# THE KERNEL OF THE BIRMAN–CRAGGS–JOHNSON HOMOMORPHISM

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Abstract. We give a simple generating set for the kernel of the Birman–Craggs–Johnson homomorphism. Using this generating set, we give a new proof of Johnson's calculation of the abelianization of the Torelli group.

## 1. Introduction

Let $\Sigma_g^b$ be a compact oriented genus $g$ surface with $b \in \{0,1\}$ boundary components and let $\mathrm{Mod}_g^b$ be its mapping class group, i.e., the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_g^b$ that fix $\partial\Sigma_g^b$ pointwise. We will omit $b$ from our notation when $b = 0$. The group $\mathrm{Mod}_g^b$ acts on $\mathrm{H}_1(\Sigma_g^b) \cong \mathbb{Z}^{2g}$ and preserves the algebraic intersection form. This gives a homomorphism $\mathrm{Mod}_g^b \to \mathrm{Sp}_{2g}(\mathbb{Z})$ whose kernel $\mathcal{I}_g^b$ is the Torelli group. These all fit into the short exact sequence

$$1 \longrightarrow \mathcal{I}_g^b \longrightarrow \mathrm{Mod}_g^b \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

Building on seminal work of Birman–Craggs [3], Johnson [13] used the Rochlin invariant of homology 3-spheres to construct the Birman–Craggs–Johnson (BCJ) homomorphisms $\sigma\colon \mathcal{I}_g^1 \to \mathrm{B}_3(g)$ and $\sigma\colon \mathcal{I}_g \to \mathrm{B}_3^0(g)$. Here $\mathrm{B}_3(g)$ and $\mathrm{B}_3^0(g)$ are abelian 2-groups we will describe in more detail below. Johnson proved in [19] that for $g \geq 3$ the maps $\sigma$ induce isomorphisms

$$\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{F}_2) \cong \mathrm{B}_3(g) \quad \text{and} \quad \mathrm{H}_1(\mathcal{I}_g; \mathbb{F}_2) \cong \mathrm{B}_3^0(g).$$

Earlier in [18] he had proved that $\mathrm{H}_1(\mathcal{I}_g^b)/2$-torsion is a free abelian group detected by the Johnson homomorphism (see §1.8 below), so this completely determines $\mathrm{H}_1(\mathcal{I}_g^b)$.

1.1. **Goal.** In this paper, we give a simple generating set for the kernel of the BCJ homomorphism. This can be viewed as a 2-torsion analogue of another theorem of Johnson [18] saying that the kernel of the Johnson homomorphism is generated by Dehn twists about separating curves. Using our theorem, we give a new proof that the BCJ homomorphism detects $\mathrm{H}_1(\mathcal{I}_g^b; \mathbb{F}_2)$, and thus a new approach to calculating $\mathrm{H}_1(\mathcal{I}_g^b)$.

1.2. **Rochlin invariant.** The Rochlin invariant ([7]; see also [29, pp. 224-227]) is an invariant $\mu(M^3) \in \mathbb{F}_2$ of a homology 3-sphere $M^3$. It can be calculated as follows. Let $W^4$ be a smooth spin 4-manifold with $\partial W^4 = M^3$. For number-theoretic reasons,[1] the signature $\mathrm{sig}(W^4)$ is divisible by 8. The Rochlin invariant of $M^3$ is $\mu(M^3) \equiv \mathrm{sig}(W^4)/8$ modulo 2. The fact that this is independent of $W^4$ follows from the classical theorem of Rochlin [28] saying that the signature of a closed smooth spin 4-manifold is divisible by 16.

1.3. **Birman–Craggs homomorphism.** Let $\iota\colon \Sigma_g \hookrightarrow \mathbb{S}^3$ be an embedding such that $\iota(\Sigma_g)$ is a Heegaard surface, so $\iota(\Sigma_g)$ separates $\mathbb{S}^3$ into two solid genus-$g$ handlebodies.[2] For

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[1] By van der Blij's Lemma [10, Corollary 9.31], the signature of a unimodular even quadratic form is divisible by 8.

[2] Waldhausen [30, 31] proved that any two genus-$g$ Heegaard surfaces in $\mathbb{S}^3$ are isotopic as unparameterized surfaces, but there are multiple isotopy classes of parameterized Heegaard surfaces $\iota\colon \Sigma_g \hookrightarrow \mathbb{S}^3$.

$f \in \mathrm{Mod}_g$, let $M_{\iota,f}$ be the closed oriented 3-manifold obtained by cutting $\mathbb{S}^3$ open along $\iota(\Sigma_g)$ and then regluing after twisting by $f$. For $f \in \mathcal{I}_g$, the Mayer–Vietoris exact sequence implies that $M_{\iota,f}$ is a homology 3-sphere. We can therefore define a set map $\mu_\iota \colon \mathcal{I}_g \to \mathbb{F}_2$ via the formula

$$\mu_\iota(f) = \mu(M_{\iota,f}) \quad \text{for } f \in \mathcal{I}_g.$$

A fundamental theorem of Birman–Craggs [3] says that $\mu_\iota$ is a homomorphism.

1.4. **Birman–Craggs–Johnson (BCJ) homomorphism.** Johnson [13] characterized how $\mu_\iota$ depends on $\iota$ and used this to package all the different $\mu_\iota$ together into a single homomorphism. He also showed how to extend this homomorphism to $\mathcal{I}_g^1$. We give a brief description of this homomorphism here; see §2 below for more details.

Fix a symplectic basis $S = \{a_1, b_1, \ldots, a_g, b_g\}$ for $\mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2) \cong \mathbb{F}_2^{2g}$. Regard the elements of $S$ as formal variables. For $n \geq 0$, let $\mathrm{B}_n(g)$ be the set of all square-free polynomials $\phi \in \mathbb{F}_2[S]$ such that the degree of $\phi$ is at most $n$. Here by "square-free" we mean that none of the monomials appearing in $\phi$ have any variable $s \in S$ to a power greater than 1. For instance, $a_1 b_1 a_2 \in \mathrm{B}_3(g)$ but $a_1 b_1^2 \notin \mathrm{B}_3(g)$.

The Birman–Craggs–Johnson (BCJ) homomorphism is a surjective homomorphism $\sigma \colon \mathcal{I}_g^1 \to \mathrm{B}_3(g)$. The conjugation action of $\mathrm{Mod}_g^1$ on $\mathcal{I}_g^1$ descends to an action of $\mathrm{Sp}_{2g}(\mathbb{Z}) = \mathrm{Mod}_g^1 / \mathcal{I}_g^1$ on $\mathrm{B}_3(g)$. We warn the reader that this action on $\mathrm{B}_3(g)$ is not the obvious one that preserves the degree of a square-free polynomial; see §2 for more details. On a closed surface, the BCJ homomorphism takes the form $\sigma \colon \mathcal{I}_g \to \mathrm{B}_3^0(g)$ where $\mathrm{B}_3^0(g)$ is a quotient of $\mathrm{B}_3(g)$ by a $(2g+1)$-dimensional subspace we will describe in §2.

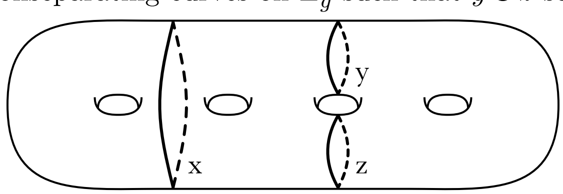
*Remark* 1.1. From its description above, we see that the dimension of the $\mathbb{F}_2$-vector space $\mathrm{B}_3(g)$ is

$$\binom{2g}{0} + \binom{2g}{1} + \binom{2g}{2} + \binom{2g}{3} = \frac{1}{3}(4g^3 + 5g + 3).$$

To form $\mathrm{B}_3^0(g)$, we quotient $\mathrm{B}_3(g)$ by a $(2g+1)$-dimensional subspace. It follows that the dimension of $\mathrm{B}_3^0(g)$ is

$$\frac{1}{3}(4g^3 + 5g + 3) - (2g+1) = \frac{1}{3}(4g^3 - g). \qquad \square$$

1.5. **Separating twists and bounding pairs.** Before giving generators for the kernel of the BCJ homomorphism, we give generators for the Torelli group. For a simple closed curve $x$ on $\Sigma_g^b$, let $T_x \in \mathrm{Mod}_g^b$ be the left Dehn twist about $x$. A *separating twist* is a Dehn twist $T_x$ about a simple closed separating curve $x$. A *bounding pair (BP) map* is a product $T_y T_z^{-1}$ with $y$ and $z$ disjoint nonseparating curves on $\Sigma_g^b$ such that $y \cup z$ separates $\Sigma_g^b$:

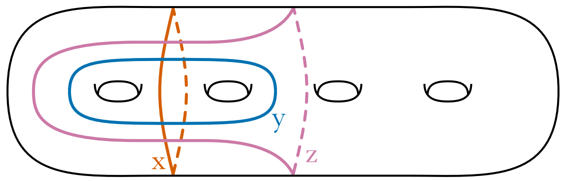



Separating twists and BP maps lie in $\mathcal{I}_g^b$. Building on work of Birman [2], Powell [22] proved that $\mathcal{I}_g^b$ is generated by separating twists and BP maps. See [12, 23] for alternate proofs. For $g \geq 3$, Johnson [17] later proved that $\mathcal{I}_g^b$ is actually generated by finitely many BP maps. Johnson's generating set is enormous – see [24] for a smaller one.

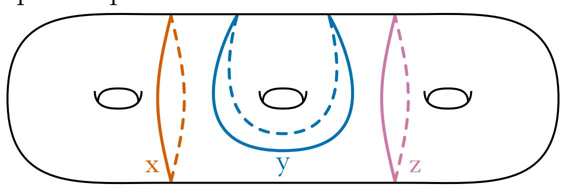
1.6. **Elements of kernel.** Let $\sigma$ be the BCJ homomorphism on $\mathcal{I}_g^b$. We now describe some elements of $\ker(\sigma)$. Since the target of $\sigma$ is an abelian 2-group, all squares of elements of $\mathcal{I}_g^b$ lie in its kernel. In particular, squares of separating twists and squares of BP maps lie in $\ker(\sigma)$. Also, all commutators of elements of $\mathcal{I}_g^b$ lie in $\ker(\sigma)$. A *basic commutator* in $\mathcal{I}_g^b$ is an element of the form $[T_x, T_y T_z^{-1}] \in \ker(\sigma)$, where:

- $T_x$ is a separating twist; and
- $T_y T_z^{-1}$ is a bounding pair map; and
- $x$ intersects both $y$ and $z$ twice.[3]

Here is an example:




For our final element of $\ker(\sigma)$, recall that a *pair of pants* is a 3-holed sphere. A *tri-separating pants map* is a product $T_x T_y T_z$, where $x$ and $y$ and $z$ are disjoint separating curves such that $x \cup y \cup z$ bounds a pair of pants:




We will prove in §2 that tri-separating pants maps lie in $\ker(\sigma)$; see Lemma 3.1.

1.7. **Main theorem.** We can now state our main theorem:

**Theorem A.** *For $g \geq 3$ and $b \in \{0, 1\}$, the kernel of the BCJ homomorphism on $\mathcal{I}_g^b$ is generated by squares of separating twists, squares of BP maps, basic commutators, and tri-separating pants maps.*

As we said above, Johnson [19] proved that for $g \geq 3$ and $b \in \{0, 1\}$ the BCJ homomorphism detects $\mathrm{H}_1(\mathcal{I}_g^b; \mathbb{F}_2)$, so

$$\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{F}_2) \cong \mathrm{B}_3(g) \quad \text{and} \quad \mathrm{H}_1(\mathcal{I}_g; \mathbb{F}_2) \cong \mathrm{B}_3^0(g).$$

Theorem A can be used to give a new proof of this in the following way. For a group $G$, the *mod-2 commutator subgroup* of $G$, denoted $[G, G]_{\mathbb{F}_2}$, is the kernel of the natural map $G \to \mathrm{H}_1(G; \mathbb{F}_2)$. It is generated by commutators $[g_1, g_2]$ and squares $g^2$ with $g, g_1, g_2 \in G$. It is obvious that squares of separating twists, squares of BP maps, and basic commutators lie in $[\mathcal{I}_g^b, \mathcal{I}_g^b]_{\mathbb{F}_2}$. We will prove that tri-separating pants maps lie in $[\mathcal{I}_g^b, \mathcal{I}_g^b]_{\mathbb{F}_2}$; see Lemma 3.2.

For some $g \geq 3$ and $b \in \{0, 1\}$, let $\sigma$ be the BCJ homomorphism on $\mathcal{I}_g^b$. Theorems A and the above discussion imply that $\ker(\sigma) \subset [\mathcal{I}_g^b, \mathcal{I}_g^b]_{\mathbb{F}_2}$. Conversely, since the target of $\sigma$ is an $\mathbb{F}_2$-vector space we have $[\mathcal{I}_g^b, \mathcal{I}_g^b]_{\mathbb{F}_2} \subset \ker(\sigma)$. We deduce:

**Corollary B** (Johnson, [19])**.** *For $g \geq 3$ and $b \in \{0, 1\}$, the kernel of the BCJ homomorphism is $[\mathcal{I}_g^b, \mathcal{I}_g^b]_{\mathbb{F}_2}$. Consequently, the BCJ homomomorphisms induce isomorphisms $\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{F}_2) \cong \mathrm{B}_3(g)$ and $\mathrm{H}_1(\mathcal{I}_g; \mathbb{F}_2) \cong \mathrm{B}_3^0(g)$.*

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[3] If $x$ is disjoint from $y$ and $z$, then $[T_x, T_y T_z^{-1}] = 1$. Since $x$ is separating, if $x$ intersects both $y$ and $z$ then it must intersect each of them at least twice.

1.8. **Johnson homomorphism.** Fix some $g \geq 3$ and $b \in \{0,1\}$. We now describe how Johnson uses Corollary B to calculate $\mathrm{H}_1(\mathcal{I}_g^b)$. The remaining ingredient is the Johnson homomorphism. Let[4] $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^b) \cong \mathbb{Z}^{2g}$. The Johnson homomorphisms are surjective homomorphisms

$$\tau\colon \mathcal{I}_g^1 \to \wedge^3 H_{\mathbb{Z}} \quad \text{and} \quad \tau\colon \mathcal{I}_g \to (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}}.$$

Here $H_{\mathbb{Z}}$ is embedded in $\wedge^3 H_{\mathbb{Z}}$ via the map $h \mapsto h \wedge \omega$, where $\omega \in \wedge^2 H_{\mathbb{Z}} \cong \wedge^2 H_{\mathbb{Z}}^*$ is the symplectic form. See [14] for the original construction, [16] for a survey of applications and other constructions, and [9, §6.6] for a textbook reference. The *Johnson kernel* is the kernel $\mathcal{K}_g^b$ of the Johnson homomorphism $\tau$. We therefore have short exact sequences

$$0 \longrightarrow \mathcal{K}_g^1 \longrightarrow \mathcal{I}_g^1 \longrightarrow \wedge^3 H_{\mathbb{Z}} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{I}_g \longrightarrow (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \longrightarrow 0.$$

Since the target of $\tau$ is abelian, we have $[\mathcal{I}_g^b, \mathcal{I}_g^b] \subset \mathcal{K}_g^b$. Johnson ([18], see [25] for an alternate proof) showed that $\mathcal{K}_g^b$ is generated by separating twists.[5] In addition, for $g \geq 3$ Johnson showed in [18] that for a separating twist $T_x$ we have $T_x^2 \in [\mathcal{I}_g^b, \mathcal{I}_g^b]$. This calculation also appears in [25]. It implies that $\mathcal{K}_g^b/[\mathcal{I}_g^b, \mathcal{I}_g^b]$ is an abelian 2-group. Moreover, since the target of $\tau$ is free abelian we deduce that the torsion subgroup of $\mathrm{H}_1(\mathcal{I}_g^b)$ is exactly the abelian 2-group $\mathcal{K}_g^b/[\mathcal{I}_g^b, \mathcal{I}_g^b]$.

