilbert thought no. Benson and I are working on this!

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds. However:

There are no known nontrivial lower bounds on RD_k.

Conjecture (Hilbert)

1. $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) = 2$.
2. *(Hilbert's 13th Problem)* $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_7 \longrightarrow \mathcal{P}_7) = 3$.

Question

Is $\mathrm{RD}_{\mathbb{C}}(-) \equiv 1$*?*

Hilbert thought no. Benson and I are working on this!
Keywords: "braid Galois group."

Remarks, cont.

For RD_k, there exist techniques for obtaining upper bounds. However:

There are no known nontrivial lower bounds on RD_k.

Conjecture (Hilbert)

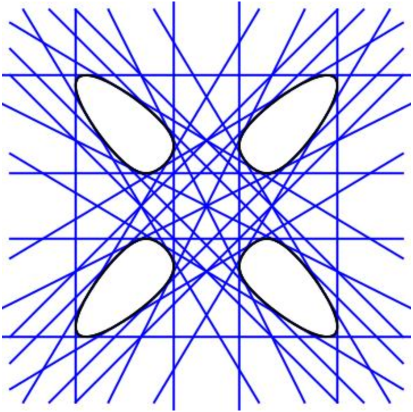
1. $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) = 2$.
2. *(Hilbert's 13th Problem)* $\mathrm{RD}_{\mathbb{C}}(\widetilde{\mathcal{P}}_7 \longrightarrow \mathcal{P}_7) = 3$.

Question

Is $\mathrm{RD}_{\mathbb{C}}(-) \equiv 1$*?*

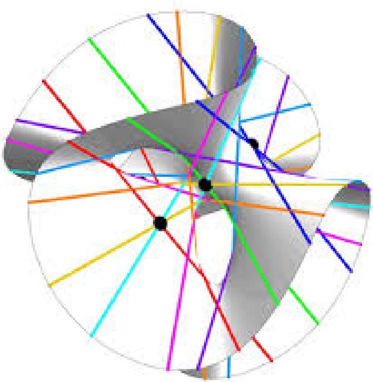
Hilbert thought no. Benson and I are working on this!
Keywords: "braid Galois group." Hopefully, more soon!

RD and Classical Enumerative Problems



A plane quartic with its 28 bitangents

RD and Classical Enumerative Problems



A cubic surface with its 27 lines

RD and Classical Enumerative Problems

Theorem (Cayley–Salmon, 1856)

There exist 27 lines on every smooth cubic surface.

RD and Classical Enumerative Problems

Theorem (Cayley–Salmon, 1856)

There exist 27 lines on every smooth cubic surface.

Question

Given a cubic, how hard is it to find one line? All 27?

RD and Classical Enumerative Problems

Theorem (Cayley–Salmon, 1856)

There exist 27 lines on every smooth cubic surface.

Question

Given a cubic, how hard is it to find one line? All 27?

Let $\mathcal{M}_{3,3}(n)$ denote the moduli of cubic surfaces with a choice of n (ordered) lines.

RD and Classical Enumerative Problems

Theorem (Cayley–Salmon, 1856)

There exist 27 lines on every smooth cubic surface.

Question

Given a cubic, how hard is it to find one line? All 27?

Let $\mathcal{M}_{3,3}(n)$ denote the moduli of cubic surfaces with a choice of n (ordered) lines.

Theorem (Jordan, 1870; Harris, 1979)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}^{disj.}(3)) = 1.$

RD and Classical Enumerative Problems

Theorem (Cayley–Salmon, 1856)

There exist 27 lines on every smooth cubic surface.

Question

Given a cubic, how hard is it to find one line? All 27?

Let $\mathcal{M}_{3,3}(n)$ denote the moduli of cubic surfaces with a choice of n (ordered) lines.

Theorem (Jordan, 1870; Harris, 1979)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}^{disj.}(3)) = 1.$

Observation (Farb–W.)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1)) = \mathrm{RD}_k(\widetilde{\mathcal{P}}_5 \longrightarrow \mathcal{P}_5) = 1.$

Lines on cubics, cont.

