REPRESENTATION STABILITY IN THE LEVEL 4 BRAID GROUP

KEVIN KORDEK AND DAN MARGALIT

Abstract. We investigate the cohomology of the level 4 subgroup of the braid group, namely, the kernel of the mod 4 reduction of the Burau representation at $t = -1$. This group is also equal to the kernel of the mod 2 abelianization of the pure braid group. We give an exact formula for the first Betti number; it is a quartic polynomial in the number of strands. We also show that, like the pure braid group, the first homology satisfies uniform representation stability in the sense of Church and Farb. Unlike the pure braid group, the group of symmetries—the quotient of the braid group by the level 4 subgroup—is one for which the representation theory has not been well studied; we develop its representation theory. This group is a non-split extension of the symmetric group.

As applications of our main results, we show that the rational cohomology ring of the level 4 braid group is not generated in degree 1 when the number of strands is at least 15, and we compute all Betti numbers of the level 4 braid group when the number of strands is at most 4. We also derive a new lower bound on the first rational Betti number of the hyperelliptic Torelli group and on the top rational Betti number of the level 4 mapping class group in genus 2. Finally, we apply our results to locate all of the 2-torsion points on the characteristic varieties of the pure braid group.

1. Introduction

For an integer $m \geq 0$, the *level m braid group* $\mathrm{B}_n[m]$ is a subgroup of the braid group B_n. It is defined as the kernel of the composition

$$\mathrm{B}_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}]) \to \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m)$$

where the first map is the (unreduced) Burau representation (see Birman's book [4]), the next map is evaluation at $t = -1$ and the last map is given by reducing entries mod m.

The group $\mathrm{B}_n[0]$ is the kernel of the Burau representation at $t = -1$. This group is called the braid Torelli group, and we denote it by $\mathcal{BI}_n$. The braid Torelli group arises in algebraic geometry: it is the product of $\mathbb{Z}$ with fundamental group of any component of the branch locus of the period map on Torelli space [20]. Brendle, Putman, and the second author found a set of generators for $\mathcal{BI}_n$ [6].

Arnol'd [2] showed that $\mathrm{B}_n[2]$ is equal to the pure braid group PB_n. Brendle and the second author showed that $\mathrm{B}_n[4]$ is equal to PB_n^2, the subgroup of PB_n generated by all squares of elements [8]. Equivalently, $\mathrm{B}_n[4]$ is the kernel of the mod 2 abelianization map

$$\mathrm{PB}_n \to H_1(\mathrm{PB}_n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\binom{n}{2}}.$$

They additionally showed that $\mathrm{B}_n[4]$ is equal to the subgroup of PB_n generated by all squares of Dehn twists [8]. Little else is known about the algebraic structure of $\mathrm{B}_n[m]$ when $m > 2$. Indeed, there are no explicit finite generating sets known for these groups (in principle one can obtain a finite generating set from the Reidemeister–Schreier process, but such a generating set would not be geometrically meaningful).

The group $\mathrm{B}_n[4]$ is also of interest in algebraic geometry. It is isomorphic to the fundamental group of the mod 2 congruence cover of the complement of the braid arrangement

X_n; see [8]. Arithmetic aspects of this cover were studied by Yu [24] in connection with Cohen–Lenstra heuristic for functions fields; see also [15]. The group $\mathrm{B}_n[4]$ and the moduli space X_n also play an important role in forthcoming work of Rudenko on scissor congruence problems [30].

Our main result is Theorem 2.5, which states that $H_1(\mathrm{B}_n[4];\mathbb{C})$ satisfies uniform representation stability in the sense of Church and Farb. The group of symmetries is $\mathcal{Z}_n = \mathrm{B}_n / \mathrm{B}_n[4]$, which is a non-split extension of the symmetric group S_n by $H_1(\mathrm{PB}_n;\mathbb{Z}/2)$. Theorem 2.5 gives the explicit decomposition of $H_1(\mathrm{B}_n[4];\mathbb{C})$ into irreducible $\mathcal{Z}_n$-modules:

$$H_1(\mathrm{B}_n[4];\mathbb{C}) \cong \begin{cases} V_2(1,(0)) & n=2 \\ V_3(1,(0)) \oplus V_3(1,(1)) \oplus V_3(\rho_3,(0)) & n=3 \\ V_n(1,(0)) \oplus V_n(1,(1)) \oplus V_n(1,(2)) \oplus V_n(\rho_3,(0)) \oplus V_n(\rho_4,(0)) & n \geq 4. \end{cases}$$

Each summand here is of the form

$$V_n(\rho,\lambda) = \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n}\left(V_m(\rho) \boxtimes V_{n-m}(\lambda)\right)$$

where $\mathcal{Z}_n^I$ is the stabilizer in $\mathcal{Z}_n$ of a set I of pairs of elements of $\{1,\ldots,n\}$, $V_m(\rho)$ is an irreducible representation of $\mathcal{Z}_m^I$, and $V_{n-m}(\lambda)$ is an irreducible representation of S_{n-m}. See Theorem 2.4 and the preceding discussion for the precise definitions.