*Remark* 1.2. Recall that Johnson [17] proved that $\mathcal{I}_g^b$ is finitely generated for $g \geq 3$. Recent work of Ershov–He [8] and Church–Ershov–Putman [6] shows that $\mathcal{K}_g^b$ is finitely generated for $g \geq 4$. □

1.9. **Abelianization.** Combined with Corollary B, the above discussion implies that $\mathrm{H}_1(\mathcal{I}_g^b)$ is detected by the Johnson homomorphism $\tau$ (which detects the torsion-free part) and the BCJ homomorphism $\sigma$ (which detects the 2-torsion). Moreover, the torsion subgroup of $\mathrm{H}_1(\mathcal{I}_g^b)$ is the image of $\mathcal{K}_g^b$ under $\sigma$. Johnson proved that there are commutative diagrams

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{K}_g^1 \arrow[r] \arrow[d, two heads, "\sigma"] & \mathcal{I}_g^1 \arrow[r, "\tau"] \arrow[d, two heads, "\sigma"] & \wedge^3 H_{\mathbb{Z}} \arrow[r] \arrow[d, two heads] & 1 \\
0 \arrow[r] & \mathrm{B}_2(g) \arrow[r] & \mathrm{B}_3(g) \arrow[r, "\tau"] & \mathrm{B}_3(g)/\mathrm{B}_2(g) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{K}_g \arrow[r] \arrow[d, two heads, "\sigma"] & \mathcal{I}_g \arrow[r, "\tau"] \arrow[d, two heads, "\sigma"] & (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \arrow[r] \arrow[d, two heads] & 1 \\
0 \arrow[r] & \mathrm{B}_2^0(g) \arrow[r] & \mathrm{B}_3^0(g) \arrow[r, "\tau"] & \mathrm{B}_3^0(g)/\mathrm{B}_2^0(g) \arrow[r] & 0
\end{tikzcd}$$

with the surjective maps $\wedge^3 H_{\mathbb{Z}} \to \mathrm{B}_3(g)/\mathrm{B}_2(g)$ and $(\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \to \mathrm{B}_3^0(g)/\mathrm{B}_2^0(g)$ the maps that reduce the free $\mathbb{Z}$-modules $\wedge^3 H_{\mathbb{Z}}$ and $(\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}}$ modulo 2. From this, we see that $\mathrm{B}_2(g)$ and $\mathrm{B}_2^0(g)$ are isomorphic to the 2-torsion in $\mathrm{H}_1(\mathcal{I}_g^b)$, so:

**Corollary C** (Johnson, [19])**.** *For $g \geq 3$, we have short exact sequences*

$$0 \longrightarrow \mathrm{B}_2(g) \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1) \longrightarrow \wedge^3 H_{\mathbb{Z}} \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{B}_2^0(g) \longrightarrow \mathrm{H}_1(\mathcal{I}_g) \longrightarrow (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \longrightarrow 0.$$

[4] We call this $H_{\mathbb{Z}}$ since in this paper $H$ will always mean $\mathrm{H}_1(\Sigma_g; \mathbb{F}_2) \cong \mathbb{F}_2^{2g}$.

[5] One can view Theorem A as an analogue of this for the BCJ homomorphism.

1.10. **Comments on the proof.** Our proof of Theorem A uses a mixture of topological and representation-theoretic reasoning. For simplicity, we will make a few comments about the proof in the special case of surfaces with one boundary component.

In this case, recall that the theorem says that the kernel of the BCJ homomorphism $\sigma\colon \mathcal{I}_g^1 \to \mathrm{B}_3(g)$ is generated by squares of separating twists, squares of BP maps, basic commutators, and tri-separating pants maps. These purported generators lie in $\ker(\sigma)$ and generate a normal subgroup $N$ of $\mathcal{I}_g^1$. Let $\mathcal{QI}_g^1 = \mathcal{I}_g^1/N$. There is an induced map $\mathcal{QI}_g^1 \to \mathrm{B}_3(g)$, and our theorem can be restated as saying that this induced map is an isomorphism. The proof of this has three steps:

**Step 1**. As we described in §1.9 above there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{K}_g^1 \arrow[r] \arrow[d, two heads, "\sigma"] & \mathcal{I}_g^1 \arrow[r, "\tau"] \arrow[d, two heads, "\sigma"] & \wedge^3 H_{\mathbb{Z}} \arrow[r] \arrow[d, two heads] & 1 \\
0 \arrow[r] & \mathrm{B}_2(g) \arrow[r] & \mathrm{B}_3(g) \arrow[r, "\tau"] & \mathrm{B}_3(g)/\mathrm{B}_2(g) \arrow[r] & 0
\end{tikzcd}$$

with $\mathcal{K}_g^1$ the Johnson kernel subgroup. Let $\mathcal{QK}_g^1$ be the image of $\mathcal{K}_g^1$ in $\mathcal{QI}_g^1$. From the above commutative diagram we have an induced map $\mathcal{QK}_g^1 \to \mathrm{B}_2(g)$. The first step of the proof will be to show that it is enough to prove that this map induces an isomorphism $\mathcal{QK}_g^1 \cong \mathrm{B}_2(g)$.

**Step 2**. The second step establishes some properties of $\mathcal{QK}_g^1$ that mimic properties of $\mathrm{B}_2(g)$:
- $\mathcal{QK}_g^1$ is an abelian 2-group; and
- $\mathcal{QK}_g^1$ has a natural action of $\mathrm{Sp}_{2g}(\mathbb{F}_2)$.

These two properties can be summarized as saying that $\mathcal{QK}_g^1$ is a representation of $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ over the field $\mathbb{F}_2$. The arguments in this step have a topological and group-theoretic flavor, and make use of some ideas from van den Berg's unpublished PhD thesis [1].

**Step 3**. The final step constructs generators and relations for $\mathcal{QK}_g^1$ as a representation of $\mathrm{Sp}_{2g}(\mathbb{F}_2)$. We then show that these generators and relations imply an isomorphism $\mathcal{QK}_g^1 \cong \mathcal{QK}_g^1$ of $\mathrm{Sp}_{2g}(\mathbb{F}_2)$-representations. We use two key tools:
- The theory of central stability [26, 27] from Church–Farb's theory of representation stability [5].
- Recent work of Minahan–Putman [20] giving a framework for establishing presentations of representations of this sort.

To verify some low-complexity cases, we rely on computer calculations.

1.11. **Outline of paper.** The paper is divided into three parts, roughly corresponding to the three steps described above. The first two parts each reduce Theorem A to a different result. Part 1 reduces Theorem A to Theorem A′, and Part 2 reduces Theorem A′ to Theorem A″. Finally, Part 3 proves Theorem A″. Each part begins with an outline.

## Part 1. Preliminary results, and a reduction to a quotient of the Johnson kernel

We start in §2 by giving more details about the BCJ homomorphism. Next, in §3 we prove that tri-separating pants maps lie in both the kernel of the BCJ homomorphism and in the mod-2 commutator subgroup of the Torelli group. Finally, in §4 we reduce Theorem A to Theorem A′, which is a result about a certain quotient of the Johnson kernel subgroup.

## 2. Preliminaries on the Birman–Craggs–Johnson (BCJ) homomorphism

Fix $g \geq 0$. This section surveys Johnson's construction of the BCJ homomorphism [13].

**2.1. Quadratic forms.** This requires some algebraic preliminaries. Let $H = \mathrm{H}_1(\Sigma_g; \mathbb{F}_2) = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$ and let $\widehat{i}(-,-)$ be the $\mathbb{F}_2$-valued algebraic intersection form on $H$. A *quadratic form* on $H$ is a set map $q\colon H \to \mathbb{F}_2$ such that[6]

$$q(x+y) = q(x) + q(y) + \widehat{i}(x,y) \quad \text{for all } x, y \in H.$$

If $S = \{a_1, b_1, \ldots, a_g, b_g\}$ is a symplectic basis for $H$, then for all $\lambda_1, \lambda_1', \ldots, \lambda_g, \lambda_g' \in \mathbb{F}_2$ there exists a unique quadratic form $q\colon H \to \mathbb{F}_2$ with

$$q(a_i) = \lambda_i \quad \text{and} \quad q(b_i) = \lambda_i' \quad \text{for all } 1 \leq i \leq g.$$

Let $\Omega_g$ be the set of quadratic forms on $H$. The sum of two quadratic forms is not a quadratic form. However, let $H^* = \mathrm{Hom}(H, \mathbb{F}_2)$ be the dual of $H$. For a fixed $q_0 \in \Omega_g$, we then have

$$\Omega_g = \{q_0 + \phi \mid \phi \in H^*\}.$$

In other words, $\Omega_g$ is a torsor for $H^*$.

**2.2. Arf invariant.** Consider $q \in \Omega_g$. The *Arf invariant* of $q$, denoted $\mathrm{Arf}(q)$, is defined as follows. Choose a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H$. Then

$$\mathrm{Arf}(q) = q(a_1)q(b_1) + \cdots + q(a_g)q(b_g) \in \mathbb{F}_2.$$

It is not obvious that this is independent of the symplectic basis, but it turns out that it is independent of the symplectic basis. For $c \in \mathbb{F}_2$, let $\Omega_g(c) = \{q \in \Omega_g \mid \mathrm{Arf}(q) = c\}$. The group $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ acts on $\Omega_g$. This action preserves the Arf invariant, and $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ acts transitively on both $\Omega_g(0)$ and $\Omega_g(1)$. In other words, the Arf invariant is a complete invariant for the isomorphism class of a quadratic form on $q$.

*Remark* 2.1. Fix a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H$. Let $q_0 \in \Omega_g(0)$ and $q_1 \in \Omega_g(1)$ be the quadratic forms defined on this basis as follows:

$$q_j(a_1) = q_j(b_1) = j \quad \text{and} \quad q_j(a_i) = q_j(b_i) = 0 \quad \text{for } 2 \leq i \leq g.$$

By what we said above, each $q \in \Omega_g(0)$ is isomorphic to $q_0$ and each $q \in \Omega_g(1)$ is isomorphic to $q_1$. □

**2.3. Polynomial functions.** Let $\Omega_g^*$ be the ring of functions $\mu\colon \Omega_g \to \mathbb{F}_2$. For each $c \in \mathbb{F}_2$, we have the constant function $c \in \Omega_g^*$. Also, for $h \in H$, we have $\overline{h} \in \Omega_g^*$ defined by

$$\overline{h}(q) = q(h) \quad \text{for } q \in \Omega_g.$$

With this notation, we have

$$\overline{h_1 + h_2} = \overline{h}_1 + \overline{h}_2 + \widehat{i}(h_1, h_2) \quad \text{for } h_1, h_2 \in H.$$

For $\mu \in \Omega_g^*$, we have $\mu^2 = \mu$. Motivated by all of this, let $\mathbb{F}_2[\overline{H}]$ be the ring of polynomials in the formal variables $\{\overline{h} \mid h \in H\}$ and let $\mathrm{B}(g)$ be the quotient of $\mathbb{F}_2[\overline{H}]$ by the ideal generated by:[7]

- $f^2 - f$ for $f \in \mathbb{F}_2[\overline{H}]$; and
- $\overline{h_1 + h_2} - (\overline{h}_1 + \overline{h}_2 + \widehat{i}(h_1, h_2))$ for all $h_1, h_2 \in H$.

[6] This could be generalized by replacing $\widehat{i}(-,-)$ with another bilinear form $\omega(-,-)$ on $H$, and then $q$ is said to *refine* the bilinear form $\omega(-,-)$. To avoid complicating our terminology, we will ignore this.

[7] The signs are not necessary since we are working over $\mathbb{F}_2$, but we include them to clarify what is going on.

We then have a well-defined injection $\mathrm{B}(g) \hookrightarrow \Omega_g^*$ taking $\overline{h} \in \overline{H}$ to $\overline{h} \in \Omega_g^*$. For a fixed symplectic basis $S = \{a_1, b_1, \ldots, a_g, b_g\}$, elements of $\mathrm{B}(g)$ can be viewed as "square-free polynomials" in the variables $\overline{S} = \{\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g\}$. Elements of $\mathrm{B}(g)$ do not have a well-defined degree since the relations we impose mix polynomials of degree 0 and 1, but it does make sense to say that $f \in \mathrm{B}(g)$ has degree at most $n$. Let $\mathrm{B}_n(g)$ be the set of $f \in \mathrm{B}(g)$ of degree at most $n$, so we have an increasing chain of vector spaces

$$0 = \mathrm{B}_{-1}(g) \subsetneq \mathrm{B}_0(g) \subsetneq \mathrm{B}_1(g) \subsetneq \cdots \subsetneq \mathrm{B}_{2g}(g) = \mathrm{B}(g).$$

Let $\Omega_g(0)^*$ be the ring of functions $\mu\colon \Omega_g(0) \to \mathbb{F}_2$. Finally, let $\mathrm{B}_n^0(g)$ be the image of $\mathrm{B}_n(g)$ under the restriction map $\Omega_g^* \to \Omega_g^*(0)^*$. For later use, we make the following observation:

**Lemma 2.2.** *Let $g \geq 1$ and let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H = \mathrm{H}_1(\Sigma_g; \mathbb{F}_2)$. Then the kernel of the map $\mathrm{B}_2(g) \to \mathrm{B}_2^0(g)$ is isomorphic to $\mathbb{F}_2$ and is generated by*

$$\kappa = \overline{a}_1 \overline{b}_1 + \cdots + \overline{a}_g \overline{b}_g.$$

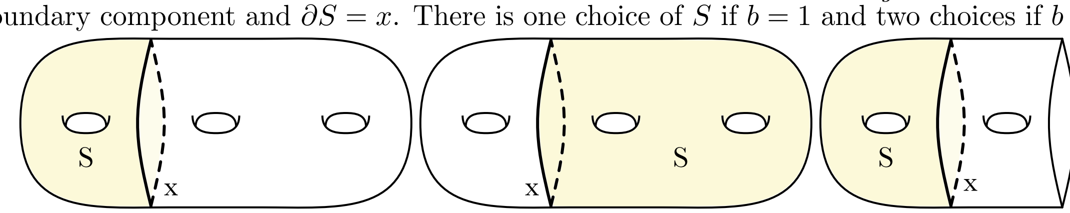
*Proof.* Considered as an element of $\Omega_g^*$, the element $\kappa \in \mathrm{B}_2(g)$ takes $q \in \Omega_g$ to $\mathrm{Arf}(q) \in \mathbb{F}_2$. This implies that $\kappa \neq 0$ and that $\kappa$ lies in the kernel of the map $\mathrm{B}_2(g) \to \mathrm{B}_2^0(g)$. That it generates the kernel is immediate. □

2.4. **BCJ homomorphism.** We now return to the BCJ homomorphism. Recall from §1.3 that the Birman–Craggs homomorphism $\mu_\iota\colon \mathcal{I}_g \to \mathbb{F}_2$ depends on the choice of an embedding $\iota\colon \Sigma_g \hookrightarrow \mathbb{S}^3$ such that $\iota(\Sigma_g)$ is a Heegaard surface. The self-linking form on $\iota(\Sigma_g)$ pulls back to a quadratic form $q_\iota$ on $H = \mathrm{H}_1(\Sigma_g; \mathbb{F}_2)$ with Arf invariant 0, and Johnson [13] proved that $\mu_\iota$ is entirely determined by $q_\iota$. Using this, he proved that there is a homomorphism $\sigma\colon \mathcal{I}_g \to \Omega_g(0)^*$ taking $f \in \mathcal{I}_g$ to the function $\sigma(f) \in \Omega_g(0)^*$ defined as follows:

- Consider $q \in \Omega_g(0)$. Let $\iota\colon \Sigma_g \hookrightarrow \mathbb{S}^3$ be an embedding whose image is a Heegaard surface with associated quadratic form $q_\iota = q$. Then $\sigma(f)(q) = \mu_\iota(f)$.

Finally, Johnson proved that the image of $\sigma$ is exactly $\mathrm{B}_3^0(g)$. This gives the BCJ homomorphism $\sigma\colon \mathcal{I}_g \to \mathrm{B}_3^0(g)$ on a closed surface. For a surface $\Sigma_g^1$ with one boundary component, Johnson embeds $\Sigma_g^1$ into $\Sigma_{g+1}$ and performs the above construction on $\Sigma_{g+1}$. The pulled-back quadratic form on $\mathrm{H}_1(\Sigma_{g+1}; \mathbb{F}_2)$ has Arf invariant 0, but its restriction to $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$ can have any Arf invariant. In this way, Johnson defines a BCJ homomorphism $\sigma\colon \mathcal{I}_g^1 \to \mathrm{B}_3(g)$.