Observation (Farb–W.)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1)) = \mathrm{RD}_k(\widetilde{\mathcal{P}}_5 \longrightarrow \mathcal{P}_5) = 1.$

This follows from a beautiful classical trick:

Lines on cubics, cont.

Observation (Farb–W.)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1)) = \mathrm{RD}_k(\widetilde{\mathcal{P}}_5 \longrightarrow \mathcal{P}_5) = 1.$

This follows from a beautiful classical trick:
Given $L \subset S$, each plane V in the pencil containing L has

$$V \cap S = L \cup C$$

for some conic C.

Lines on cubics, cont.

Observation (Farb–W.)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1)) = \mathrm{RD}_k(\widetilde{\mathcal{P}}_5 \longrightarrow \mathcal{P}_5) = 1.$

This follows from a beautiful classical trick:
Given $L \subset S$, each plane V in the pencil containing L has

$$V \cap S = L \cup C$$

for some conic C. Moreover, C degenerates to a pair of lines $L_1 \cup L_2$ at the roots of the discriminant of the pencil of conics, and **this discriminant has degree 5.**

Lines on cubics, cont.

Observation (Farb–W.)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1)) = \mathrm{RD}_k(\widetilde{\mathcal{P}}_5 \longrightarrow \mathcal{P}_5) = 1.$

This follows from a beautiful classical trick:

Given $L \subset S$, each plane V in the pencil containing L has

$$V \cap S = L \cup C$$

for some conic C. Moreover, C degenerates to a pair of lines $L_1 \cup L_2$ at the roots of the discriminant of the pencil of conics, and **this discriminant has degree 5.** Solving this quintic, we get 5 pairs of disjoint lines on S. By a theorem of Harris, we get the remaining 17 lines by adjoining radicals. □

Lines on cubics, cont.

Observation (Farb–W.)

$\mathrm{RD}_k(\mathcal{M}_{3,3}(27) \longrightarrow \mathcal{M}_{3,3}(1)) = \mathrm{RD}_k(\widetilde{\mathcal{P}}_5 \longrightarrow \mathcal{P}_5) = 1.$

This follows from a beautiful classical trick:
Given $L \subset S$, each plane V in the pencil containing L has

$$V \cap S = L \cup C$$

for some conic C. Moreover, C degenerates to a pair of lines $L_1 \cup L_2$ at the roots of the discriminant of the pencil of conics, and **this discriminant has degree 5.** Solving this quintic, we get 5 pairs of disjoint lines on S. By a theorem of Harris, we get the remaining 17 lines by adjoining radicals. □

Question

What is $\mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3})$*?*

RD and Roots of Polynomials

RD and Roots of Polynomials

"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date." *(Sylvester, Hammond 1887)*

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* (Sylvester, Hammond 1887)

To bring this up to 2017, add:

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to 2017, add:

- Hilbert 1927

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* (Sylvester, Hammond 1887)

To bring this up to 2017, add:

- Hilbert 1927
- Segre 1947, 1955,

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to 2017, add:

- Hilbert 1927
- Segre 1947, 1955,
- Brauer 1975.

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to 2017, add:

- Hilbert 1927
- Segre 1947, 1955,
- Brauer 1975.

That's it!

RD and Roots of Polynomials

> *"The theory has been a 'plant of slow growth'. The Lund Thesis [Bring] of December, 1786 (a matter of a couple of pages), Hamilton's report of 1836, with the tract of Mr. Jerrard referred to therein, and the memoire [Sylvester] of 'Crelle' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date."* *(Sylvester, Hammond 1887)*

To bring this up to 2017, add:

- Hilbert 1927
- Segre 1947, 1955,
- Brauer 1975. (**0 citations!**)

That's it!