Our first step towards proving Theorem 2.5 is to prove Theorem 2.1, which gives an explicit basis for $H_1(\mathrm{B}_n[4];\mathbb{Q})$. From this basis we obtain a formula for the first Betti number of $\mathrm{B}_n[4]$, which is a quartic polynomial in n:

$$\dim H_1(\mathrm{B}_n[4];\mathbb{Q}) = 3\binom{n}{4} + 3\binom{n}{3} + \binom{n}{2}.$$

In order to prove this equality, there are two steps. We first construct new abelian quotients of $\mathrm{B}_n[4]$; these are defined in terms of double covers of the disk with n punctures (Section 3.1). Then we construct a basis by first giving an infinite spanning set (Section 5) and whittling it down to a basis, using the squared lantern relation, a Jacobi identity, the Witt–Hall identity, and the Artin relations for PB_n (Sections 6–7); this is the technical heart of the computation of $\dim H_1(\mathrm{B}_n[4];\mathbb{Q})$. It is perhaps surprising that the end result here is a polynomial in n; for instance, for the free group F_n the dimension of $H_1(F_n^2;\mathbb{Q})$ is exponential in n.

Church and Farb introduced the theory of representation stability and proved that $H_k(\mathrm{PB}_n;\mathbb{Q})$ satisfies uniform representation stability [13]. By passing to PB_n-invariants, our Theorem 2.5 recovers their result for $k=1$. Church and Farb take advantage of the explicit basis for $H_k(\mathrm{PB}_n;\mathbb{Q})$ provided by Arnol'd; our Theorem 2.1 plays the role of the Arnol'd result. They also employ the representation theory of S_n. As part of our work we develop the representation theory of $\mathcal{Z}_n$ from the ground up (Section 8). Our results suggest that it is an interesting problem to study the stability of $H_k(\mathrm{B}_n[m])$ as k, n, and m vary.

One other thing that makes the proof of Theorem 2.5 difficult is that our generators for the irreducible components of $H_1(\mathrm{B}_n[4];\mathbb{Q})$ do not seem to have simple expressions. The components $V_n(\rho_3,(0))$ and $V_n(\rho_4,(0))$ are the spans of the orbits of the elements

$$x_3 = (1-T_{13})\prod_{4\leq j\leq n}(1+T_{1j})(1+T_{2j})\tau_{12} \quad \text{and} \quad x_4 = (1-T_{14})(1-T_{23})\tau_{12},$$

respectively, where each T_{ij} is an Artin generator for PB_n and each τ_{ij} is the image of T_{ij}^2 in $H_1(\mathrm{B}_n[4];\mathbb{Q})$. In order to show that the actions of $\mathcal{Z}_n$ on these spans agree with the definitions of $V_n(\rho_3,(0))$ and $V_n(\rho_4,(0))$ requires deep understanding of the algebraic structure of $H_1(\mathrm{B}_n[4];\mathbb{Q})$.

We derive a number of consequences of our methods and results, about the level 4 hyperelliptic mapping class group $\mathrm{SMod}_g[4]$, the braid Torelli group $\mathcal{BI}_n$, the level 4 mapping class group $\mathrm{Mod}_g[4]$, the characteristic varieties $V_d(X_n)$ of the braid arrangement X_n, and $\mathrm{B}_n[4]$ itself. Specifically, we give the following applications.

(1) $H^*(\mathrm{B}_n[4];\mathbb{Q})$ is not generated in degree 1 (Theorem 2.8).
(2) $\mathrm{B}_n[4]$ is not generated by 4th powers of half-twists (Theorem 2.3).
(3) All 2-torsion on $V_1(X_n)$ lies on central components, and outside $V_2(X_n)$ (Theorem 2.14).

We further give:

(4) a new lower bound for the first Betti number of $\mathcal{BI}_n$ (Theorem 2.11),
(5) a new lower bound for the top Betti number of $\mathrm{Mod}_2[4]$ (Proposition 2.7), and
(6) computations of all Betti numbers of $\mathrm{B}_n[4]$ for $n \leq 4$ (Theorem 2.2).

Finally, we obtain analogues of some of our results about $\mathrm{B}_n[4]$ for $\mathrm{SMod}_g[4]$:

(7) we determine the first Betti number of $\mathrm{SMod}_g[4]$ (Corollary 2.6), and
(8) we show $H^*(\mathrm{SMod}_g[4];\mathbb{Q})$ is not generated in degree 1 (Theorem 2.9).