2.5. **Separating twists.** Let $b \in \{0, 1\}$ and let $T_x$ be a separating twist. Johnson proved that $\sigma(T_x)$ has the following description. Let $S$ be a subsurface of $\Sigma_g^b$ with exactly one boundary component and $\partial S = x$. There is one choice of $S$ if $b = 1$ and two choices if $b = 0$:



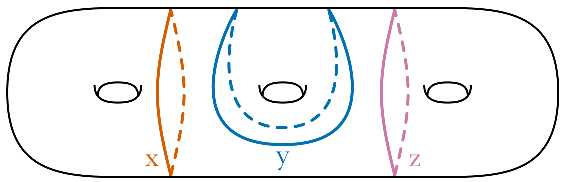

We have a symplectic subspace $\mathrm{H}_1(S; \mathbb{F}_2)$ of $H$. We can restrict any quadratic form $q$ on $H$ to $\mathrm{H}_1(S; \mathbb{F}_2)$, and Johnson [13] proved that $\sigma(f)$ is the map taking a quadratic form $q$ (of Arf invariant 0 if $b = 0$) to the Arf invariant of the restriction of $q$ to $\mathrm{H}_1(S; \mathbb{F}_2)$.

We can write down a formula for this as follows: letting $\{x_1, y_1, \ldots, x_h, y_h\}$ be a symplectic basis for $\mathrm{H}_1(S; \mathbb{F}_2)$, we have $\sigma(f) = \overline{x}_1 \overline{y}_1 + \cdots + \overline{x}_h \overline{y}_h$. Recalling that the Johnson kernel $\mathcal{K}_g^b$ is the subgroup of the Torelli group generated by separating twists, Johnson uses this calculation to show that $\sigma(\mathcal{K}_g^1) = \mathrm{B}_2(g)$ and $\sigma(\mathcal{K}_g) = \mathrm{B}_2^0(g)$.

*Remark* 2.3. Johnson [13] also gave a formula for the value of $\sigma$ on a bounding pair (BP) map. This turns out to lie in either $\mathrm{B}_3(g)$ or $\mathrm{B}_3^0(g)$, and is why $\sigma$ takes $\mathcal{I}_g^1$ onto $\mathrm{B}_3(g)$ and $\mathcal{I}_g$ onto $\mathrm{B}_3^0(g)$. The formula is slightly complicated and we will not need it. □

## 3. Tri-separating pants maps and the BCJ homomorphism

Fix some $g \geq 0$ and $b \in \{0,1\}$. Recall that a *pair of pants* is a 3-holed sphere. A *tri-separating pants map* is a product $T_x T_y T_z \in \mathcal{I}_g^b$, where $x$ and $y$ and $z$ are disjoint separating curves such that $x \cup y \cup z$ bounds a pair of pants. See the figure here:



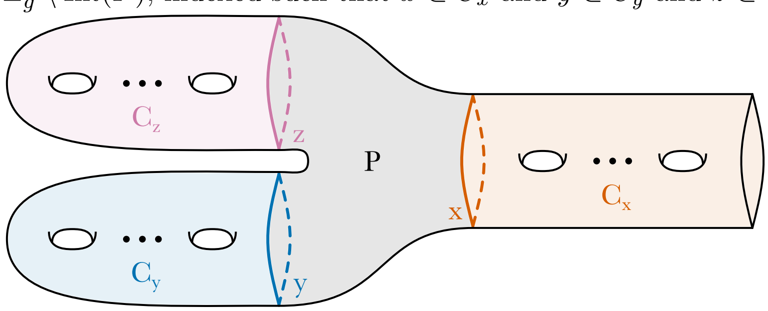

This section proves that tri-separating pants maps lie in the kernel of the BCJ homomorphism, and then proves that they also lie in the mod-2 commutator subgroup of $\mathcal{I}_g^b$.

### 3.1. Kernel of the BCJ homomorphism.

Our first result is as follows:

**Lemma 3.1.** *For some $g \geq 0$ and $b \in \{0,1\}$, let $T_x T_y T_z \in \mathcal{I}_g^b$ be a tri-separating pants map. Then $T_x T_y T_z$ lies in the kernel of the BCJ homomorphism.*

*Proof.* Isotoping $x$ and $y$ and $z$, we can assume that all three curves lie in $\mathrm{Int}(\Sigma_g^b)$. Let $P \subset \Sigma_g^b$ be the pair of pants such that $\partial P = x \sqcup y \sqcup z$. Let $C_x$ and $C_y$ and $C_z$ be the three components of $\Sigma_g^b \setminus \mathrm{Int}(P)$, indexed such that $x \subset C_x$ and $y \subset C_y$ and $z \subset C_z$:




If $b = 1$, then exactly one of $C_x$ and $C_y$ and $C_z$ contains $\partial \Sigma_g^b \cong \mathbb{S}^1$. In that case, just like in the above figure we can reorder $x$ and $y$ and $z$ to ensure that $\partial \Sigma_g^b \subset C_x$. Since $T_x$ and $T_y$ and $T_z$ commute, this reordering does not affect the value of $T_x T_y T_z$.

Let $S = P \cup C_y \cup C_z$. Since $\partial \Sigma_g^b \subset C_x$ if $b = 1$, the following hold:

- $C_y \cong \Sigma_{h_1}^1$ and $C_z \cong \Sigma_{h_2}^1$ and $S \cong \Sigma_{h_1+h_2}^1$; and
- $x, y, z \subset S$ with $x = \partial S$.

We can identify $\mathrm{H}_1(C_y; \mathbb{F}_2) \cong \mathbb{F}_2^{2h_1}$ and $\mathrm{H}_1(C_z; \mathbb{F}_2) \cong \mathbb{F}_2^{2h_2}$ and $\mathrm{H}_1(S; \mathbb{F}_2) \cong \mathbb{F}_2^{2h_1+2h_2}$ with subspaces of $\mathrm{H}_1(\Sigma_g^b; \mathbb{F}_2)$. Under this identification, all three are symplectic subspaces. Moreover, $\mathrm{H}_1(C_y; \mathbb{F}_2)$ and $\mathrm{H}_1(C_z; \mathbb{F}_2)$ are orthogonal symplectic subspaces with $\mathrm{H}_1(S; \mathbb{F}_2) = \mathrm{H}_1(C_y; \mathbb{F}_2) \oplus \mathrm{H}_1(C_z; \mathbb{F}_2)$. Let

$$B = \{x_1, y_1, \ldots, x_{h_1}, y_{h_1}\} \quad \text{and} \quad B' = \{x'_1, y'_1, \ldots, x'_{h_2}, y'_{h_2}\}$$

be symplectic bases for $\mathrm{H}_1(C_y; \mathbb{F}_2)$ and $\mathrm{H}_1(C_z; \mathbb{F}_2)$, respectively. It follows that $B \sqcup B'$ is a symplectic basis for $\mathrm{H}_1(S; \mathbb{F}_2)$. Using the formulas for the BCJ homomorphism from §2.5, we see that

$$\begin{aligned}\sigma(T_xT_yT_z) =&(\overline{x}_1\overline{y}_1 + \cdots + \overline{x}_{h_1}\overline{y}_{h_1} + \overline{x}'_1\overline{y}'_1 + \cdots + \overline{x}'_{h_2}\overline{y}'_{h_2}) \\ &+ (\overline{x}_1\overline{y}_1 + \cdots + \overline{x}_{h_1}\overline{y}_{h_1}) + (\overline{x}'_1\overline{y}'_1 + \cdots + \overline{x}'_{h_2}\overline{y}'_{h_2}) \\ =&2\overline{x}_1\overline{y}_1 + \cdots + 2\overline{x}_{h_1}\overline{y}_{h_1} + 2\overline{x}'_1\overline{y}'_1 + \cdots + 2\overline{x}'_{h_2}\overline{y}'_{h_2} = 0,\end{aligned}$$

as desired. □

3.2. **Tri-separating pants maps and the mod-2 commutator subgroup.** For a group $G$, recall that the mod-2 commutator subgroup of $G$, denoted $[G, G]_{\mathbb{F}_2}$, is the kernel of the natural map $G \to \mathrm{H}_1(G; \mathbb{F}_2)$. It is generated by commutators $[g_1, g_2]$ and squares $g^2$ with $g, g_1, g_2 \in G$. Our second main result in this section is as follows:

**Lemma 3.2.** *For some $g \geq 0$ and $b \in \{0, 1\}$, let $T_xT_yT_z \in \mathcal{I}_g^b$ be a tri-separating pants map. Then $T_xT_yT_z \in [\mathcal{I}_g^b, \mathcal{I}_g^b]_{\mathbb{F}_2}$.*

It is enlightening to prove this directly by using relations in the mapping class group to write $T_xT_yT_z$ as a product of commutators and squares. We give an alternate calculation-free approach.

*Proof of Lemma 3.2.* As in the proof of Lemma 3.1, after possibly reordering $x$ and $y$ and $z$ we can find a subsurface $S$ of $\Sigma_g^b$ such that the following hold:

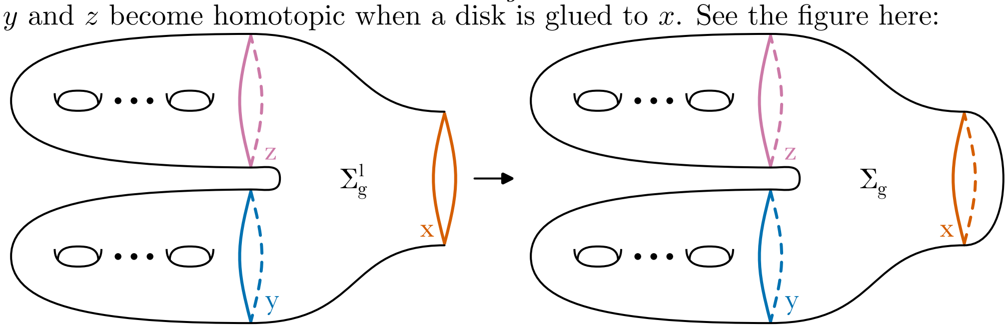
- $S \cong \Sigma_h^1$ for some $h \leq g$; and
- $x, y, z \subset S$ with $x = \partial S$.

Replacing $\Sigma_g^b$ with $S$, we can therefore assume without loss of generality that $b = 1$ and $x = \partial\Sigma_g^1$.

If $y$ bounds a disk in $\Sigma_g^1$, then $T_y = 1$ and $T_x = T_z$. This implies that $T_xT_yT_z = T_x^2$, which lies in $[\mathcal{I}_g^1, \mathcal{I}_g^1]_{\mathbb{F}_2}$. Similarly, if $z$ bounds a disk in $\Sigma_g^1$ then $T_xT_yT_z = T_x^2 \in [\mathcal{I}_g^1, \mathcal{I}_g^1]_{\mathbb{F}_2}$. We can therefore assume that neither $y$ nor $z$ bounds a disk in $\Sigma_g^1$. This implies that $g \geq 2$.

Since $T_z^2 \in [\mathcal{I}_g^1, \mathcal{I}_g^1]_{\mathbb{F}_2}$ and $T_xT_yT_z = (T_xT_yT_z^{-1})T_z^2$, it is enough to prove that $T_xT_yT_z^{-1} \in [\mathcal{I}_g^1, \mathcal{I}_g^1]_{\mathbb{F}_2}$. As notation, for a group $\Gamma$ and $\lambda \in \Gamma$ let $[\lambda]_2$ be the image of $\lambda$ in $\mathrm{H}_1(\Gamma; \mathbb{F}_2)$. Our goal is to prove that the element $T_xT_yT_z^{-1} \in \mathcal{I}_g^1$ satisfies $[T_xT_yT_z^{-1}]_2 = 0$.

Let $\phi\colon \mathrm{Mod}_g^1 \to \mathrm{Mod}_g$ be the map that glues a disk to $x = \partial\Sigma_g^1$ and extends mapping classes over that disk by the identity. Set $\mathcal{D}_g^1 = \ker(\phi)$. The element $T_xT_yT_z^{-1}$ lies in $\mathcal{D}_g^1$ since $y$ and $z$ become homotopic when a disk is glued to $x$. See the figure here:




The group $\mathcal{D}_g^1$ is called the *disk-pushing* subgroup of $\mathrm{Mod}_g^1$. See [9, §4.2.5] for a discussion of it. Elements in $\mathcal{D}_g^1$ "push" the glued-on disk around the surface while allowing it to rotate. Letting $U\Sigma_g$ be the unit tangent bundle of $\Sigma_g$, we have[8] $\mathcal{D}_g^1 \cong \pi_1(U\Sigma_g)$. Moreover, elements of $\mathcal{D}_g^1$ act trivially on $\mathrm{H}_1(\Sigma_g^1)$, so $\mathcal{D}_g^1 \subset \mathcal{I}_g^1$. For $\lambda \in \mathcal{D}_g^1$, let $[\lambda]_2^D$ be the corresponding element of $\mathrm{H}_1(\mathcal{D}_g^1; \mathbb{F}_2)$. From the above, we see that to prove that $[T_xT_yT_z^{-1}]_2 = 0$ in $\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{F}_2)$, it is enough to prove that $[T_xT_yT_z^{-1}]_2^D = 0$.

[8]This is where we use the fact that $g \geq 2$.

Let $\Sigma_{g,1}$ be a genus-$g$ surface with 1 puncture. Let $\mathrm{Mod}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$. The map $\phi\colon \mathrm{Mod}_g^1 \to \mathrm{Mod}_g$ factors as

$$\mathrm{Mod}_g^1 \xrightarrow{\phi'} \mathrm{Mod}_{g,1} \xrightarrow{\phi''} \mathrm{Mod}_g,$$

where $\phi'$ glues a punctured disk to $x = \partial\Sigma_g^1$ and extends mapping classes over that punctured disk by the identity and $\phi''\colon \mathrm{Mod}_{g,1} \to \mathrm{Mod}_g$ fills in the puncture.

Set $\mathcal{P}_{g,1} = \ker(\phi'')$. The group $\mathcal{P}_{g,1}$ is called the *point-pushing* subgroup of $\mathrm{Mod}_{g,1}$. Elements in $\mathcal{P}_{g,1}$ "push" the puncture around the surface, and $\mathcal{P}_{g,1} \cong \pi_1(\Sigma_g)$. The disk-pushing subgroup $\mathcal{D}_g^1$ and the point-pushing subgroup $\mathcal{P}_{g,1}$ fit into the following commutative diagram with exact rows:

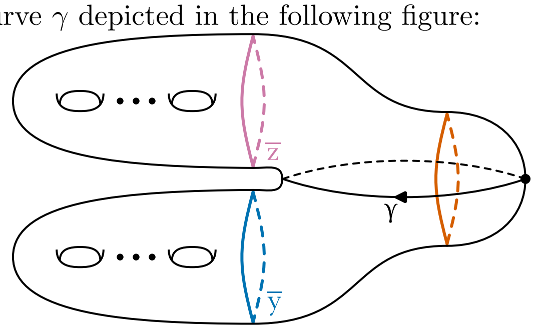
$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathcal{D}_g^1 \arrow[r] \arrow[d, equal] & \mathcal{P}_{g,1} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_1(U\Sigma_g) \arrow[r] & \pi_1(\Sigma_g) \arrow[r] & 1.
\end{tikzcd}$$

Here the rows are central extensions, and the central $\mathbb{Z}$ subgroup of $\mathcal{D}_g^1$ is generated by[9] $T_x$. Passing to $\mathrm{H}_1$ with $\mathbb{F}_2$-coefficients, we get an exact sequence

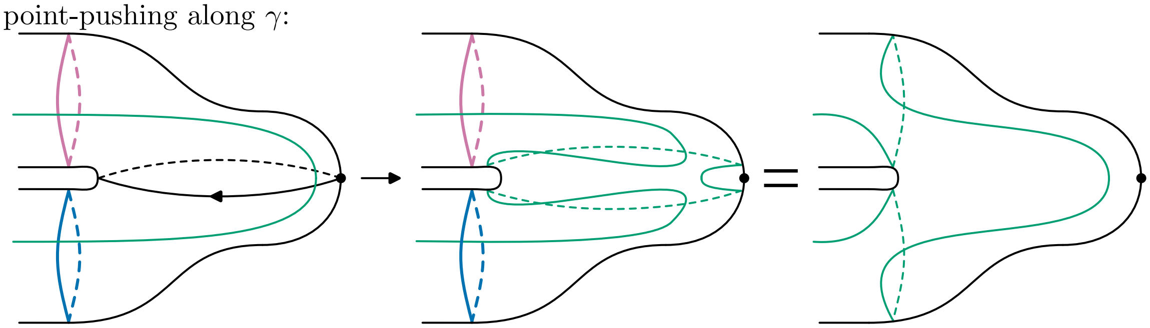
$$\mathbb{F}_2 \longrightarrow \mathrm{H}_1(\mathcal{D}_g^1;\mathbb{F}_2) \longrightarrow \mathrm{H}_1(\mathcal{P}_{g,1};\mathbb{F}_2) \longrightarrow 0, \tag{3.1}$$

where the image of $\mathbb{F}_2$ in $\mathrm{H}_1(\mathcal{D}_g^1;\mathbb{F}_2)$ is generated by $[T_x]_2^D \in \mathrm{H}_1(\mathcal{D}_g^1;\mathbb{F}_2)$.