What's Known

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

What's Known

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H: \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

Hamilton computed the initial values of H:

r	4	5	6	7	8	9
H(r)	5	11	47	923	409,619	83,763,206,255

What's Known

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

Hamilton computed the initial values of H:

r	4	5	6	7	8	9
H(r)	5	11	47	923	409,619	83,763,206,255

Sylvester–Hammond, 1887 - generating function for $H(r)$

What's Known, cont.

By mid-20th century, Hamilton's Theorem appears to have been forgotten:

What's Known, cont.

By mid-20th century, Hamilton's Theorem appears to have been forgotten:

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

What's Known, cont.

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

By mid-20th century, Hamilton's Theorem appears to have been forgotten:

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

What's Known, cont.

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

By mid-20th century, Hamilton's Theorem appears to have been forgotten:

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

Segre and Brauer reproved the theorem soon after, but without an explicit formula.

What's Known, cont.

Theorem (Hamilton, 1836)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

By mid-20th century, Hamilton's Theorem appears to have been forgotten:

Conjecture (Segre, Annals 1947)

There exists a monotone increasing function $H\colon \mathbb{N} \longrightarrow \mathbb{N}$ *such that for* $n > H(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

Segre and Brauer reproved the theorem soon after, but without an explicit formula.

Theorem (Brauer, 1975)

Let $B(r) := (r-1)!$ *For* $n > B(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$.

Improving on Hamilton, Brauer

Improving on Hamilton, Brauer

- Let $\mathcal{M}_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$.

Improving on Hamilton, Brauer

- Let $\mathcal{M}_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$.
- Let $\mathcal{M}^r_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$ with a choice of r-plane lying on them.

Improving on Hamilton, Brauer

- Let $\mathcal{M}_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$.
- Let $\mathcal{M}_{3,N}^r$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$ with a choice of r-plane lying on them.
- Let $L_3(r, N) := \mathrm{RD}_k(\mathcal{M}_{3,N}^r \longrightarrow \mathcal{M}_{3,N})$.

Improving on Hamilton, Brauer

- Let $\mathcal{M}_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$.
- Let $\mathcal{M}_{3,N}^r$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$ with a choice of r-plane lying on them.
- Let $L_3(r, N) := \mathrm{RD}_k(\mathcal{M}_{3,N}^r \longrightarrow \mathcal{M}_{3,N})$.

Theorem (Farb–W.)

There exist polynomial functions $f, g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $n \geq \frac{(d+k)!}{d!}$,

$$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq \max\{n - (d + k + 1), L_3(f(d, k), g(d, k))\}.$$

Improving on Hamilton, Brauer

- Let $\mathcal{M}_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$.
- Let $\mathcal{M}_{3,N}^r$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$ with a choice of r-plane lying on them.
- Let $L_3(r,N) := \mathrm{RD}_k(\mathcal{M}_{3,N}^r \longrightarrow \mathcal{M}_{3,N})$.

Theorem (Farb–W.)

There exist polynomial functions $f, g\colon \mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N}$ such that for all $n \geq \frac{(d+k)!}{d!}$,

$$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq \max\{n-(d+k+1), L_3(f(d,k), g(d,k))\}.$$

Corollary

There exist monotone increasing functions $FW, \varphi\colon \mathbb{N}\longrightarrow\mathbb{N}$ s.t.

Improving on Hamilton, Brauer

- Let $\mathcal{M}_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$.
- Let $\mathcal{M}^r_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$ with a choice of r-plane lying on them.
- Let $L_3(r,N) := \mathrm{RD}_k(\mathcal{M}^r_{3,N} \longrightarrow \mathcal{M}_{3,N})$.

Theorem (Farb–W.)

There exist polynomial functions $f, g \colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $n \geq \frac{(d+k)!}{d!}$,

$$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq \max\{n-(d+k+1), L_3(f(d,k),g(d,k))\}.$$

Corollary

There exist monotone increasing functions $FW, \varphi \colon \mathbb{N} \longrightarrow \mathbb{N}$ s.t.