Representation stability has been studied for representations of Weyl groups (such as the symmetric groups and the hyperoctahedral groups), certain linear groups, and certain wreath products, among others; see the surveys by Farb, Khomenko–Kesari, and Wilson [16, 25, 34]. The group $\mathcal{Z}_n$ seems to have not appeared before in the theory. The general trend has to obtain representation stability from the finite generation of a module over a category associated to a sequence of groups. The category for the groups $\{\mathcal{Z}_n\}$ is the subject of a forthcoming paper with Miller and Patzt. With the current technology it does not appear to be possible to obtain our uniform representation stability from this categorical viewpoint.

Representation stability has also been studied extensively for various types of configuration spaces, beginning with the work of Church, Church–Farb, and Church–Ellenberg–Farb [11, 12, 13]. However, there is no general theory for the representation stability of the homology of covers of configuration spaces. On the other hand, the representation stability for the homology of specific covers, such as orbit configuration spaces, have been studied, for example, by Bibby–Gadish and Casto [3, 10]. Congruence covers of complements of hyperplane arrangements are well studied; see, for example, the survey by Suciu [32]. However, we are not aware of any previously known general closed formulas for the Betti numbers of congruence covers of X_n.

The uniform representation stability of the $\{H_1(\mathrm{B}_n[4];\mathbb{C})\}$ can also be interpreted as a result about the twisted homology of B_n with coefficients in the $V_n(\rho,\lambda)$. Wahl and Randal-Williams have proven a general stability result for the homology of braid groups with twisted coefficients [29]. Their theorem applies to certain coefficient systems, called polynomial coefficient systems. It seems to be an interesting (and difficult) problem to determine if the $V_n(\rho,\lambda)$ are polynomial in this sense. Even if this were the case, their result would not imply our Theorem 2.5; it would only imply that the multiplicities stabilize.

2. Statements of results

In this section, we explain our results in detail and give an outline for the paper. We begin with a discussion of Theorem 2.1. This theorem can in theory be obtained from our main result, Theorem 2.5 (and the dimension count in Theorem 2.1 does indeed follow immediately). However, our entire approach to Theorem 2.5 is predicated on Theorem 2.1.

An explicit basis. In this paper we identify B_n with the mapping class group of the 2-dimensional disk $\mathbb{D}_n$ with n marked points in the interior [17, Chapter 9]. In general, for a surface S, possibly with boundary and possibly with marked points, we define the mapping class group $\mathrm{Mod}(S)$ as the group of homotopy classes of orientation-preserving homeomorphisms of S that preserve the set of marked points and fix the boundary pointwise.

We label the marked points by $[n] = \{1, \dots, n\}$ and denote by T_{ij} the (left) Dehn twist about the curve in $\mathbb{D}_n$ indicated in Figure 1 and by τ_{ij} the image of T_{ij}^2 in $H_1(\mathrm{B}_n[4]; \mathbb{Q})$. The group PB_n acts on $H_1(\mathrm{B}_n[4]; \mathbb{Q})$ by conjugation; for $f \in \mathrm{PB}_n$ we denote by $f\tau_{ij}$ the image of τ_{ij} under the action of f.

For $n \geq 4$ we define the set $\mathcal{S}$ to be $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ where

$$\begin{aligned}
\mathcal{S}_1 &= \{\tau_{ij} \mid i < j\}, \\
\mathcal{S}_2 &= \{T_{ik}\tau_{ij},\ T_{jk}\tau_{ik},\ T_{ij}\tau_{jk} \mid i < j < k\}, \text{ and} \\
\mathcal{S}_3 &= \{T_{il}T_{jk}\tau_{ij},\ T_{ij}T_{k\ell}\tau_{ik},\ T_{ik}T_{j\ell}\tau_{i\ell} \mid i < j < k < \ell\}.
\end{aligned}$$

For $n < 4$ we define $\mathcal{S}$ in the same way, except that we declare $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ to be empty when n is less than 2, 3, and 4, respectively. In this paper we compose elements of B_n from right to left (functional notation).

Theorem 2.1. *For all $n \geq 1$ the set $\mathcal{S}$ is a basis for $H_1(\mathrm{B}_n[4]; \mathbb{Q})$. In particular,*

$$\dim H_1(\mathrm{B}_n[4]; \mathbb{Q}) = 3\binom{n}{4} + 3\binom{n}{3} + \binom{n}{2}.$$

We do not know if the abelianization of $\mathrm{B}_n[4]$ is torsion free, which is to say that we do not know a complete description of $H_1(\mathrm{B}_n[4]; \mathbb{Z})$. On the other hand, the proof of Theorem 2.1 also works with $\mathbb{Z}/p\mathbb{Z}$ coefficients for any odd prime p, implying that any non-trivial torsion in the abelianization would have to be 2-primary.

As applications of Theorem 2.1 we prove the following two theorems in Section 11.3 and 10, respectively. The first gives all Betti numbers for $\mathrm{B}_3[4]$ and $\mathrm{B}_4[4]$.