Recall that our goal is to prove that $[T_xT_yT_z^{-1}]_2^D = 0$ in $\mathrm{H}_1(\mathcal{D}_g^1;\mathbb{F}_2)$. Let $\overline{y}$ and $\overline{z}$ be the images of $y$ and $z$ in $\Sigma_{g,1}$. The image of $T_xT_yT_z^{-1} \in \mathcal{D}_g^1$ in $\mathcal{P}_{g,1}$ is thus $T_{\overline{y}}T_{\overline{z}}^{-1}$. Write $[T_{\overline{y}}T_{\overline{z}}^{-1}]_2^P$ for its image in $\mathrm{H}_1(\mathcal{P}_{g,1};\mathbb{F}_2) = \mathrm{H}_1(\Sigma_g;\mathbb{F}_2)$. We claim that $[T_{\overline{y}}T_{\overline{z}}^{-1}]_2^P = 0$. To see this, since separating curves are null-homologous it is enough to prove that $T_{\overline{y}}T_{\overline{z}}^{-1} \in \mathcal{P}_{g,1} = \pi_1(\Sigma_g)$ equals the separating curve $\gamma$ depicted in the following figure:




This follows from the following picture, which illustrates the effect on a transverse arc of point-pushing along $\gamma$:



[9] Here are some more details. Viewing $\mathcal{D}_g^1$ as the fundamental group of the unit tangent bundle, the central $\mathbb{Z}$ is the loop that rotates the base unit tangent vector around the fiber. The associated mapping class rotates the boundary component, and is thus $T_x$.

Since $[T_x T_y T_z^{-1}]_2^D$ lies in the kernel of the map $\mathrm{H}_1(\mathcal{D}_g^1; \mathbb{F}_2) \to \mathrm{H}_1(\mathcal{P}_{g,1}; \mathbb{F}_2)$ from (3.1), we conclude that either

$$[T_x T_y T_z^{-1}]_2^D = [T_x]_2^D \quad \text{or} \quad [T_x T_y T_z^{-1}]_2^D = 0.$$

However, we cannot have $[T_x T_y T_z^{-1}]_2^D = [T_x]_2$ since $T_x T_y T_z^{-1}$ lies in the kernel of the BCJ homomorphism (cf. Lemma[10] 3.1), while $T_x$ does not (see the formulas in §2.5). We conclude that $[T_x T_y T_z^{-1}]_2^D = 0$, as desired. □

## 4. Quotients and a reduction

Fix some $g \geq 0$ and $b \in \{0, 1\}$. Recall that Theorem A says that for $g \geq 3$ the kernel of the BCJ homomorphism on $\mathcal{I}_g^b$ is generated by squares of separating twists, squares of BP maps, basic commutators,[11] and tri-separating pants maps. In this section, we reduce this to a statement about a certain quotient of the Johnson kernel.

4.1. **Johnson homomorphisms and kernel.** Let $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^b) \cong \mathbb{Z}^{2g}$. Embed $H_{\mathbb{Z}}$ into $\wedge^2 H_{\mathbb{Z}}$ via the map $h \mapsto h \wedge \omega$, where $\omega \in \wedge^2 H_{\mathbb{Z}} \cong \wedge^2 H_{\mathbb{Z}}^*$ is the symplectic form. Johnson ([14]; see [9, §6.6] for a textbook reference) constructed the Johnson homomorphisms

$$\tau \colon \mathcal{I}_g^1 \to \wedge^3 H_{\mathbb{Z}} \quad \text{and} \quad \tau \colon \mathcal{I}_g \to (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}}.$$

The Johnson kernel $\mathcal{K}_g^b$ is the kernel of the Johnson homomorphism on $\mathcal{I}_g^b$. We therefore have exact sequences

$$0 \longrightarrow \mathcal{K}_g^1 \longrightarrow \mathcal{I}_g^1 \xrightarrow{\ \tau\ } \wedge^3 H_{\mathbb{Z}} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{I}_g \xrightarrow{\tau} (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \longrightarrow 0.$$

Johnson [18] proved that $\mathcal{K}_g^b$ is the subgroup of $\mathcal{I}_g^b$ generated by separating twists.

4.2. **Quotients of Torelli and the Johnson kernel.** Let $\mathcal{QI}_g^b$ be the quotient of $\mathcal{I}_g^b$ by the subgroup generated by squares of separating twists, squares of BP maps, basic commutators, and tri-separating pants maps. Note that this is a normal subgroup, so $\mathcal{QI}_g^b$ is a group. Let $\mathcal{QK}_g^b$ be the image of $\mathcal{K}_g^b$ in $\mathcal{QI}_g^b$. The group $\mathcal{QK}_g^b$ is a normal subgroup of $\mathcal{QI}_g^b$.

4.3. **Nature of quotients.** Let $\tau$ be the Johnson homomorphism on $\mathcal{I}_g^b$, so $\mathcal{K}_g^b = \ker(\tau)$. Since the target of $\tau$ is abelian, all commutators (and in particular, all basic commutators) lie in $\mathcal{K}_g^b$. Also, $\mathcal{K}_g^b$ is generated by separating twists, and in particular $\mathcal{K}_g^b$ contains all squares of separating twists and all tri-separating pants maps. Finally, Johnson's calculations in [14] also show that $\tau$ takes the subgroup generated by squares of bounding pair maps to

$$2 \wedge^3 H_{\mathbb{Z}} \quad \text{or} \quad 2(\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}},$$

depending on whether we are working in $\mathcal{I}_g^1$ or $\mathcal{I}_g$. Setting $H = \mathrm{H}_1(\Sigma_g^b; \mathbb{F}_2)$, we have

$$\left(\wedge^3 H_{\mathbb{Z}}\right)/2\left(\wedge^3 H_{\mathbb{Z}}\right) \cong \wedge^3 H,$$

and similarly for $(\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}}$. We deduce the following lemma:

[10] That lemma states that $T_x T_y T_z$ lies in the kernel of the BCJ homomorphism $\sigma$, but since $T_z^{-2} \in \ker(\sigma)$ it follows that $T_x T_y T_z^{-1} \in \ker(\sigma)$ as well.

[11] We will remind the reader of the definition of a basic commutator when we make use of them.

**Lemma 4.1.** *Let $g \geq 0$. Set $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1) = \mathrm{H}_1(\Sigma_g)$ and $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2) = \mathrm{H}_1(\Sigma_g; \mathbb{F}_2)$. We then have commutative diagrams with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_g^1 \arrow[r] \arrow[d] & \mathcal{I}_g^1 \arrow[r] \arrow[d] & \wedge^3 H_{\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{QK}_g^1 \arrow[r] & \mathcal{QI}_g^1 \arrow[r] & \wedge^3 H \arrow[r] & 0 \end{tikzcd}$$

*and*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_g \arrow[r] \arrow[d] & \mathcal{I}_g \arrow[r] \arrow[d] & (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{QK}_g \arrow[r] & \mathcal{QI}_g \arrow[r] & (\wedge^3 H)/H \arrow[r] & 0. \end{tikzcd}$$

4.4. **Induced BCJ homomorphisms.** Recall that the BCJ homomorphisms are of the form $\sigma\colon \mathcal{I}_g^1 \to \mathrm{B}_3(g)$ and $\sigma\colon \mathcal{I}_g \to \mathrm{B}_3^0(g)$. The kernel of $\sigma$ contains squares of separating twists, squares of BP maps, basic commutators, and tri-separating pants maps (see Lemma 3.1). It follows that the BCJ homomorphisms factor through maps $\sigma\colon\ \mathcal{QI}_g^1 \to \mathrm{B}_3(g)$ and $\sigma\colon\ \mathcal{QI}_g \to \mathrm{B}_3^0(g)$ that we will call the *induced BCJ homomorphisms*. These are related to the exact sequences in Lemma 4.1 as follows:

**Lemma 4.2.** *Let $g \geq 0$. Set $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2) = \mathrm{H}_1(\Sigma_g; \mathbb{F}_2)$. We then have commutative diagrams with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{QK}_g^1 \arrow[r] \arrow[d, "\sigma"] & \mathcal{QI}_g^1 \arrow[r] \arrow[d, "\sigma"] & \wedge^3 H \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{B}_2(g) \arrow[r] & \mathrm{B}_3(g) \arrow[r] & \mathrm{B}_3(g)/\mathrm{B}_2(g) \arrow[r] & 0 \end{tikzcd}$$

*and*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{QK}_g \arrow[r] \arrow[d, "\sigma"] & \mathcal{QI}_g \arrow[r] \arrow[d, "\sigma"] & (\wedge^3 H)/H \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{B}_2^0(g) \arrow[r] & \mathrm{B}_3^0(g) \arrow[r] & \mathrm{B}_3^0(g)/\mathrm{B}_2^0(g) \arrow[r] & 0. \end{tikzcd}$$

*Proof.* Immediate from Lemma 4.1 and Johnson's work in [13]. □

4.5. **Alternate main theorem.** Parts 2 and 3 of this paper are devoted to proving the following theorem:

**Theorem A′.** *For $g \geq 3$, the induced BCJ homomorphism $\sigma\colon\ \mathcal{QI}_g^1 \to \mathrm{B}_3(g)$ restricts to an isomorphism $\sigma\colon\ \mathcal{QK}_g^1 \to \mathrm{B}_2(g)$.*

The rest of this section shows how to use Theorem A′ to prove Theorem A.

4.6. **Deleting the boundary.** The first step is to prove that Theorem A′ implies a similar theorem for closed surfaces:

**Lemma 4.3.** *For some $g \geq 3$, assume that the induced BCJ homomorphism $\sigma\colon\ \mathcal{QI}_g^1 \to \mathrm{B}_3(g)$ restricts to an isomorphism $\sigma\colon\ \mathcal{QK}_g^1 \to \mathrm{B}_2(g)$. Then the induced BCJ homomorphism $\sigma\colon\ \mathcal{QI}_g \to \mathrm{B}_3^0(g)$ also restricts to an isomorphism $\sigma\colon\ \mathcal{QK}_g \to \mathrm{B}_2^0(g)$.*

*Proof.* We have a commutative diagram

$$\begin{tikzcd}
\mathcal{QK}_g^1 \arrow[r, "\sigma", "\cong"'] \arrow[d, two heads] & \mathrm{B}_2(g) \arrow[d, two heads] \\
\mathcal{QK}_g \arrow[r, "\sigma"] & \mathrm{B}_2^0(g).
\end{tikzcd}$$

To prove the lemma, we must therefore prove that the elements of $\mathcal{QK}_g^1$ corresponding to elements of the kernel of the projection $\mathrm{B}_2(g) \twoheadrightarrow \mathrm{B}_2^0(g)$ lie in the kernel of the projection $\mathcal{QK}_g^1 \twoheadrightarrow \mathcal{QK}_g$. Fix a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$. Lemma 2.2 says that the kernel of the projection $\mathrm{B}_2(g) \twoheadrightarrow \mathrm{B}_2^0(g)$ is isomorphic to $\mathbb{F}_2$ and is generated by the element

$$\kappa = \overline{a}_1\overline{b}_1 + \cdots + \overline{a}_g\overline{b}_g.$$

Letting $x = \partial\Sigma_g^1$, the formulas in §2 imply that $\sigma(T_x) = \kappa$. The lemma therefore follows from the fact that $T_x$ is in the kernel of the map $\mathcal{K}_g^1 \to \mathcal{K}_g$. □

4.7. **Proof of main theorem.** We now show how to use Theorem A′ to prove Theorem A, whose statement we recall:

**Theorem A.** *For $g \geq 3$ and $b \in \{0, 1\}$, the kernel of the BCJ homomorphism on $\mathcal{I}_g^b$ is generated by squares of separating twists, squares of BP maps, basic commutators, and tri-separating pants maps.*

*Proof, assuming Theorem A′.* It is enough to prove that the induced BCJ homomorphisms $\sigma\colon \mathcal{QI}_g^1 \to \mathrm{B}_3(g)$ and $\sigma\colon \mathcal{QI}_g \to \mathrm{B}_3^0(g)$ are isomorphisms. Letting $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$, Lemma 4.2 implies that there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{QK}_g^1 \arrow[r] \arrow[d, "\sigma"] & \mathcal{QI}_g^1 \arrow[r] \arrow[d, "\sigma"] & \wedge^3 H \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{B}_2(g) \arrow[r] & \mathrm{B}_3(g) \arrow[r] & \mathrm{B}_3(g)/\mathrm{B}_2(g) \arrow[r] & 0.
\end{tikzcd}$$

Theorem A′ implies $\sigma\colon \mathcal{QK}_g^1 \to \mathrm{B}_2(g)$ is an isomorphism, so by the five lemma we conclude that $\sigma\colon \mathcal{QI}_g^1 \to \mathrm{B}_3(g)$ is an isomorphism, as desired. The proof that $\sigma\colon \mathcal{QI}_g \to \mathrm{B}_3^0(g)$ is an isomorphism is similar, using Lemma 4.3 together with Theorem A′ to see that $\sigma\colon \mathcal{QK}_g \to \mathrm{B}_2^0(g)$ is an isomorphism. □

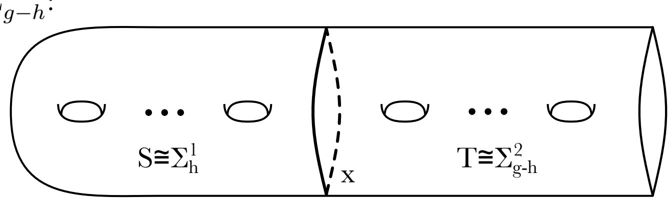
## Part 2. The nature of the quotient

It remains to prove Theorem A′. This part of the paper studies properties of $\mathcal{QK}_g^1$ and reduces Theorem A′ to Theorem A″, which is a description of the target of the BCJ homomorphism by generators and relations. We start in §5 by proving that $\mathcal{QK}_g^1$ is an abelian 2-group equipped with an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. We then prove in §6 that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{QK}_g^1$ factors through $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$. Using this, we reduce Theorem A′ to Theorem A″ in §7.

### 5. The group $\mathcal{QK}_g^1$ is an abelian 2-group

Our main goal in this section is to prove that $\mathcal{QK}_g^1$ is an abelian 2-group for $g \geq 3$.