- *For $n > FW(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$,*

Improving on Hamilton, Brauer

- Let $\mathcal{M}_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$.
- Let $\mathcal{M}^r_{3,N}$ denote the moduli of cubic hypersurfaces in $\mathbb{P}^N$ with a choice of r-plane lying on them.
- Let $L_3(r, N) := \mathrm{RD}_k(\mathcal{M}^r_{3,N} \longrightarrow \mathcal{M}_{3,N})$.

Theorem (Farb–W.)

There exist polynomial functions $f, g\colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $n \geq \frac{(d+k)!}{d!}$,

$$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq \max\{n - (d + k + 1), L_3(f(d, k), g(d, k))\}.$$

Corollary

There exist monotone increasing functions $FW, \varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ s.t.

- *For $n > FW(r)$, $\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq n - r$,*
- *For all $d \geq 0$, $r \geq \varphi(d)$, then $B(r)/FW(r) \geq d!$.*

Improving on Hamilton, Brauer, cont.

Idea comes from:

Improving on Hamilton, Brauer, cont.

Idea comes from:

Theorem (Hilbert, 1927)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_9 \longrightarrow \mathcal{P}_9) \leq \max\{4, \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3})\}.$

Improving on Hamilton, Brauer, cont.

Idea comes from:

Theorem (Hilbert, 1927)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_9 \longrightarrow \mathcal{P}_9) \leq \max\{4, \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3})\}$.

i.e. Hilbert used a line on a cubic surface to simplify the solution of the general degree 9 polynomial!

Improving on Hamilton, Brauer, cont.

Idea comes from:

Theorem (Hilbert, 1927)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_9 \longrightarrow \mathcal{P}_9) \leq \max\{4, \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3})\}.$

i.e. Hilbert used a line on a cubic surface to simplify the solution of the general degree 9 polynomial!

His proof suggests two things:

Improving on Hamilton, Brauer, cont.

Idea comes from:

Theorem (Hilbert, 1927)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_9 \longrightarrow \mathcal{P}_9) \leq \max\{4, \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3})\}$.

i.e. Hilbert used a line on a cubic surface to simplify the solution of the general degree 9 polynomial!
His proof suggests two things:

1. A method for finding special points on Fano varieties of complete intersections.

Improving on Hamilton, Brauer, cont.

Idea comes from:

Theorem (Hilbert, 1927)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_9 \longrightarrow \mathcal{P}_9) \leq \max\{4, \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3})\}$.

i.e. Hilbert used a line on a cubic surface to simplify the solution of the general degree 9 polynomial!

His proof suggests two things:

1. A method for finding special points on Fano varieties of complete intersections.
2. A method for using linear subspaces on complete intersections to improve bounds on resolvent degree.

Improving on Hamilton, Brauer, cont.

Idea for 2 actually goes back to the beginnings of the subject:

Improving on Hamilton, Brauer, cont.

Idea for 2 actually goes back to the beginnings of the subject:

Theorem (Bring, 1786)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq \max\{n-4, \mathrm{RD}_k(\mathcal{M}^1_{2,2} \longrightarrow \mathcal{M}_{2,2})\}$ $(= n-4)$, *for* $n \geq 5$.

Improving on Hamilton, Brauer, cont.

Idea for 2 actually goes back to the beginnings of the subject:

Theorem (Bring, 1786)

$\mathrm{RD}_k(\widetilde{\mathcal{P}}_n \longrightarrow \mathcal{P}_n) \leq \max\{n-4, \mathrm{RD}_k(\mathcal{M}^1_{2,2} \longrightarrow \mathcal{M}_{2,2})\}$ $(= n-4)$, *for* $n \geq 5$.

Question

By combining Hamilton's method with that of Bring–Hilbert, can we go further?

RD and Congruence Subgroups

Theme in Kähler geometry: study/obstruct compressions to a smaller dimensional variety.

RD and Congruence Subgroups

Theme in Kähler geometry: study/obstruct compressions to a smaller dimensional variety.