Theorem 2.2. *For $n = 3$ and $n = 4$, the dimensions of $H_k(\mathrm{B}_n[4]; \mathbb{Q})$ are as follows:*

$$\dim H_k(\mathrm{B}_3[4]; \mathbb{Q}) = \begin{cases} 1 & k = 0 \\ 6 & k = 1 \\ 5 & k = 2 \\ 0 & k \geq 3 \end{cases} \qquad \dim H_k(\mathrm{B}_4[4]; \mathbb{Q}) = \begin{cases} 1 & k = 0 \\ 21 & k = 1 \\ 103 & k = 2 \\ 83 & k = 3 \\ 0 & k \geq 4 \end{cases}$$

The second application shows that, even though $H_1(\mathrm{B}_n[4]; \mathbb{Q})$ is generated by the images of 4th powers of half-twists, the group $\mathrm{B}_n[4]$ is not. This is in contrast with B_n and PB_n, each of which is generated by its simplest elements, half-twists and squares of half-twists.

Theorem 2.3. *Let $n \geq 3$ and suppose that G is a subgroup of $\mathrm{B}_n[4]$ that contains $\mathcal{BI}_n$. Then G does not have a generating set consisting entirely of even powers of Dehn twists about curves surrounding 2 points. In particular $\mathrm{B}_n[4]$ is not generated by 4th powers of half-twists.*

We have the following related question.

Question 1. *Does $\mathrm{B}_n[4]$ have a generating set whose cardinality is equal to the dimension of $H_1(\mathrm{B}_n[4]; \mathbb{Q})$ (or even a generating set whose cardinality is a quartic polynomial in n)?*

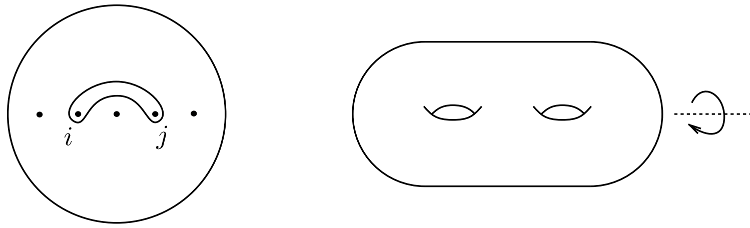

FIGURE 1. *Left:* T_{ij} is the Dehn twist about the indicated curve; *Right:* A hyperelliptic involution

Irreducible representations of $\mathcal{Z}_n$. In order to state our main theorem, Theorem 2.5, we must first describe the irreducible representations of the group $\mathcal{Z}_n = \mathrm{B}_n \,/\, \mathrm{B}_n[4]$. The end result of our discussion here is Theorem 2.4 below, which gives a naming system for the irreducible representations.

In order to state our classification of irreducible $\mathcal{Z}_n$-representations we require several definitions. Let $[n]$ denote $\{1, \ldots, n\}$ and let $[n]^{\underline{2}}$ denote the set of unordered pairs of elements of $[n]$. The standard action of B_n on $[n]$ (via the symmetric group S_n) induces an action on $[n]^{\underline{2}}$. We say that a subset I of $[n]^{\underline{2}}$ is full if the union of the elements of I is $[n]$. The symmetric group S_n acts on the set of full subsets of $[n]^{\underline{2}}$; let $\mathbb{I}_n$ be a set of orbit representatives.

Let $I \in \mathbb{I}_m$. For $n \geq m$ we may regard I as a subset of $[n]^{\underline{2}}$. We denote by B_n^I the stabilizer in B_n of the set I and by $\mathcal{Z}_n^I$ the image in $\mathcal{Z}_n$. We prove in Section 8.1 that there is a natural surjective map

$$\mathcal{Z}_n^I \to \mathcal{Z}_m^I \times S_{n-m}.$$

Next, let $\mathcal{PZ}_n$ denote the image of PB_n in $\mathcal{Z}_n$. This group is isomorphic to $(\mathbb{Z}/2)^{\binom{n}{2}}$, and the irreducible representations of $\mathcal{PZ}_n$ are in bijection with subsets of $[n]^{\underline{2}}$; see Section 8.2. We denote the representation corresponding to $I \subseteq [n]^{\underline{2}}$ by V_I. We say that a representation of $\mathcal{PZ}_n$ is I-isotypic if it decomposes as a direct sum of copies of V_I.

We are ready to describe the irreducible representations of $\mathcal{Z}_n$ that appear in our classification. The input for one of these representations consists of two pieces of data, an I-isotypic irreducible representation ρ of some $\mathcal{Z}_m^I$ with $I \in \mathbb{I}_m$ and an irreducible representation of S_{n-m}; as usual we label the latter by its corresponding padded partition of $[n-m]$, call it λ. With these in hand, we define a $\mathcal{Z}_n$-representation $V_n(\rho, \lambda)$ by the formula

$$V_n(\rho, \lambda) = \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} \left(V_m(\rho) \boxtimes V_{n-m}(\lambda) \right)$$

where $V_m(\rho)$ and $V_{n-m}(\lambda)$ are the representations corresponding to ρ and λ and $\mathcal{Z}_n^I$ acts via the surjection to $\mathcal{Z}_n^I \to \mathcal{Z}_m^I \times S_{n-m}$. Of course if ρ is isomorphic to ρ' then $V_m(\rho, \lambda)$ is isomorphic to $V_m(\rho', \lambda)$. In order to obtain a unique name for each representation, we fix one representative from each equivalence class once and for all.