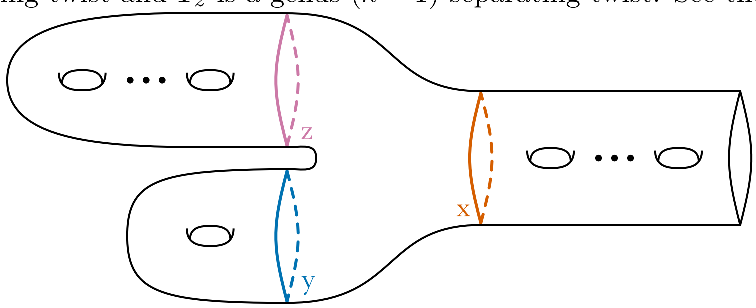
5.1. **Generation by genus-1 separating twists.** This requires two lemmas. A *genus-$h$ separating twist* in $\mathrm{Mod}_g^1$ is a separating twist $T_x$ such that $x$ separates $\Sigma_g^1$ into subsurfaces $S \cong \Sigma_h^1$ and $T \cong \Sigma_{g-h}^2$:




For $\lambda \in \mathcal{K}_g^1$, let $[\lambda]_Q$ be its image in $\mathcal{QK}_g^1$. Our first lemma is as follows:

**Lemma 5.1.** *For $g \geq 1$, the group $\mathcal{QK}_g^1$ is generated by the set of all $[T_x]_Q$ such that $T_x$ is a genus-1 separating twist in $\mathrm{Mod}_g^1$.*

*Proof.* Johnson [18] proved that $\mathcal{K}_g^1$ is generated by all separating twists. Letting $T_x$ be a genus-$h$ separating twist, it is enough to prove that $[T_x]_Q$ can be written as a product of terms of the form $[T_{x'}]_Q$ with $T_{x'}$ a genus-1 separating twist. If $h = 1$ then we are done, so assume that $h \geq 2$. We can then find a tri-separating pants map $T_x T_y T_z$ such that $T_y$ is a genus-1 separating twist and $T_z$ is a genus-$(h-1)$ separating twist. See the figure here:




When we formed $\mathcal{QK}_g^1$, we quotiented $\mathcal{K}_g^1$ by a subgroup that included all tri-separating pants maps. It follows that $[T_x T_y T_z]_Q = 1$, so

$$[T_x]_Q = [T_z]_Q^{-1}[T_y]_Q^{-1} = [T_z]_Q[T_y]_Q.$$

The second equality[12] uses the fact that we quotiented by squares of separating twists when we formed $\mathcal{QK}_g^1$, so for a separating twist $T_w$ we have $[T_w^2]_Q = 1$ and hence $[T_w]_Q^{-1} = [T_w]_Q$. The mapping class $T_y$ is a genus-1 separating twist, and by induction $[T_z]_Q$ can be written as a product of terms of the form $[T_{z'}]_Q$ with $T_{z'}$ a genus-1 separating twist. The lemma follows. $\square$

5.2. **Trivial Torelli action.** The conjugation action of $\mathrm{Mod}_g^1$ on its normal subgroup $\mathcal{K}_g^1$ descends to an action of $\mathrm{Mod}_g^1$ on $\mathcal{QK}_g^1$. We next prove that the restriction of this action to $\mathcal{I}_g^1$ is trivial:

**Lemma 5.2.** *For $g \geq 3$, the action of $\mathcal{I}_g^1$ on $\mathcal{QK}_g^1$ is trivial.*

*Proof.* Let $T_x$ be a genus-1 separating twist. By Lemma 5.1 it is enough to prove that $\mathcal{I}_g^1$ acts trivially on $[T_x]_Q$. Let $\Gamma < \mathcal{I}_g^1$ be the stabilizer of $[T_x]_Q$, i.e., the subgroup of all $f \in \mathcal{I}_g^1$ such that $[fT_xf^{-1}]_Q = [T_x]_Q$, or equivalently such that $[T_xfT_x^{-1}f^{-1}]_Q = 1$. Our goal is to prove that $\Gamma = \mathcal{I}_g^1$. To do this, we will prove that $\Gamma$ contains a generating set for $\mathcal{I}_g^1$.

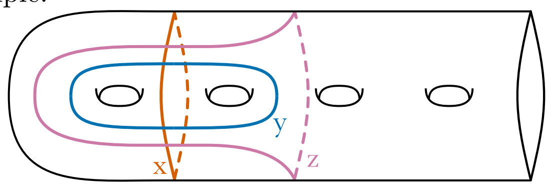
We first enumerate some elements of $\Gamma$:

[12] Strictly speaking the second equality is unnecessary, but it seems strange to not acknowledge it.

- If $f \in \mathcal{I}_g^1$ satisfies $f(x) = x$, then

$$[fT_x f^{-1}]_Q = [T_{f(x)}]_Q = [T_x]_Q$$

  and thus $f \in \Gamma$. In particular, $\Gamma$ contains all BP maps $T_y T_z^{-1}$ such that $y$ and $z$ are both disjoint from $x$.
- Now assume that $T_y T_z^{-1}$ is a BP map such that $x$ intersects both $y$ and $z$ twice. Here is an example:




  By definition, the element

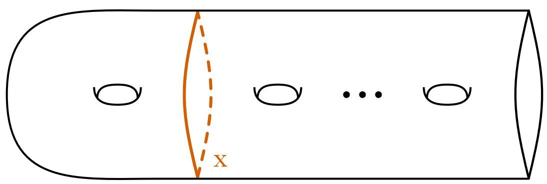
$$T_x(T_y T_z^{-1})T_x^{-1}(T_y T_z^{-1})^{-1} \in \mathcal{I}_g^1$$

  is a basic commutator. When we formed $\mathcal{QK}_g^1$, we quotiented $\mathcal{K}_g^1$ by a subgroup that included all the basic commutators. It follows that

$$[T_x(T_y T_z^{-1})T_x^{-1}(T_y T_z^{-1})^{-1}]_Q = 1,$$

  so $[(T_y T_z^{-1})T_x(T_y T_z^{-1})^{-1}]_Q = [T_x]_Q$. It follows that $T_y T_z^{-1} \in \Gamma$.

Summarizing, the group $\Gamma < \mathcal{I}_g^1$ contains the set $S$ of all BP maps $T_y T_z^{-1}$ such that $y$ and $z$ are either both disjoint from $x$ or both intersect $x$ twice. It follows from work of Johnson [17] that $S$ generates $\mathcal{I}_g^1$, so $\Gamma = \mathcal{I}_g^1$. In fact, what Johnson proved is as follows. Draw $T_x$ as follows:




Johnson [17] constructed a finite generating set $S'$ for $\mathcal{I}_g^1$ consisting of BP maps $T_y T_z^{-1}$ that intersect $x$ as described above, so $S' < S$. The lemma follows. □

5.3. **Abelian group.** The above lemma has the following corollary:

**Corollary 5.3.** *For $g \geq 3$, the following hold:*
- *The group $\mathcal{QK}_g^1$ is an abelian 2-group.*
- *The action of $\mathrm{Mod}_g^1$ on $\mathcal{QK}_g^1$ factors through an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

*Proof.* That the action of $\mathrm{Mod}_g^1$ on $\mathcal{QK}_g^1$ factors through an action of

$$\mathrm{Mod}_g^1 / \mathcal{I}_g^1 = \mathrm{Sp}_{2g}(\mathbb{Z})$$

is immediate from Lemma 5.2. That lemma also implies that the subgroup $\mathcal{K}_g^1$ of $\mathcal{I}_g^1$ acts trivially on $\mathcal{QK}_g^1$. For $x, y \in \mathcal{QK}_g^1$, this implies that $xyx^{-1} = y$, so $xy = yx$. We conclude that $\mathcal{QK}_g^1$ is abelian. We will henceforth write elements of it using additive notation.

Lemma 5.1 says that $\mathcal{K}_g^1$ is generated by elements of the form $[T_x]_Q$, where $T_x$ is a genus-1 separating twist. When we formed $\mathcal{QK}_g^1$, we quotiented $\mathcal{K}_g^1$ by a subgroup that included all squares of separating twists. For a separating twist $T_x$, this implies that $2[T_x]_Q = 0$. We conclude that $\mathcal{QK}_g^1$ is an abelian 2-group, as desired. □

## 6. The group $\mathcal{QK}_g^1$ is a representation of $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$

Fix some $g \geq 3$. Our next goal is to prove that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{QK}_g^1$ factors through $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$. As notation, let $H_\mathbb{Z} = \mathrm{H}_1(\Sigma_g^1)$ and let $\omega$ be the $\mathbb{Z}$-valued algebraic intersection form on $H_\mathbb{Z}$.

6.1. **Symplectic summands and separating twists.** A *symplectic summand* of $H_\mathbb{Z}$ is a direct summand $V_\mathbb{Z}$ of $H_\mathbb{Z}$ such that the restriction of $\omega$ to $V_\mathbb{Z}$ is symplectic, i.e., such that $\omega$ identifies $V_\mathbb{Z}$ with its dual $V_\mathbb{Z}^* = \mathrm{Hom}(V, \mathbb{Z})$. Equivalently, letting $\perp$ be the orthogonal complement with respect to $\omega$ we have $H_\mathbb{Z} = V_\mathbb{Z} \oplus V_\mathbb{Z}^\perp$. We remark that if $V_\mathbb{Z}$ is a subgroup of $H_\mathbb{Z}$ such that the restriction of $\omega$ to $V_\mathbb{Z}$ is symplectic, then $V_\mathbb{Z}$ is automatically a direct summand and $H_\mathbb{Z} = V_\mathbb{Z} \oplus V_\mathbb{Z}^\perp$.

One way these arise is as follows. Let $T_x$ be a genus-$h$ separating twist in $\mathcal{I}_g^1$. The curve $x$ separates $\Sigma_g^1$ into a subsurface $S$ with $S \cong \Sigma_h^1$ and a subsurface $T$ with $T \cong \Sigma_{g-h}^2$. It is immediate that $\mathrm{H}_1(S)$ is a symplectic summand of $H_\mathbb{Z}$; indeed, letting $B < \mathrm{H}_1(T)$ be the subspace spanned by the boundary components, we have a decomposition[13]

$$H_\mathbb{Z} = \mathrm{H}_1(S) \oplus (\mathrm{H}_1(T)/B)$$

that is orthogonal with respect to $\omega$. We will call $\mathrm{H}_1(S)$ the symplectic summand *associated* to $T_x$.

6.2. **Symplectic summand generators.** Let $V_\mathbb{Z}$ be a nonzero symplectic summand of $H_\mathbb{Z}$. Johnson [15] proved that there is a separating twist $T_x$ such that $V_\mathbb{Z}$ is the symplectic summand associated to $T_x$. Johnson [15] also proved that if $T_{x'}$ is another separating twist such that $V_\mathbb{Z}$ is the symplectic summand associated to $T_{x'}$, then there exists some $f \in \mathcal{I}_g^1$ such that $f(x) = x'$. Since $\mathcal{I}_g^1$ acts trivially on $\mathcal{QK}_g^1$ (see Lemma 5.2), we have

$$[T_x]_Q = [f T_x f^{-1}]_Q = [T_{f(x)}]_Q = [T_{x'}]_Q.$$

It follows that the element $[T_x]_Q \in \mathcal{QK}_g^1$ only depends on $V_\mathbb{Z}$. We will denote it by $[V_\mathbb{Z}]_Q$. We have $V_\mathbb{Z} \cong \mathbb{Z}^{2h}$ for some $h \geq 1$, and we will call $h$ the *genus* of $V_\mathbb{Z}$. The following is immediate from Lemma 5.1:

**Lemma 6.1.** *For $g \geq 3$, the group $\mathcal{QK}_g^1$ is generated by the set of all $[V_\mathbb{Z}]_Q$ such that $V_\mathbb{Z}$ is a genus-1 symplectic summand of $H_\mathbb{Z} = \mathrm{H}_1(\Sigma_g^1)$.*

6.3. **Relations between symplectic summand generators.** These elements satisfy the following relation. Say that symplectic summands $V_\mathbb{Z}, W_\mathbb{Z} < H_\mathbb{Z}$ are *orthogonal* if $\omega(v, w) = 0$ for all $v \in V_\mathbb{Z}$ and $w \in W_\mathbb{Z}$. This implies that $V_\mathbb{Z} \oplus W_\mathbb{Z}$ is a symplectic summand of $H_\mathbb{Z}$.

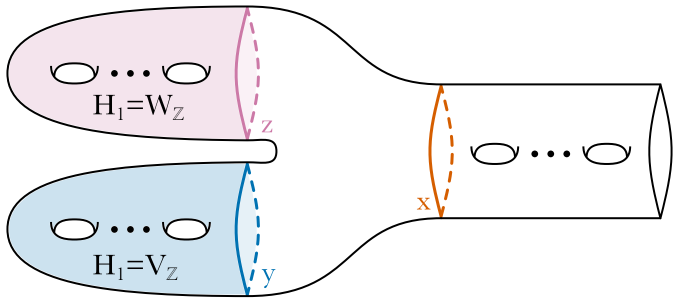
**Lemma 6.2.** *Let $g \geq 3$. Set $H_\mathbb{Z} = \mathrm{H}_1(\Sigma_g^1)$, and let $V_\mathbb{Z}, W_\mathbb{Z} < H_\mathbb{Z}$ be nonzero orthogonal symplectic summands. We then have $[V_\mathbb{Z}]_Q + [W_\mathbb{Z}]_Q = [V_\mathbb{Z} \oplus W_\mathbb{Z}]_Q$.*

*Proof.* The same argument Johnson used in [15] to prove that there is a separating twist whose associated symplectic summand is a given symplectic summand shows more generally that there exist disjoint simple closed separating curves $y$ and $z$ such that:

- $V_\mathbb{Z}$ is the symplectic summand associated to $T_y$; and
- $W_\mathbb{Z}$ is the symplectic summand associated to $T_z$.

We can then find a simple closed separating curve $x$ that is disjoint from $y$ and $z$ such that $x \cup y \cup z$ bounds a pair of pants. See the following figure:

[13]The reason we quotient by $B$ here is that the map $\mathrm{H}_1(T) \to \mathrm{H}_1(\Sigma_g^1)$ is not injective.




As this figure shows, the symplectic summand associated to $T_x$ is $V_{\mathbb{Z}} \oplus W_{\mathbb{Z}}$. When we formed $\mathcal{QK}_g^1$, we quotiented $\mathcal{I}_g^1$ by a subgroup contains all tri-separating pants maps. The image in $\mathcal{QK}_g^1$ of the tri-separating pants map $T_x T_y T_z$ is thus 0. It follows that

$$0 = [T_x T_y T_z]_Q = [T_x]_Q + [T_y]_Q + [T_z]_Q = [V_{\mathbb{Z}} \oplus W_{\mathbb{Z}}]_Q + [V_{\mathbb{Z}}]_Q + [W_{\mathbb{Z}}]_Q.$$

Since $\mathcal{QK}_g^1$ is an abelian 2-group, this implies that that $[V_{\mathbb{Z}}]_Q + [W_{\mathbb{Z}}]_Q = [V_{\mathbb{Z}} \oplus W_{\mathbb{Z}}]_Q$, as desired. ☐

6.4. **Factoring through** $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$**.** We now prove that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{QK}_g^1$ factors through $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$. The calculation underlying our proof first appeared in van den Berg's unpublished thesis [1].