- Gromov–Schoen

RD and Congruence Subgroups

Theme in Kähler geometry: study/obstruct compressions to a smaller dimensional variety.

- Gromov–Schoen
- Beauville–Siu

RD and Congruence Subgroups

Theme in Kähler geometry: study/obstruct compressions to a smaller dimensional variety.

- Gromov–Schoen
- Beauville–Siu

  ⋮

RD and Congruence Subgroups

Theme in Kähler geometry: study/obstruct compressions to a smaller dimensional variety.

- Gromov–Schoen
- Beauville–Siu

  ⋮

Resolvent degree suggests a natural refinement:

Question

Given a cover $\widetilde{X} \longrightarrow X$, can we (virtually) compress the cover to a cover of a lower dimensional variety?

RD and Congruence Subgroups

Theme in Kähler geometry: study/obstruct compressions to a smaller dimensional variety.

- Gromov–Schoen
- Beauville–Siu

$\vdots$

Resolvent degree suggests a natural refinement:

Question

Given a cover $\widetilde{X} \longrightarrow X$, can we (virtually) compress the cover to a cover of a lower dimensional variety?

More precisely:

Question

Given arithmetic locally symmetric space $X = \Gamma \setminus G/K$, and $\Gamma' \subset \Gamma$ finite index, what is $\mathrm{RD}_k(X(\Gamma') \longrightarrow X(\Gamma))$?

Example

Let $\mathcal{E} = \mathbb{Z}[\omega]$ denote the Eisenstein integers.

Example

Let $\mathcal{E} = \mathbb{Z}[\omega]$ denote the Eisenstein integers.
Let $\Gamma_4 = \mathrm{PU}(4,1)(\mathcal{E})$.

Example

Let $\mathcal{E} = \mathbb{Z}[\omega]$ denote the Eisenstein integers.
Let $\Gamma_4 = \mathrm{PU}(4,1)(\mathcal{E})$.
$\Gamma_4 \twoheadrightarrow W(E_6)$. Denote kernel by Γ_4'.

Example

Let $\mathcal{E} = \mathbb{Z}[\omega]$ denote the Eisenstein integers.
Let $\Gamma_4 = \mathrm{PU}(4,1)(\mathcal{E})$.
$\Gamma_4 \twoheadrightarrow W(E_6)$. Denote kernel by Γ_4'.
$\Gamma_4 \circlearrowleft \mathbb{CH}^4$. Get a congruence cover

$$X(\Gamma_4') \longrightarrow X(\Gamma_4)$$

Example

Let $\mathcal{E} = \mathbb{Z}[\omega]$ denote the Eisenstein integers.
Let $\Gamma_4 = \mathrm{PU}(4,1)(\mathcal{E})$.
$\Gamma_4 \twoheadrightarrow W(E_6)$. Denote kernel by Γ_4'.
$\Gamma_4 \circlearrowleft \mathbb{CH}^4$. Get a congruence cover

$$X(\Gamma_4') \longrightarrow X(\Gamma_4)$$

By work of Clemens–Griffiths and Allcock–Carlson–Toledo:

Example

Let $\mathcal{E} = \mathbb{Z}[\omega]$ denote the Eisenstein integers.
Let $\Gamma_4 = \mathrm{PU}(4,1)(\mathcal{E})$.
$\Gamma_4 \twoheadrightarrow W(E_6)$. Denote kernel by Γ_4'.
$\Gamma_4 \circlearrowleft \mathbb{CH}^4$. Get a congruence cover

$$X(\Gamma_4') \longrightarrow X(\Gamma_4)$$

By work of Clemens–Griffiths and Allcock–Carlson–Toledo:

- $X(\Gamma_4)$ is a moduli space of certain intermediate Jacobians,

Example

Let $\mathcal{E} = \mathbb{Z}[\omega]$ denote the Eisenstein integers.
Let $\Gamma_4 = \mathrm{PU}(4,1)(\mathcal{E})$.
$\Gamma_4 \twoheadrightarrow W(E_6)$. Denote kernel by Γ_4'.
$\Gamma_4 \circlearrowleft \mathbb{CH}^4$. Get a congruence cover

$$X(\Gamma_4') \longrightarrow X(\Gamma_4)$$

By work of Clemens–Griffiths and Allcock–Carlson–Toledo:

- $X(\Gamma_4)$ is a moduli space of certain intermediate Jacobians,
- $X(\Gamma_4')$ is the moduli space of these intermediate Jacobians equipped with a $W(E_6)$-level structure.