We observe that if we take $I = \emptyset$, then $V_n(\rho, \lambda)$ is the representation of $\mathcal{Z}_n$ that factors through the S_n-representation $V_n(\lambda)$. We denote such a representation as $V_n(1, \lambda)$.

Theorem 2.4. *The $V_n(\rho, \lambda)$ are irreducible $\mathcal{Z}_n$-representations. Further, every irreducible $\mathcal{Z}_n$-representation is isomorphic to exactly one $V_n(\rho, \lambda)$.*

The usual map $\mathrm{B}_n \to S_n$ induces a short exact sequence

$$1 \to \mathcal{PZ}_n \to \mathcal{Z}_n \to S_n \to 1.$$

If this sequence were split, we could hope to understand the representation theory of $\mathcal{Z}_n$ via the representation theory of S_n. We prove, however, in Proposition 8.6 that it is not split.

Statement of the main theorem: representation stability. The conjugation action of B_n on $\mathrm{B}_n[4]$ induces an action of B_n on $H_1(\mathrm{B}_n[4];\mathbb{C})$. Since this restricts to a trivial action of $\mathrm{B}_n[4]$, we have that $H_1(\mathrm{B}_n[4];\mathbb{C})$ is in a natural way a representation of $\mathcal{Z}_n$.

Church and Farb defined representation stability for sequences of representations of $S_n = \mathrm{B}_n / \mathrm{PB}_n$. We extend their definition to our setting and show that the $H_1(\mathrm{B}_n[4];\mathbb{C})$ are uniformly representation stable.

For each n there is a standard inclusion $\mathrm{B}_n \to \mathrm{B}_{n+1}$ obtained by adding a strand. We show in Lemma 9.6 that this induces inclusions $\mathrm{B}_n[4] \to \mathrm{B}_{n+1}[4]$ and $\mathcal{Z}_n \to \mathcal{Z}_{n+1}$. Suppose we have a sequence of $\mathcal{Z}_n$-representations V_n and maps $\varphi_n : V_n \to V_{n+1}$. Following Church–Farb, we say that the sequence $\{V_n\}$ is consistent if for each n the map φ_n is equivariant with respect to the $\mathcal{Z}_n$-action. The inclusions $\mathrm{B}_n[4] \to \mathrm{B}_{n+1}[4]$ induce maps $H_1(\mathrm{B}_n[4];\mathbb{C}) \to H_1(\mathrm{B}_{n+1}[4];\mathbb{C})$. With respect to these maps the $H_1(\mathrm{B}_n[4];\mathbb{C})$ form a consistent sequence of $\mathcal{Z}_n$-representations (Lemma 9.7).

Further following Church–Farb, we say that a consistent sequence of $\mathcal{Z}_n$-representations V_n satisfies *representation stability* if

(1) the maps $\varphi_n : V_n \to V_{n+1}$ are injective,
(2) the span of the $\mathcal{Z}_{n+1}$-orbit of $\varphi_n(V_n)$ is equal to V_{n+1}, and
(3) if we decompose each V_n into irreducible $\mathcal{Z}_n$-representations

$$V_n \cong \bigoplus_{(\rho,\lambda)} c_{\rho,\lambda,n} V_n(\rho,\lambda)$$

then each of the sequences of multiplicities $c_{\rho,\lambda,n} \geq 0$ is independent of n for n large.

We say that the V_n satisfy *uniform representation stability* if there is some N so that every $c_{\rho,\lambda,n}$ is independent of n for $n \geq N$.

Let I_3 and I_4 be the subsets of $[3]^{\underline{2}}$ and $[4]^{\underline{2}}$ given by

$$I_3 = \{\{1,3\},\{2,3\}\}, \quad \text{and} \quad I_4 = \{\{1,3\},\{2,3\},\{1,4\},\{2,4\}\}.$$

We may assume that $I_3 \in \mathbb{I}_3$ and $I_4 \in \mathbb{I}_4$. Let μ_2 denote the multiplicative group $\{\pm 1\}$; we can regard μ_2 as a subgroup of $\mathrm{GL}(\mathbb{C})$. In Section 9.1 we define specific homomorphisms $\rho_k : \mathcal{Z}_k^{I_k} \to \mu_2$, and hence representations of $\mathcal{Z}_k^{I_k}$ for $k \in \{3,4\}$. Each ρ_k is the sum of the winding numbers of the pairs of strands corresponding to the elements of I_k.