**Lemma 6.3.** *Let $g \geq 3$. The action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{QK}_g^1$ factors through $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$.*

*Proof.* Set $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1)$, and let $\omega$ be the algebraic intersection pairing on $H_{\mathbb{Z}}$. For $v \in H_{\mathbb{Z}}$, let $\tau_v \in \mathrm{Sp}_{2g}(\mathbb{Z})$ be the associated symplectic transvection, so

$$\tau_v(h) = h + \omega(v,h)v \quad \text{for all } h \in H_{\mathbb{Z}}.$$

The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by the set of symplectic transvections $\tau_v$ such that $v \in H_{\mathbb{Z}}$ is primitive, that is, not divisible by any $d \geq 2$. We start with the following relation in $\mathcal{QK}_g^1$:

**Claim.** *Let $V_{\mathbb{Z}}$ be a nonzero symplectic summand of $H_{\mathbb{Z}}$. Let $a_1, a_2 \in V$ be elements such that $\omega(a_1, a_2) = 0$ and such that their span $\langle a_1, a_2 \rangle$ is a rank-2 direct summand of $V_{\mathbb{Z}}$. Also, let $w \in V_{\mathbb{Z}}^{\perp}$. Then for all $n, m \in \mathbb{Z}$ we have*

$$[\tau_{na_1+ma_2+w}(V_{\mathbb{Z}})]_Q + [\tau_{na_1+w}(V_{\mathbb{Z}})]_Q + [\tau_{ma_2+w}(V_{\mathbb{Z}})]_Q + [V_{\mathbb{Z}}]_Q = 0.$$

*Proof of claim.* Complete the $a_i$ to a symplectic basis $\{a_1, b_1, \ldots, a_h, b_h\}$ for $V_{\mathbb{Z}}$. Set $I = \{a_3, b_3, \ldots, a_h, b_h\}$. Using the relation from Lemma 6.2, we have

$$\begin{aligned}
(6.1) \qquad & [\tau_{na_1+ma_2+w}(V_{\mathbb{Z}})]_Q \\
&= [\langle a_1, b_1 + n(na_1 + ma_2 + w), a_2, b_2 + m(na_1 + ma_2 + w), I \rangle]_Q \\
&= [\langle a_1, b_1 + nw, a_2, b_2 + mw, I \rangle]_Q \\
&= [\langle a_1, b_1 + nw \rangle]_Q + [\langle a_2, b_2 + mw, I \rangle]_Q.
\end{aligned}$$

Similarly, we have

$$[\tau_{na_1+w}(V_{\mathbb{Z}})]_Q = [\langle a_1, b_1 + nw \rangle]_Q + [\langle a_2, b_2, I \rangle]_Q, \tag{6.2}$$

$$[\tau_{ma_2+w}(V_{\mathbb{Z}})]_Q = [\langle a_1, b_1 \rangle]_Q + [\langle a_2, b_2 + mw, I \rangle]_Q, \tag{6.3}$$

$$[V_{\mathbb{Z}}]_Q = [\langle a_1, b_1 \rangle]_Q + [\langle a_2, b_2, I \rangle]_Q. \tag{6.4}$$

Using the fact that $\mathcal{QK}_g^1$ is an abelian 2-group, we can add (6.1) and (6.2) and (6.3) and (6.4) and deduce that

$$\begin{aligned}
&[\tau_{na_1+ma_2+w}(V_{\mathbb{Z}})]_Q + [\tau_{na_1+w}(V_{\mathbb{Z}})]_Q + [\tau_{ma_2+w}(V_{\mathbb{Z}})]_Q + [V_{\mathbb{Z}}]_Q \\
&= 2[\langle a_1, b_1 + nw \rangle]_Q + 2[\langle a_2, b_2 + mw, I \rangle]_Q + 2[\langle a_1, b_1 \rangle]_Q + 2[\langle a_2, b_2, I \rangle]_Q = 0,
\end{aligned}$$

as desired. □

We now turn to the proof of the lemma. We must prove that the kernel of the mod-2 reduction map $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/2)$ acts trivially on $\mathcal{QK}_g^1$. This kernel is called the *level-2 congruence subgroup* of $\mathrm{Sp}_{2g}(\mathbb{Z})$, and is denoted $\mathrm{Sp}_{2g}(\mathbb{Z}, 2)$. It is generated by transvections $\tau_{2v}$ with $v \in H_{\mathbb{Z}}$ primitive; see [21]. Fixing a primitive $v \in H_{\mathbb{Z}}$, we must therefore prove that $\tau_{2v}$ acts trivially on $\mathcal{QK}_g^1$.

Let $V_{\mathbb{Z}}$ be a symplectic summand of $H_{\mathbb{Z}}$. By Lemma 6.1, it is enough to prove that $\tau_{2v}$ fixes $[V_{\mathbb{Z}}]_Q$. Below we will prove this when $V_{\mathbb{Z}}$ has genus at least 2. To see that this implies the general case, assume that $V_{\mathbb{Z}}$ has genus 1. Using the relation from Lemma 6.2, we have

$$[V_{\mathbb{Z}}]_Q + [V_{\mathbb{Z}}^{\perp}]_Q = [H_{\mathbb{Z}}]_Q.$$

Since $g \geq 3$, both $V_{\mathbb{Z}}^{\perp}$ and $H_{\mathbb{Z}}$ have genus at least 2. If $\tau_{2v}$ fixes $[V_{\mathbb{Z}}^{\perp}]_Q$ and $[H_{\mathbb{Z}}]_Q$, it will also fix $[V_{\mathbb{Z}}]_Q$.

We have now reduced ourselves to the case where $V_{\mathbb{Z}}$ has genus at least 2. Write $2v = 2na_1 + w$ with $a_1 \in V_{\mathbb{Z}}$ primitive and $n \in \mathbb{Z}$ and $w \in V_{\mathbb{Z}}^{\perp}$. Complete $\{a_1\}$ to a symplectic basis $\{a_1, b_1, \ldots, a_h, b_h\}$ for $V_{\mathbb{Z}}$. By assumption we have $h \geq 2$, so it makes sense to discuss $a_2$. Using the above claim three times, we have

$$\begin{aligned}
&[\tau_{2na_1+w}(V_{\mathbb{Z}})]_Q + [\tau_{a_2+w}(V_{\mathbb{Z}})]_Q + [V_{\mathbb{Z}}]_Q \\
&= [\tau_{2na_1+a_2+w}(V_{\mathbb{Z}})]_Q \\
&= [\tau_{na_1+(na_1+a_2)+w}(V_{\mathbb{Z}})]_Q \\
&= [\tau_{na_1+w}(V_{\mathbb{Z}})]_Q + [\tau_{na_1+a_2+w}(V_{\mathbb{Z}})]_Q + [V_{\mathbb{Z}}]_Q \\
&= [\tau_{na_1+w}(V_{\mathbb{Z}})]_Q + [\tau_{na_1+w}(V_{\mathbb{Z}})]_Q + [\tau_{a_2+w}(V_{\mathbb{Z}})]_Q + [V_{\mathbb{Z}}]_Q + [V_{\mathbb{Z}}]_Q \\
&= [\tau_{a_2+w}(V_{\mathbb{Z}})]_Q.
\end{aligned}$$

Subtracting $[\tau_{a_2+w}(V_{\mathbb{Z}})]_Q$ from both sides of this, we get that $[\tau_{2na_1+w}(V_{\mathbb{Z}})]_Q + [V_{\mathbb{Z}}]_Q = 0$. Since $2v = 2na_1 + w$, it follows that $[\tau_{2v}(V_{\mathbb{Z}})]_Q = [V_{\mathbb{Z}}]_Q$, as desired. □

## 7. Reduction to generators and relations for $\mathrm{B}_2(g)$

Fix $g \geq 3$. This section reduces Theorem A′ to Theorem A″, which gives a description of the target of the BCJ homomorphism by generators and relations.

7.1. **Symplectic subspaces and orthogonality.** Set $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$. This is equipped with an $\mathbb{F}_2$-valued symplectic form $\widehat{i}(-,-)$. We will use $\widehat{i}(-,-)$ to define orthogonal complements and orthogonality for subspaces of $H$. A *symplectic subspace* of $H$ is a subspace $V < H$ on which the symplectic form restricts to a symplectic form. Since $H \cong \mathbb{F}_2^{2g}$ is a vector space, a symplectic subspace $V$ of $H$ is automatically a direct summand and satisfies $H = V \oplus V^{\perp}$. We have $V \cong \mathbb{F}_2^{2h}$ for some $h \leq g$ called the *genus* of $V$.

7.2. **Mod-2 symplectic summand generators and relations.** Let $V$ be a nonzero symplectic subspace of $H$. We can lift $V$ to a symplectic summand $V_{\mathbb{Z}}$ of $H_{\mathbb{Z}}$ that projects to $V$ under the map $H_{\mathbb{Z}} \to H$. This gives us a generator $[V_{\mathbb{Z}}]_Q$ of $\mathcal{QK}_g^1$. If $V'_{\mathbb{Z}}$ is another symplectic summand of $H_{\mathbb{Z}}$ that projects to $V$, then there exists some

$$f \in \mathrm{Sp}_{2g}(\mathbb{Z}, 2) = \ker(\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/2))$$

such that $f(V_{\mathbb{Z}}) = V'_{\mathbb{Z}}$. Since the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{QK}_g^1$ factors through $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$ (Lemma 6.3), we have

$$[V_{\mathbb{Z}}]_Q = [f(V_{\mathbb{Z}})]_Q = [V'_{\mathbb{Z}}]_Q.$$

In other words, the element $[V_{\mathbb{Z}}]_Q$ of $\mathcal{QK}_g^1$ only depends on $V$. We will denote it by $[V]_Q$. We have the following

**Lemma 7.1.** *Let $g \geq 3$ and let $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$. The following hold:*

*(i) The group $\mathcal{QK}_g^1$ is generated by elements of the form $[V]_Q$ with $V$ a genus-1 symplectic subspace of $H$.*

*(ii) Let $V, W < H$ be nonzero orthogonal symplectic subspaces. We then have $[V]_Q + [W]_Q = [V \oplus W]_Q$.*

*Proof.* These follow from Lemmas 6.1 and 6.2, which are the corresponding results for symplectic summands of $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1)$. □

7.3. **Vector space given by generators and relations.** Recall that $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$. Define $\mathfrak{QK}_g^1$ to be the $\mathbb{F}_2$-vector space given by generators and relations as follows:

- The generators of $\mathfrak{QK}_g^1$ are formal symbols $[\![V]\!]$ with $V < H$ a nonzero symplectic subspace.
- The relations of $\mathfrak{QK}_g^1$ are as follows. Let $V, W < H$ be nonzero orthogonal symplectic subspaces. We then have the relation $[\![V]\!] + [\![W]\!] = [\![V \oplus W]\!]$.

By Lemma 7.1, there is a map $\rho\colon \mathfrak{QK}_g^1 \to \mathcal{QK}_g^1$ defined on generators as follows:

$$\rho([\![V]\!]) = [V]_Q.$$

It will follow from our results below that this is an isomorphism.

Recall that the induced BCJ homomorphism is of the form $\sigma\colon \mathcal{QK}_g^1 \to \mathrm{B}_2(g)$. We will recall the form of its target $\mathrm{B}_2(g)$ later when we start doing explicit calculations. We will call the map $\beta = \sigma \circ \rho$ from $\mathfrak{QK}_g^1$ to $\mathrm{B}_2(g)$ the *lifted BCJ homomorphism.*

7.4. **Alternate main theorem, II.** Part 3 of this paper is devoted to proving the following theorem:[14]

**Theorem A″.** *For $g \geq 2$, the lifted BCJ homomorphism $\beta\colon \mathfrak{QK}_g^1 \to \mathrm{B}_2(g)$ is an isomorphism.*

As we will see, this theorem will quickly imply Theorem A′.

7.5. **Proof of main theorem, II.** We recall the statement of Theorem A′:

**Theorem A′.** *For $g \geq 3$, the induced BCJ homomorphism $\sigma\colon \mathcal{QI}_g^1 \to \mathrm{B}_3(g)$ restricts to an isomorphism $\sigma\colon \mathcal{QK}_g^1 \to \mathrm{B}_2(g)$.*

*Proof, assuming Theorem A″.* Let $\rho\colon \mathfrak{QK}_g^1 \to \mathcal{QK}_g^1$ defined on generators as follows:

$$\rho([\![V]\!]) = [V]_Q.$$

The map $\rho$ is surjective, and by Theorem A″ the composition

$$\mathfrak{QK}_g^1 \xrightarrow{\rho} \mathcal{QK}_g^1 \xrightarrow{\sigma} \mathrm{B}_2(g).$$

is an isomorphism. This implies that $\sigma$ is an isomorphism, as desired. □

## Part 3. Generators and relations

This final part of the paper proves Theorem A″, which by our work over the past two parts implies Theorem A. There are four sections. In §8, we recall some details about the BCJ homomorphism and its target. Next, in §9 we prove Theorem A″ in the special cases $g = 2$ and $g = 3$. These two cases will form the base of our inductive proof of the general case. We then describe an inductive presentation of the target of the BCJ homomorphism in §10. Finally, we prove Theorem A″ in §11.

[14] Our previous results required $g \geq 3$, but this theorem is true when $g = 2$ and it will be convenient to start our induction at $g = 2$.

## 8. Preliminaries on the lifted BCJ homomorphism

Fix some $g \geq 0$. The lifted BCJ homomorphism is of the form $\beta\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathrm{B}_2(g)$. This section recalls the structure of $\mathrm{B}_2(g)$ and establishes a few preliminary results. Set $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$ and let $\widehat{i}(-,-)$ be the $\mathbb{F}_2$-valued intersection form on $H$.

8.1. **Reminder of target of BCJ homomorphism.** Let $\mathbb{F}_2[\overline{H}]$ be the ring of polynomials in the formal variables $\{\overline{h} \mid h \in H\}$ and let $\mathrm{B}(g)$ be the quotient of $\mathbb{F}_2[\overline{H}]$ by the ideal generated by:

- $f^2 - f$ for $f \in \mathbb{F}_2[\overline{H}]$; and
- $\overline{h_1 + h_2} - (\overline{h}_1 + \overline{h}_2 + \widehat{i}(h_1, h_2))$ for all $h_1, h_2 \in H$.

For a fixed symplectic basis $S = \{a_1, b_1, \ldots, a_g, b_g\}$, elements of $\mathrm{B}(g)$ can be viewed as "square-free polynomials" in the variables $\overline{S} = \{\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g\}$. Elements of $\mathrm{B}(g)$ do not have a well-defined degree since the relations we impose mix polynomials of degree 0 and 1, but it does make sense to say that $f \in \mathrm{B}(g)$ has degree at most $n$. Let $\mathrm{B}_n(g)$ be the set of $f \in \mathrm{B}(g)$ of degree at most $n$, so we have an increasing chain of vector spaces

$$0 = \mathrm{B}_{-1}(g) \subsetneq \mathrm{B}_0(g) \subsetneq \mathrm{B}_1(g) \subsetneq \cdots \subsetneq \mathrm{B}_{2g}(g) = \mathrm{B}(g).$$

The dimension of $\mathrm{B}_n(g)$ is

$$\binom{2g}{0} + \binom{2g}{1} + \cdots + \binom{2g}{n}.$$

In particular, $\mathrm{B}_2(g)$ has dimension

$$\binom{2g}{0} + \binom{2g}{1} + \binom{2g}{2} = 1 + 2g + \frac{2g(2g-1)}{2} = 2g^2 + g + 1.$$

8.2. **Reminder about $\mathfrak{Q}\mathfrak{K}_g^1$.** Recall that $\mathfrak{Q}\mathfrak{K}_g^1$ is the $\mathbb{F}_2$-vector space given by generators and relations as follows:

- The generators of $\mathfrak{Q}\mathfrak{K}_g^1$ are formal symbols $[\![V]\!]$ with $V < H$ a nonzero symplectic subspace.
- The relations of $\mathfrak{Q}\mathfrak{K}_g^1$ are as follows. Let $V, W < H$ be nonzero orthogonal symplectic subspaces. We then have the relation $[\![V]\!] + [\![W]\!] = [\![V \oplus W]\!]$.

Consider a generator $[\![V]\!]$ of $\mathfrak{Q}\mathfrak{K}_g^1$. Let $\{x_1, y_1, \ldots, x_h, y_h\}$ be a symplectic basis for $V$. From the formulas for the BCJ homomorphism from §2.5, we see that the lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathrm{B}_2(g)$ is given by the following formula:

$$\beta([\![V]\!]) = \overline{x}_1\overline{y}_1 + \cdots + \overline{x}_h\overline{y}_h. \tag{8.1}$$

8.3. **Surjectivity.** We close this section with the following observation:

**Lemma 8.1.** *For $g \geq 2$, the lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathrm{B}_2(g)$ is surjective.*

It is enlightening to prove Lemma 8.1 directly using the formula (8.1). We explain how to derive it from Johnson's work.

*Proof of Lemma 8.1.* Johnson [13] proved that the ordinary BCJ homomorphism $\sigma\colon \mathcal{K}_g^1 \to \mathrm{B}_2(g)$ is surjective. This implies that the induced BCJ homomorphism $\sigma\colon \mathcal{QK}_g^1 \to \mathrm{B}_2(g)$ is surjective. The lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathrm{B}_2(g)$ equals the composition

$$\mathfrak{Q}\mathfrak{K}_g^1 \xrightarrow{\rho} \mathcal{QK}_g^1 \xrightarrow{\sigma} \mathrm{B}_2(g),$$

where $\rho\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathcal{QK}_g^1$ is the map described in §7.3. The map $\rho$ is surjective; see Lemma 7.1. The lemma follows. $\square$

## 9. Theorem A$''$ when $g = 2$ and $g = 3$

Recall that Theorem A$''$ says that the lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathrm{B}_2(g)$ is an isomorphism for $g \geq 2$. This section establishes this for $g = 2$ and $g = 3$.