Sample Theorem

Sample Theorem

Theorem (Farb–W.)

Hilbert's Sextic Conjecture $\Rightarrow \mathrm{RD}_{\mathbb{C}}(X(\Gamma_4') \longrightarrow X(\Gamma_4)) \geq 2$.

Sample Theorem

Theorem (Farb–W.)

Hilbert's Sextic Conjecture $\Rightarrow \mathrm{RD}_{\mathbb{C}}(X(\Gamma_4') \longrightarrow X(\Gamma_4)) \geq 2$.

Proof.

Two steps:

Sample Theorem

Theorem (Farb–W.)

Hilbert's Sextic Conjecture $\Rightarrow \mathrm{RD}_{\mathbb{C}}(X(\Gamma_4') \longrightarrow X(\Gamma_4)) \geq 2$.

Proof.

Two steps:

1. A variant of Hilbert's trick for the degree 9 shows that

$$\mathrm{RD}_k(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) \leq \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}).$$

Sample Theorem

Theorem (Farb–W.)

Hilbert's Sextic Conjecture $\Rightarrow \mathrm{RD}_{\mathbb{C}}(X(\Gamma_4') \longrightarrow X(\Gamma_4)) \geq 2$.

Proof.

Two steps:

1. A variant of Hilbert's trick for the degree 9 shows that

$$\mathrm{RD}_k(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) \leq \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}).$$

2. Allcock–Carlson–Toledo's uniformization theorem implies that

$$\mathrm{RD}_{\mathbb{C}}(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}) = \mathrm{RD}_{\mathbb{C}}(X(\Gamma_4') \longrightarrow X(\Gamma_4)).$$

Sample Theorem

Theorem (Farb–W.)

Hilbert's Sextic Conjecture $\Rightarrow \mathrm{RD}_{\mathbb{C}}(X(\Gamma_4') \longrightarrow X(\Gamma_4)) \geq 2.$

Proof.

Two steps:

1. A variant of Hilbert's trick for the degree 9 shows that

$$\mathrm{RD}_k(\widetilde{\mathcal{P}}_6 \longrightarrow \mathcal{P}_6) \leq \mathrm{RD}_k(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}).$$

2. Allcock–Carlson–Toledo's uniformization theorem implies that

$$\mathrm{RD}_{\mathbb{C}}(\mathcal{M}_{3,3}(1) \longrightarrow \mathcal{M}_{3,3}) = \mathrm{RD}_{\mathbb{C}}(X(\Gamma_4') \longrightarrow X(\Gamma_4)).$$

□

.

Sample Theorem, Cont.

Two things I like about this theorem:

Sample Theorem, Cont.

Two things I like about this theorem:

- It relates three seemingly very different problems.

Sample Theorem, Cont.

Two things I like about this theorem:

- It relates three seemingly very different problems.
- It uses a uniformization theorem to obtain nontrivial relations between algebraic problems.

Sample Theorem, Cont.

Two things I like about this theorem:

- It relates three seemingly very different problems.
- It uses a uniformization theorem to obtain nontrivial relations between algebraic problems.

No Boundaries!

Conclusion

"The study of [resolvent degree], far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians."

(Sylvester, Hammond 1887)

Conclusion

"The study of [resolvent degree], far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians."

(Sylvester, Hammond 1887)

Sylvester and Hammond's words apply just as much today!

Happy Birthday, Benson!