Theorem 2.5. *There are $\mathcal{Z}_n$-equivariant isomorphisms*

$$H_1(\mathrm{B}_n[4];\mathbb{C}) \cong \begin{cases} V_2(1,(0)) & n=2 \\ V_3(1,(0)) \oplus V_3(1,(1)) \oplus V_3(\rho_3,(0)) & n=3 \\ V_n(1,(0)) \oplus V_n(1,(1)) \oplus V_n(1,(2)) \oplus V_n(\rho_3,(0)) \oplus V_n(\rho_4,(0)) & n \geq 4. \end{cases}$$

Further, the sequence $\{H_1(\mathrm{B}_n[4];\mathbb{C})\}$ of $\mathcal{Z}_n$-modules is uniformly representation stable.

The $\mathcal{PZ}_n$-invariants of $H_1(\mathrm{B}_n[4];\mathbb{C})$ is exactly $H_1(\mathrm{PB}_n;\mathbb{C})$; this subspace corresponds to the summands $V_n(\rho,\lambda)$ in the statement of Theorem 2.5 with $\rho = 1$. Thus, the first statement of Theorem 2.5 recovers Church–Farb's description of $H_1(\mathrm{PB}_n;\mathbb{C})$ as an S_n-representation.

Also, representation stability for a sequence V_n of $\mathcal{Z}_n$-modules implies representation stability for the sequence of $\mathcal{PZ}_n$-invariants, which are S_n-modules. In this way, the second statement of Theorem 2.5 recovers the representation stability of $H_1(\mathrm{PB}_n;\mathbb{C})$ discovered by Church–Farb [13].

It appears to be an interesting problem to determine the character table of $\mathcal{Z}_n$.

Level 4 hyperelliptic mapping class groups. Let Σ_g be a closed orientable surface of genus g. Let Mod_g denote its mapping class group. This group is, for example, the (orbifold) fundamental group of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g.

The hyperelliptic mapping class group SMod_g is the centralizer in Mod_g of some fixed hyperelliptic involution; see Figure 1. The level m mapping class group $\mathrm{Mod}_g[m]$ is the subgroup of Mod_g consisting of all elements that act trivially on $H_1(\Sigma_g;\mathbb{Z}/m)$. The level m hyperelliptic mapping class group $\mathrm{SMod}_g[m]$ is the intersection $\mathrm{SMod}_g \cap \mathrm{Mod}_g[m]$.

As we will explain in Section 11.1, there are isomorphisms $\mathrm{B}_{2g+1}[4] \cong \mathrm{SMod}_g[4] \times \mathbb{Z}$ for all $g \geq 1$. Even in the absence of an inclusion $\Sigma_g \to \Sigma_{g+1}$, these isomorphisms give rise to maps $\mathrm{SMod}_g[4] \to \mathrm{SMod}_{g+1}[4]$ as follows:

$$\mathrm{SMod}_g[4] \to \mathrm{B}_{2g+1}[4] \to \mathrm{B}_{2g+3}[4] \to \mathrm{SMod}_{g+1}[4],$$

where the first map is inclusion into the first factor, the second map is the standard inclusion, and the third map is projection onto the first factor. The induced maps $H_1(\mathrm{SMod}_g[4];\mathbb{C}) \to H_1(\mathrm{SMod}_{g+1}[4];\mathbb{C})$ are injective and equivariant with respect to the $\mathcal{Z}_{2g+1}$- and $\mathcal{Z}_{2g+3}$-actions. Since the $\mathbb{Z}$-factor of $\mathrm{B}_{2g+1}[4]$ is central, it follows that this factor corresponds to the trivial representation $V_{2g+1}(1,(0))$ in Theorem 2.5. We thus obtain the following consequence of Theorem 2.5.

Corollary 2.6. *For $g \geq 1$, there are $\mathcal{Z}_{2g+1}$-equivariant isomorphisms*

$$H_1(\mathrm{SMod}_g[4];\mathbb{C}) \cong \begin{cases} V_3(1,(1)) \oplus V_3(\rho_3,(0)) & g = 1 \\ V_{2g+1}(1,(1)) \oplus V_{2g+1}(1,(2)) \oplus V_{2g+1}(\rho_3,(0)) \oplus V_n(\rho_4,(0)) & g \geq 2. \end{cases}$$

In particular, we have

$$\dim H_1(\mathrm{SMod}_g[4];\mathbb{Q}) = 3\binom{2g+1}{4} + 3\binom{2g+1}{3} + \binom{2g+1}{2} - 1.$$

Further, the sequence $\{H_1(\mathrm{SMod}_g[4];\mathbb{C})\}$ of $\mathcal{Z}_{2g+1}$-modules is uniformly representation stable.

In contrast to Corollary 2.6, $H_1(\mathrm{Mod}_g[m];\mathbb{Q}) = 0$ for $g \geq 3$ and $m \geq 1$; see the paper by Hain [22].

Under the map $\mathrm{B}_{2g+1}[4] \to \mathrm{SMod}_g[4]$ from Section 11.1, the basis elements from Theorem 2.1 map to the classes of 4th powers of Dehn twists about nonseparating curves.