### 9.1. Genus two.

We start with $g = 2$:

**Lemma 9.1.** *The lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_2^1 \to \mathrm{B}_2(2)$ is an isomorphism.*

*Proof.* The dimension of $\mathrm{B}_2(2)$ is

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 1 + 4 + 6 = 11.$$

Since $\beta$ is surjective (Lemma 8.1), it is enough to prove that $\mathfrak{Q}\mathfrak{K}_2^1$ has dimension at most 11. The $\mathbb{F}_2$-vector space $\mathfrak{Q}\mathfrak{K}_2^1$ has two kinds of generators:

- Letting $H = \mathrm{H}_1(\Sigma_2^1; \mathbb{F}_2)$, a single generator $[\![H]\!]$.
- For each genus-1 symplectic subspace $V \cong \mathbb{F}^2$ of $H$, a generator $[\![V]\!]$. The group $\mathrm{Sp}_4(\mathbb{F}_2)$ acts transitively on such symplectic subspaces, and the stabilizer of $V$ is isomorphic to $\mathrm{Sp}_2(\mathbb{F}_2) \times \mathrm{Sp}_2(\mathbb{F}_2)$ since the stabilizer preserves the decomposition $H = V \oplus V^{\perp}$. It follows that there are[15]
$$\frac{|\mathrm{Sp}_4(\mathbb{F}_2)|}{|\mathrm{Sp}_2(\mathbb{F}_2)|^2} = \frac{(2^4-1)2^3(2^2-1)2}{((2^2-1)2)^2} = \frac{15 \times 8 \times 3 \times 2}{6^2} = \frac{720}{36} = 20$$
such generators.

For each genus-1 symplectic subspace $V$ of $H$, we have a single relation

$$[\![V]\!] + [\![V^{\perp}]\!] = [\![H]\!].$$

These are all the relations in $\mathfrak{Q}\mathfrak{K}_2^1$. They allow us to eliminate exactly one of $[\![V]\!]$ or $[\![V^{\perp}]\!]$ for each genus-1 symplectic subspace $V$. After doing this, half of the 20 generators $[\![V]\!]$ survive. We conclude that the dimension of $\mathfrak{Q}\mathfrak{K}_2^1$ is at most $10 + 1 = 11$, as desired. $\square$

### 9.2. Genus three.

We next handle the case $g = 3$:

**Lemma 9.2.** *The lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_3^1 \to \mathrm{B}_2(3)$ is an isomorphism.*

*Proof.* The dimension of $\mathrm{B}_2(3)$ is

$$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} = 1 + 6 + 15 = 22.$$

Since $\beta$ is surjective (Lemma 8.1), it is enough to prove that $\mathfrak{Q}\mathfrak{K}_3^1$ is at most 22-dimensional. As we will explain below, we verified this on a computer. The $\mathbb{F}_2$-vector space $\mathfrak{Q}\mathfrak{K}_3^1$ has three kinds of generators:

- Letting $H = \mathrm{H}_1(\Sigma_3^1; \mathbb{F}_2)$, a single generator $[\![H]\!]$.
- For each genus-1 symplectic subspace $V \cong \mathbb{F}^2$ of $H$, a generator $[\![V]\!]$. The group $\mathrm{Sp}_6(\mathbb{F}_2)$ acts transitively on such symplectic subspaces, and the stabilizer of $V$ is isomorphic to $\mathrm{Sp}_2(\mathbb{F}_2) \times \mathrm{Sp}_4(\mathbb{F}_2)$ since the stabilizer preserves the decomposition $H = V \oplus V^{\perp}$. It follows that there are
$$\begin{aligned}\frac{|\mathrm{Sp}_6(\mathbb{F}_2)|}{|\mathrm{Sp}_2(\mathbb{F}_2)| \times |\mathrm{Sp}_4(\mathbb{F}_2)|} &= \frac{(2^6-1)2^5(2^4-1)2^3(2^2-1)2}{(2^2-1)2 \times (2^4-1)2^3(2^2-1)2} \\ &= \frac{63 \times 32 \times 15 \times 8 \times 3 \times 2}{3 \times 2 \times 15 \times 8 \times 3 \times 2} = \frac{1451520}{4320} = 336\end{aligned}$$
such generators.

[15] Here we are using the standard fact that for $g \geq 1$ and $p \geq 2$ a prime we have $|\mathrm{Sp}_{2g}(\mathbb{F}_p)| = (p^{2g} - 1)p^{2g-1}(p^{2g-2}-1)p^{2g-3} \cdots (p^2-1)p$.

- For each genus-2 symplectic subspace $W \cong \mathbb{F}^4$ of $H$, a generator $[\![W]\!]$. Since $W^\perp$ is a rank-1 symplectic subspace, these are in bijection with the rank-1 symplectic subspaces. We therefore conclude that there are 336 such generators.

Since $\beta([\![H]\!]) \neq 0$, the generator $[\![H]\!]$ is nonzero and spans a 1-dimensional subspace of $\mathfrak{QK}_3^1$. Let $\Lambda$ be the quotient of $\mathfrak{QK}_3^1$ by the subspace spanned by $[\![H]\!]$. It is enough to prove that $\Lambda$ is at most 21-dimensional. For $x \in \mathfrak{QK}_3^1$, let $\overline{x}$ be the image of $x$ in $\Lambda$.

For a genus-2 symplectic subspace $W$ of $H$, we have a relation

$$\overline{[\![W]\!]} + \overline{[\![W^\perp]\!]} = \overline{[\![H]\!]} = 0$$

in $\Lambda$. Since $W^\perp$ is a genus-1 symplectic subspace of $H$, this allows us to eliminate all the generators $\overline{[\![W]\!]}$. In other words, $\Lambda$ is spanned by the generators $\overline{[\![V]\!]}$ as $V$ ranges over the 336 genus-1 symplectic subspaces of $H$.

For an orthogonal decomposition $H = V_1 \oplus V_2 \oplus V_3$ with each $V_i$ a genus-1 symplectic subspace of $H$, we have a relation

$$\overline{[\![V_1]\!]} + \overline{[\![V_2]\!]} + \overline{[\![V_3]\!]} = \overline{[\![H]\!]} = 0$$

in $\Lambda$. Let $\Lambda'$ be the $\mathbb{F}_2$-vector space with the following presentation:

- The generators of $\Lambda'$ are formal symbols $\langle\!\langle V \rangle\!\rangle$ with $V$ a genus-1 symplectic subspace of $H$.
- The relations of $\Lambda'$ are as follows. For each orthogonal decomposition $H = V_1 \oplus V_2 \oplus V_3$ with the $V_i$ genus-1 symplectic subspaces of $H$, we have a relation $\langle\!\langle V_1 \rangle\!\rangle + \langle\!\langle V_2 \rangle\!\rangle + \langle\!\langle V_3 \rangle\!\rangle = 0$.

We have a surjection $\Lambda' \to \Lambda$ taking a generator $\langle\!\langle V \rangle\!\rangle$ to $\overline{[\![V]\!]}$, so it is enough to prove that $\Lambda'$ has dimension at most 21.

As we observed above, there are 336 generators $\langle\!\langle V \rangle\!\rangle$ of $\Lambda'$. The group $\mathrm{Sp}_6(\mathbb{F}_2)$ acts transitively on (unordered) orthogonal decompositions $H = V_1 \oplus V_2 \oplus V_3$ with each $V_i$ a genus-1 symplectic subspace of $H$. The stabilizer of one such decomposition is isomorphic to an extension of $\mathrm{Sp}_2(\mathbb{F}_2) \times \mathrm{Sp}_2(\mathbb{F}_2) \times \mathrm{Sp}_2(\mathbb{F}_2)$ by the symmetric group $\mathfrak{S}_3$ on 3 letters.[16] It follows that the following counts the number of relations in $\Lambda'$:

$$\begin{aligned}\frac{|\mathrm{Sp}_6(\mathbb{F}_2)|}{|\mathrm{Sp}_2(\mathbb{F}_2)|^3 \times |\mathfrak{S}_3|} &= \frac{(2^6-1)2^5(2^4-1)2^3(2^2-1)2}{((2^2-1)2)^3 \times 6} \\ &= \frac{63 \times 32 \times 15 \times 8 \times 3 \times 2}{6^3 \times 6} = \frac{1451520}{1296} = 1120.\end{aligned}$$

Using software written in C++ by[17] the authors [4], we enumerated these 336 generators and 1120 relations as follows:

(a) Each two-dimensional subspace of $H \cong \mathbb{F}_2^6$ can be uniquely represented as the row space of a $2 \times 6$ matrix $M$ over $\mathbb{F}_2$ that is in reduced row echelon form. In other words, for some $1 \leq p < q \leq 6$ we have:
  - The first nonzero entry of row 1 appears in position $p$ and the first nonzero entry of row 2 appears in position $q$; and
  - The entry in position $q$ of row 1 is 0.

  Here are some examples of matrices with this shape:

$$\begin{pmatrix} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

[16] The symmetric group $\mathfrak{S}_3$ appears since we are considering unordered decompositions. Reordering the $V_i$ does not change the relation.

[17] This code was written and tested entirely by the authors. No AI software was used.

Such a matrix represents a genus-1 symplectic subspace if and only if the algebraic intersection pairing between its rows is $1 \in \mathbb{F}_2$.

(b) The program first enumerates all possible matrices as in the previous bullet point. For each, it checks if the algebraic intersection pairing between its rows is 1. If so, it adds it to the list of generators.

(c) For each $1 \leq i < j < k \leq 336$, the program then checks to see if the $i^{\text{th}}$ and $j^{\text{th}}$ and $k^{\text{th}}$ generators represent orthogonal genus-1 symplectic subspaces. If so, it adds a relation to the list of relations.

The program outputs a $336 \times 1120$ matrix with entries in $\mathbb{F}_2$. Each column has exactly three nonzero entries. It then calculates the dimension of the column space of this matrix, which turns out to be 315. It follows that $\Lambda'$ is $336 - 315 = 21$ dimensional, as desired. □

## 10. Inductive presentation of $\mathrm{B}_2(g)$

Let $g \geq 2$, and set $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$. This section introduces an inductive description of $\mathrm{B}_2(g)$. Our proof of the general case of Theorem A$''$ in the next section proceeds by showing that $\mathfrak{Q}\mathfrak{K}_g^1$ has a similar inductive description for $g \geq 4$.

10.1. **Notation for $\mathrm{B}_n(g)$.** Let $U$ be an $\mathbb{F}_2$-vector space equipped with a symplectic form $\widehat{i}(-,-)$; for instance, $U$ might a symplectic subspace of $H$. Generalizing our construction of $\mathrm{B}(g)$, we define $\mathrm{B}(U)$ to be the quotient of the ring $\mathbb{F}_2[\overline{U}]$ of polynomials in the formal variables $\{\overline{u} \mid u \in U\}$ by the ideal generated by:

- $f^2 - f$ for $f \in \mathbb{F}_2[\overline{U}]$; and
- $\overline{u_1 + u_2} - (\overline{u}_1 + \overline{u}_2 + \widehat{i}(u_1, u_2))$ for all $u_1, u_2 \in U$.

Letting $h$ be the genus of $U$, this has a filtration

$$0 = \mathrm{B}_{-1}(U) \subsetneq \mathrm{B}_0(U) \subsetneq \mathrm{B}_1(U) \subsetneq \cdots \subsetneq \mathrm{B}_{2h}(U) = \mathrm{B}(U),$$

where $\mathrm{B}_n(U)$ is the image in $\mathrm{B}(U)$ of the set of polynomials in $\mathbb{F}_2[\overline{U}]$ of degree at most $n$.

This construction is functorial in the following sense. Let $U$ and $U'$ be $\mathbb{F}_2$-vector spaces equipped with symplectic forms and let $\iota \colon U \to U'$ be a linear map preserving the symplectic form. This implies that $\iota$ is injective. The map $\iota \colon U \to U'$ induces a map $\mathrm{B}_n(U) \to \mathrm{B}_n(U')$ that is also easily seen to be injective. We will use this functoriality frequently when $U$ is a symplectic subspace of $U'$, in which case the map $\iota \colon U \to U'$ is the inclusion.

10.2. **Standard symplectic decomposition and deleted subspaces.** Fix a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H$, which we will call the *standard symplectic basis*. For $1 \leq i \leq g$, let $H(i) = \langle a_i, b_i \rangle$. This is a genus-1 symplectic subspace of $H$, and we have an orthogonal decomposition

$$H = H(1) \oplus H(2) \oplus \cdots \oplus H(g).$$

For distinct $1 \leq i_1, \ldots, i_k \leq g$, let $H(\widehat{i}_1, \ldots, \widehat{i}_k)$ be the subspace of $H$ obtained by deleting the terms $H(i_j)$ from the above decomposition, so

$$H(\widehat{i}_1, \ldots, \widehat{i}_k) = \bigoplus_{\substack{1 \leq i \leq g \\ i \neq i_1, \ldots, i_g}} H(i).$$

Finally, define

$$\mathrm{B}_n(g; \widehat{i}_1, \ldots, \widehat{i}_k) = \mathrm{B}_n(H(\widehat{i}_1, \ldots, \widehat{i}_k)).$$

There is thus an induced injective map $\mathrm{B}_n(g; \widehat{i}_1, \ldots, \widehat{i}_k) \to \mathrm{B}_n(g)$.

10.3. **A potential presentation.** Since we will only need to understand $\mathrm{B}_2(g)$, we now restrict to this case. We start with the map

$$\epsilon\colon \bigoplus_{1\leq i\leq g} \mathrm{B}_2(g;\widehat{i}) \to \mathrm{B}_2(g)$$

induced by the inclusions. Our goal is to determine the kernel of this map. Consider some $1 \leq j < k \leq g$. Let $f'\colon \mathrm{B}_2(g;\widehat{j},\widehat{k}) \to \mathrm{B}_2(g;\widehat{j})$ and $f''\colon \mathrm{B}_2(g;\widehat{j},\widehat{k}) \to \mathrm{B}_2(g;\widehat{k})$ be the two inclusions. We have a commutative diagram

$$\begin{tikzcd} & \mathrm{B}_2(g;\widehat{j}) \arrow[dr] & \\ \mathrm{B}_2(g;\widehat{j},\widehat{k}) \arrow[ur, "f'"] \arrow[dr, "f''"] & & \mathrm{B}_2(g) \\ & \mathrm{B}_2(g;\widehat{k}) \arrow[ur] & \end{tikzcd}$$

Let $d_{jk}\colon \mathrm{B}_2(g;\widehat{j},\widehat{k}) \to \oplus_{i=1}^{g} \mathrm{B}_2(g;\widehat{i})$ be the composition of the map

$$(f',f'')\colon \mathrm{B}_2(g;\widehat{j},\widehat{k}) \to \mathrm{B}_2(g;\widehat{j}) \oplus \mathrm{B}_2(g;\widehat{k})$$

with the inclusion of the two-term direct sum into the $g$-term direct sum, and let

$$d = \bigoplus_{1\leq j<k\leq g} d_{jk}\colon \bigoplus_{1\leq j<k\leq g} \mathrm{B}_2(g;\widehat{j},\widehat{k}) \to \bigoplus_{1\leq i\leq g} \mathrm{B}_2(g;\widehat{i}).$$

Since we are working over $\mathbb{F}_2$, the image of $d$ is contained in the kernel of $\epsilon$, i.e., the composition

$$\bigoplus_{1\leq j<k\leq g} \mathrm{B}_2(g;\widehat{j},\widehat{k}) \xrightarrow{d} \bigoplus_{1\leq i\leq g} \mathrm{B}_2(g;\widehat{i}) \xrightarrow{\epsilon} \mathrm{B}_2(g)$$

is the zero map.

10.4. **The presentation.** Our main result in this section is that the above is a presentation of $\mathrm{B}_2(g)$. We remark that this is exactly the kind of presentation that appears in the theory of *central stability* for sequences of representations of the symmetric group. This was introduced in [26] and further developed in [27], and gives one way of formalizing Church–Farb's notion of representation stability [5].