The group SMod_g is the orbifold fundamental group of the hyperelliptic locus $\mathcal{H}_g$ in $\mathcal{M}_g$. The group $\mathrm{SMod}_g[4]$ is the fundamental group of any connected component $\mathcal{H}_g[4]$ of the hyperelliptic locus in the moduli space of genus g Riemann surfaces C with level 4 structure,[1] i.e. a symplectic basis for $H_1(C;\mathbb{Z}/4)$. In fact, $\mathcal{H}_g[4]$ is a $K(\pi,1)$ space for $\mathrm{SMod}_g[4]$. Thus $H^i(\mathrm{SMod}_g[4];\mathbb{Q}) \cong H^i(\mathcal{H}_g[4];\mathbb{Q})$ for all $j \geq 0$. As such, Corollary 2.6 gives the first Betti number of $\mathcal{H}_g[4]$.

For $g = 2$ we have $\mathrm{SMod}_g = \mathrm{Mod}_g$. In this case we have the following result.

Proposition 2.7. *We have* $\dim H_3(\mathrm{Mod}_2[4];\mathbb{Q}) \geq 3068$.

Proposition 2.7 improves on a special case of a result of Fullarton–Putman [18, Theorem A], which gives $\dim H_3(\mathrm{Mod}_2[4];\mathbb{Q}) \geq 24$.

[1]There are $\dfrac{2^{g^2}(2^{2g}-1)\cdots(2^2-1)}{(2g+2)!}$ such components; they are mutually isomorphic.

Albanese cohomology. For a finitely generated group Γ, the *Albanese cohomology* of Γ is the subalgebra $H^*_{Alb}(\Gamma;\mathbb{Q})$ of the rational cohomology algebra $H^*(\Gamma;\mathbb{Q})$ generated by $H^1(\Gamma;\mathbb{Q})$. In other words, $H^*_{Alb}(\Gamma;\mathbb{Q})$ is the image of the cup product mapping

$$\Lambda^* H^1(\Gamma;\mathbb{Q}) \to H^*(\Gamma;\mathbb{Q}).$$

The term "Albanese cohomology" was introduced by Church–Ellenberg–Farb in their work on representation stability [12].

Arnol'd [2] showed that the cohomology ring of PB_n is generated by degree 1 classes, in other words $H^*_{Alb}(\mathrm{PB}_n;\mathbb{Q}) = H^*(\mathrm{PB}_n;\mathbb{Q})$. We prove the following contrasting result.

Theorem 2.8. *Let $n \geq 15$. Then $H^*_{Alb}(\mathrm{B}_n[4];\mathbb{Q})$ is a proper subalgebra of $H^*(\mathrm{B}_n[4];\mathbb{Q})$.*

Although Theorem 2.8 asserts that there are cohomology classes in $H^*(\mathrm{B}_n[4];\mathbb{Q})$ that are not cup products of classes in $H^1(\mathrm{B}_n[4];\mathbb{Q})$, our proof does not produce examples of such classes. We will derive Theorem 2.8 from the following slightly stronger result.

Theorem 2.9. *For all $g \geq 7$ the Albanese cohomology $H^*_{Alb}(\mathrm{SMod}_g[4];\mathbb{Q})$ is a proper subalgebra of $H^*(\mathrm{SMod}_g[4];\mathbb{Q})$.*

The original proofs that the cohomology groups of the pure braid groups and the pure string motion groups are representation stable take advantage of the fact that the cohomology algebras of both are generated in degree 1; see the papers by Church–Farb and Wilson [13, 33]. One would like to emulate this in the case of $\mathrm{B}_n[4]$ to prove that its higher cohomology groups of $\mathrm{B}_n[4]$ are representation stable. However, as Theorem 2.8 shows, this approach cannot work. Instead, we propose the following.

Conjecture 2.10. *For each $k \geq 1$ the sequence of $\mathcal{Z}_n$-representations $H^k_{Alb}(\mathrm{B}_n[4];\mathbb{Q})$ is uniformly representation stable for $n \geq 4k$.*

Conjecture 2.10 may be compared with a result of Church–Ellenberg–Farb, which states that the Albanese cohomology of the Torelli group is a finitely generated FI-module, hence uniformly representation stable [12, proof of Theorem 7.2.2].

Hyperelliptic Torelli groups. The *braid Torelli group* $\mathcal{BI}_n$ is the level 0 subgroup of B_n, i.e. the kernel of the Burau representation evaluated at $t=-1$ (the latter is sometimes called the integral Burau representation). This group, an infinite-index subgroup of B_n, is even more mysterious than the $\mathrm{B}_n[m]$. For example, it is not known if this group is finitely generated when $n \geq 7$. It is, however, known that for $n = 2,3,4,5,6$ have that that $\mathcal{BI}_n$ is isomorphic to 1, $\mathbb{Z}$, F_∞, $F_\infty \times \mathbb{Z}$, and $F_\infty \ltimes F_\infty$, respectively, where F_∞ denotes the free group of countably infinite rank. It is also known that $\mathcal{BI}_7$ is not finitely presented [5, Theorem 1.3]. An appealing infinite generating set for $\mathcal{BI}_n$, consisting of all squares of Dehn twists about curves surrounding either 3 or 5 punctures, was identified by Brendle, Putman, and the second author [6].