**Lemma 10.1.** *Let the notation be as above, and assume that $g \geq 3$. The following is then exact:*

$$\bigoplus_{1\leq j<k\leq g} \mathrm{B}_2(g;\widehat{j},\widehat{k}) \xrightarrow{d} \bigoplus_{1\leq i\leq g} \mathrm{B}_2(g;\widehat{i}) \xrightarrow{\epsilon} \mathrm{B}_2(g) \to 0.$$

*Proof.* We must prove that $\epsilon$ is surjective and that $\mathrm{Im}(d) = \ker(\epsilon)$. Let $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{F}_2)$ and let $S = \{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for $H$. Define $\overline{S} = \{\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g\}$, and set

$$B = \{1\} \cup \overline{S} \cup \left\{\overline{s}\overline{t} \mid \overline{s}, \overline{t} \in \overline{S} \text{ distinct}\right\}.$$

The set $B$ is a basis for the $\mathbb{F}_2$-vector space $\mathrm{B}_2(g)$. For $1 \leq i_1 < \cdots < i_k \leq g$, set

$$\overline{S}(\widehat{i}_1, \ldots, \widehat{i}_k) = \overline{S} \setminus \{\overline{a}_{i_1}, \overline{b}_{i_1}, \ldots, \overline{a}_{i_k}, \overline{b}_{i_k}\}$$
$$B(\widehat{i}_1, \ldots, \widehat{i}_k) = \{1\} \cup \overline{S}(\widehat{i}_1, \ldots, \widehat{i}_k) \cup \left\{\overline{s}\overline{t} \mid \overline{s}, \overline{t} \in \overline{S}(\widehat{i}_1, \ldots, \widehat{i}_k) \text{ distinct}\right\}.$$

The set $B(\widehat{i}_1, \ldots, \widehat{i}_k)$ is a basis for the $\mathbb{F}_2$-vector space $\mathrm{B}_2(g;\widehat{i}_1, \ldots, \widehat{i}_k)$.

Since $g \geq 3$, each element of $B$ lies in $B(\widehat{i})$ for some $1 \leq i \leq g$. This implies that $\epsilon$ is surjective. Moreover, if $b \in B$ lies in $B(\widehat{j})$ and $B(\widehat{k})$ for some distinct $1 \leq j < k \leq g$, then $b \in B(\widehat{j}, \widehat{k})$ and

$$d(b) = (b, b) \in \mathrm{B}_2(g, \widehat{j}) \oplus \mathrm{B}_2(\widehat{k}) \subset \bigoplus_{1 \leq i \leq g} \mathrm{B}_2(g; \widehat{i}).$$

Since we are working over $\mathbb{F}_2$, we have $b = -b$. It follows that quotienting by the image of $d$ identifies $b \in \mathrm{B}_2(g, \widehat{j})$ and $b \in \mathrm{B}_2(g, \widehat{k})$. We conclude that $\mathrm{Im}(d) = \ker(\epsilon)$. □

## 11. Proof of Theorem A″

We close this paper by proving Theorem A″, whose statement we recall below. Recall that in §7.3 of Part 2 we proved that Theorem A″ implies Theorem A′, and in §4 of Part 1 we proved that Theorem A′ implies Theorem A. We remark that the proof of Theorem A″ uses the proof strategy introduced in [20].

**Theorem A″.** *For $g \geq 2$, the lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathrm{B}_2(g)$ is an isomorphism.*

*Proof.* The proof will be by induction on $g$. The base cases $g = 2$ and $g = 3$ follow from Lemmas 9.1 and 9.2. Assume, therefore, that $g \geq 4$ and that the theorem is true in all genera between 2 and $g - 1$. We will prove that $\mathfrak{Q}\mathfrak{K}_g^1$ has a presentation similar to the one we constructed for $\mathrm{B}_2(g)$ in Lemma 10.1. Our inductive hypothesis will then allow us to identify the terms in this presentation with those for $\mathrm{B}_2(g)$.

This requires some notation. Let $U$ be an $\mathbb{F}_2$-vector space equipped with a symplectic form; for instance, $U$ might a symplectic subspace of $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$. Generalizing our construction of $\mathfrak{Q}\mathfrak{K}_g^1$ from §7.3, define $\mathfrak{Q}\mathfrak{K}(U)$ to be the $\mathbb{F}_2$-vector space given by generators and relations as follows:

- The generators of $\mathfrak{Q}\mathfrak{K}_g^1$ are formal symbols $[\![V]\!]$ with $V < U$ a nonzero symplectic subspace.
- The relations of $\mathfrak{Q}\mathfrak{K}_g^1$ are as follows. Let $V, W < U$ be nonzero orthogonal symplectic subspaces. We then have the relation $[\![V]\!] + [\![W]\!] = [\![V \oplus W]\!]$.

Using the same formula (8.1) giving the lifted BCJ homomorphism $\beta\colon \mathfrak{Q}\mathfrak{K}_g^1 \to \mathrm{B}_2(g)$, there is a lifted BCJ homomorphism $\mathfrak{Q}\mathfrak{K}(U) \to \mathrm{B}_2(U)$. In fact, if $U$ has genus $h$ then $\mathfrak{Q}\mathfrak{K}(U) \cong \mathfrak{Q}\mathfrak{K}_h^1$ and $\mathrm{B}_2(U) \cong \mathrm{B}_2(h)$, and the lifted BCJ homomorphism $\mathfrak{Q}\mathfrak{K}(U) \to \mathrm{B}_2(U)$ can be identified with the usual lifted BCJ homomorphism $\mathfrak{Q}\mathfrak{K}_h^1 \to \mathrm{B}_2(h)$.

This construction is functorial in the following sense. Let $U$ and $U'$ be $\mathbb{F}_2$-vector spaces equipped with symplectic forms and let $\iota\colon U \to U'$ be a linear map preserving the symplectic form. This implies that $\iota$ is injective. The map $\iota\colon U \to U'$ induces a map $\mathfrak{Q}\mathfrak{K}(U) \to \mathfrak{Q}\mathfrak{K}(U')$. Unlike for $\mathrm{B}_2(U)$, this induced map is not obviously injective.

As in §10.2, let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{F}_2)$. For $1 \leq i \leq g$, let $H(i) = \langle a_i, b_i \rangle$. This is a genus-1 symplectic subspace of $H$, and we have an orthogonal decomposition

$$H = H(1) \oplus H(2) \oplus \cdots \oplus H(g).$$

For distinct $1 \leq i_1, \ldots, i_k \leq g$, let $H(\widehat{i_1}, \ldots, \widehat{i_k})$ be the subspace of $H$ obtained by deleting the terms $H(i_j)$ from the above decomposition, so

$$H(\widehat{i_1}, \ldots, \widehat{i_k}) = \bigoplus_{\substack{1 \leq i \leq g \\ i \neq i_1, \ldots, i_g}} H(i).$$

Finally, define

$$\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}_1, \ldots, \widehat{i}_k) = \mathfrak{Q}\mathfrak{K}(H(\widehat{i}_1, \ldots, \widehat{i}_k)).$$

Exactly like in §10.3, the maps induced by the various inclusion maps fit together into maps

$$\bigoplus_{1 \leq j < k \leq g} \mathfrak{Q}\mathfrak{K}_g^1(\widehat{j}, \widehat{k}) \xrightarrow{D} \bigoplus_{1 \leq i \leq g} \mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \xrightarrow{E} \mathfrak{Q}\mathfrak{K}_g^1$$

such that the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_{1 \leq j < k \leq g} \mathfrak{Q}\mathfrak{K}_g^1(\widehat{j}, \widehat{k}) \arrow[r, "D"] \arrow[d, "\beta'"] & \bigoplus_{1 \leq i \leq g} \mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \arrow[r, "E"] \arrow[d, "\beta''"] & \mathfrak{Q}\mathfrak{K}_g^1 \arrow[d, "\beta"] & \\
\bigoplus_{1 \leq j < k \leq g} \mathrm{B}_2(g; \widehat{j}, \widehat{k}) \arrow[r, "d"] & \bigoplus_{1 \leq i \leq g} \mathrm{B}_2(g; \widehat{i}) \arrow[r, "\epsilon"] & \mathrm{B}_2(g) \arrow[r] & 0.
\end{tikzcd}$$

Here the bottom row is the exact sequence from Lemma 10.1. The maps $\beta'$ and $\beta''$ are direct sums of the various lifted BCJ homomorphisms.

For $1 \leq i \leq g$ we have $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \cong \mathfrak{Q}\mathfrak{K}_{g-1}^1$ and $\mathrm{B}_2(g; \widehat{i}) \cong \mathrm{B}_2(g-1)$, and for $1 \leq j < k \leq g$ we have $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{j}, \widehat{k}) \cong \mathfrak{Q}\mathfrak{K}_{g-2}^1$ and $\mathrm{B}_2(g; \widehat{j}, \widehat{k}) \cong \mathrm{B}_2(g-2)$. Our inductive hypothesis therefore implies that $\beta'$ and $\beta''$ are isomorphisms. We remark that this uses the fact that $g \geq 4$ since this is needed to ensure that $g - 2 \geq 2$. To prove that $\beta$ is an isomorphism, it is therefore enough to prove that $E$ is surjective:

**Claim.** *The map* $E\colon \bigoplus_{1 \leq i \leq g} \mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \to \mathfrak{Q}\mathfrak{K}_g^1$ *is surjective.*

Let $\Lambda$ be the image of $E$. Our goal is to prove that $\Lambda = \mathfrak{Q}\mathfrak{K}_g^1$. Note that $H(1)$ is a genus-1 symplectic subspace of $H$ such that $[\![H(1)]\!] \in \mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}_2)$. It follows that $[\![H(1)]\!] \in \Lambda$. The group $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ acts transitively on genus-1 symplectic subspaces of $H$, so the $\mathrm{Sp}_{2g}(\mathbb{F}_2)$-orbit of $\Lambda$ contains all $[\![V]\!]$ with $V$ a genus-1 symplectic subspace of $H$. We claim that these generate $\mathfrak{Q}\mathfrak{K}_g^1$. Indeed, if $W$ is an arbitrary symplectic subspace of $H$ then there is an orthogonal decomposition $W = V_1 \oplus \cdots \oplus V_h$ with each $V_i$ a genus-1 symplectic subspace, so

$$[\![W]\!] = [\![V_1]\!] + \cdots + [\![V_i]\!].$$

It follows that the $\mathrm{Sp}_{2g}(\mathbb{F}_2)$-orbit of $\Lambda$ spans $\mathfrak{Q}\mathfrak{K}_g^1$. To prove that $\Lambda = \mathfrak{Q}\mathfrak{K}_g^1$, it is therefore enough to prove that $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ takes $\Lambda$ to $\Lambda$. For this, it is enough to prove the following:

- (♠) There exist generating sets $S$ for $\Lambda$ and $T$ for $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ such that for $s \in S$ and $\phi \in T$ we have $\phi(s) \in \Lambda$.

We will verify this in three steps: first we will construct $S$, then we will construct $T$, and then finally we will verify (♠).

**Step 1.** *We construct a generating set $S$ for $\Lambda$.*

By definition, $\Lambda$ is generated by the images of the maps $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \to \mathfrak{Q}\mathfrak{K}_g^1$ as $i$ ranges over $1 \leq i \leq g$. Consider some such $1 \leq i \leq g$. Our inductive hypothesis applies to $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \cong \mathfrak{Q}\mathfrak{K}_{g-1}^1$ and says that $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \cong \mathrm{B}_2(g; \widehat{i})$. Lemma 10.1 implies that the map

$$\bigoplus_{\substack{1 \leq i' \leq g \\ i' \neq i}} \mathrm{B}_2(g; \widehat{i}, \widehat{i'}) \to \mathrm{B}_2(g; \widehat{i})$$

is surjective. This map fits into a commutative diagram

$$\begin{array}{ccc}
\bigoplus\limits_{\substack{1\leq i'\leq g\\ i'\neq i}} \mathfrak{Q}\mathfrak{K}_g^1(\widehat{i},\widehat{i'}) & \longrightarrow & \mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \\
\downarrow & & \downarrow \cong \\
\bigoplus\limits_{\substack{1\leq i'\leq g\\ i'\neq i}} \mathrm{B}(g;\widehat{i},\widehat{i'}) & \twoheadrightarrow & \mathrm{B}(g;\widehat{i})
\end{array}$$

Since $g \geq 4$, our inductive hypothesis also applies to all the groups $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i},\widehat{i'}) \cong \mathfrak{Q}\mathfrak{K}_{g-2}^1$ appearing in this diagram. Applying our inductive hypothesis, we see that the left-hand vertical arrow is an isomorphism. We deduce that $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i})$ is generated by the images of the maps $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i},\widehat{i'}) \to \mathfrak{Q}\mathfrak{K}_g^1(\widehat{i})$ as $i'$ ranges over $1 \leq i' \leq g$ with $i' \neq i$.

Since $\Lambda$ is generated by the images of the maps $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \to \mathfrak{Q}\mathfrak{K}_g^1$ as $i$ ranges over $1 \leq i \leq g$, it follows that $\Lambda$ is generated by the images of the maps $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i},\widehat{i'}) \to \mathfrak{Q}\mathfrak{K}_g^1$ as $i$ and $i'$ range over distinct elements of $\{1,\ldots,g\}$. For $1 \leq i,i' \leq g$ distinct, let

$$S(i,i') = \left\{ [\![V]\!] \in \mathfrak{Q}\mathfrak{K}_g^1 \mid V \text{ a symplectic subspace of } H \text{ satisfying } V \subset H(\widehat{i},\widehat{i'}) \right\}.$$

The set $S(i,i')$ generates the image of the map $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i},\widehat{i'}) \to \mathfrak{Q}\mathfrak{K}_g^1$. We conclude that $\mathfrak{Q}\mathfrak{K}_g^1$ is generated by the set

$$S = \bigcup_{\substack{1\leq i,i'\leq g\\ i\neq i'}} S(i,i').$$

**Step 2.** *We construct a generating set $T$ for* $\mathrm{Sp}_{2g}(\mathbb{F}_2)$.

For a symplectic subspace $V$ of $H$, we can identify $\mathrm{Sp}(V)$ with the subgroup of $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ consisting of $\phi \in \mathrm{Sp}_{2g}(\mathbb{F}_2)$ that act trivially on $V^{\perp}$. Let

$$T = \bigcup_{\substack{1\leq j,j'\leq g\\ j\neq j'}} \mathrm{Sp}(H(j) \oplus H(j')) < \mathrm{Sp}_{2g}(\mathbb{F}_2).$$

This is a generating set for $\mathrm{Sp}_{2g}(\mathbb{F}_2)$; for instance, it contains all the elementary symplectic matrices (see, e.g., [11]).

**Step 3.** *We prove (♠): for $s \in S$ and $\phi \in T$ we have $\phi(s) \in \Lambda$.*

Consider $s \in S$ and $\phi \in T$. By definition, we can find:

- $1 \leq i,i' \leq g$ with $i \neq i'$ such that $s = [\![V]\!]$ with $V$ a symplectic subspace of $H$ satisfying $V \subset H(\widehat{i},\widehat{i'})$; and
- $1 \leq j,j' \leq g$ with $j \neq j'$ such that $\phi \in \mathrm{Sp}(H(j) \oplus H(j'))$.

To show that $\phi(s) \in \Lambda$, we must prove that $[\![\phi(V)]\!] \in \Lambda$.

There are two cases. The first is that $\{j,j'\} = \{i,i'\}$. Since

$$H = H(\widehat{i},\widehat{i'}) \oplus H(i) \oplus H(i') = H(\widehat{i},\widehat{i'}) \oplus H(j) \oplus H(j'),$$

the element $\phi \in \mathrm{Sp}(H(j) \oplus H(j'))$ acts trivially on $H(\widehat{i},\widehat{i'})$. It follows that $\phi(V) = V$ and there is nothing to prove.

The second case is that $\{j,j'\} \neq \{i,i'\}$. Swapping $i$ and $i'$ if necessary, we can assume that $i \notin \{j,j'\}$. This implies that $\phi \in \mathrm{Sp}(H(j) \oplus H(j'))$ acts trivially on $H(i)$. Since

$$H = H(\widehat{i}) \oplus H(i),$$

it follows that $\phi$ takes $H(\widehat{i})$ to $H(\widehat{i})$. Since $V \subset H(\widehat{i}, \widehat{i'}) \subset H(\widehat{i})$, we conclude that $\phi(V) \subset H(\widehat{i})$ and thus that $\llbracket \phi(V) \rrbracket$ lies in the image of the map $\mathfrak{Q}\mathfrak{K}_g^1(\widehat{i}) \to \mathfrak{Q}\mathfrak{K}_g^1$. This implies that $\llbracket \phi(V) \rrbracket \in \Lambda$, as desired. □

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