The *hyperelliptic Torelli group* $\mathcal{SI}_g$ is the subgroup of SMod_g whose elements act trivially on $H_1(\Sigma_g;\mathbb{Z})$. There are isomorphisms $\mathcal{BI}_{2g+1} \cong \mathcal{SI}_g \times \mathbb{Z}$; see [7]. Using Hain's description [19] of the image of the second Johnson homomorphism, one can obtain a lower bound

$$\dim H_1(\mathcal{BI}_{2g+1};\mathbb{Q}) \geq \frac{g(g-1)(4g^2+4g-3)}{3} + 1.$$

We will deduce an improved lower bound from Theorem 2.1 as follows.

Theorem 2.11. *For $g \geq 3$ we have*

$$\dim H_1(\mathcal{BI}_{2g+1};\mathbb{Q}) \geq \frac{1}{6}\left(20g^4 + 12g^3 - 5g^2 + 9g\right)$$

The finiteness properties of the $\mathcal{BI}_n$ and the $\mathrm{B}_n[m]$ are related by the following proposition, which we prove in Section 12.

Proposition 2.12. *Let n be odd. If the sequence $(\dim H_1(\mathrm{B}_n[m];\mathbb{Q}))_{m=1}^{\infty}$ is unbounded then $H_1(\mathcal{BI}_n;\mathbb{Q})$ is infinite dimensional, and in particular $\mathcal{BI}_n$ is not finitely generated.*

We make the following conjecture.

Conjecture 2.13. *For fixed $n \geq 4$, the sequence $(\dim H_1(\mathrm{B}_n[m];\mathbb{Q}))_{m=1}^{\infty}$ is unbounded. In particular, $H_1(\mathcal{BI}_n;\mathbb{Q})$ is infinite dimensional and $\mathcal{BI}_n$ is not finitely generated.*

Theorem 2.1 provides some evidence for Conjecture 2.13; with the equalities

$$\dim H_1(\mathrm{B}_n;\mathbb{Q}) = 1 \qquad \dim H_1(\mathrm{PB}_n;\mathbb{Q}) = \binom{n}{2}$$

it implies

$$\dim H_1(\mathrm{B}_n;\mathbb{Q}) \ll \dim H_1(\mathrm{B}_n[2];\mathbb{Q}) \ll \dim H_1(\mathrm{B}_n[4];\mathbb{Q}).$$

From the above descriptions of $\mathcal{BI}_n$ for $2 \leq n \leq 6$ we have that $H_1(\mathcal{BI}_n;\mathbb{Q})$ is not finite dimensional for $4 \leq n \leq 6$. Brendle, Childers, and the second author showed that $H_{g-1}(\mathcal{BI}_{2g+1};\mathbb{Q})$ is infinite dimensional [5, Theorem 1.3]. It is not known if any of the other $H_k(\mathcal{BI}_n;\mathbb{Q})$ are infinite dimensional.

The group $\mathcal{BI}_{2g+1}$ is isomorphic to the direct product of $\mathbb{Z}$ with fundamental group of any component of the branch locus of the period map on Torelli space [20]. Therefore Conjecture 2.13 implies that this fundamental group is infinitely generated.

Torsion points on the characteristic variety for the braid arrangement. Let $X_n \subset \mathbb{C}^n$ denote the complement of the braid arrangement

$$X_n = \mathbb{C}^n \setminus \{(x_1,\ldots,x_n) \in \mathbb{C}^n \mid x_i = x_j \text{ some } i,j\}$$

There is a one-to-one correspondence between homomorphisms $\mathrm{PB}_n \cong \pi_1(X_n) \to \mathbb{C}^\times$ and 1-dimensional complex local systems over X_n; to a homomorphism $\rho : \mathrm{PB}_n \to \mathbb{C}^\times$ we associate the 1-dimensional local system $\mathbb{C}_\rho$ with monodromy ρ. Thus the space of all such local systems can be identified with the algebraic torus $\mathrm{Hom}(\mathrm{PB}_n,\mathbb{C}^\times)$, which we identify with $(\mathbb{C}^\times)^{\binom{n}{2}}$ via $\rho \to (\rho(T_{12}),\ldots,\rho(T_{n-1,n}))$.

For $d \geq 1$, the dth characteristic variety $V_d(X_n)$ of X_n is the subvariety of $(\mathbb{C}^\times)^{\binom{n}{2}}$ consisting of all $\rho \in \mathrm{Hom}(\mathrm{PB}_n,\mathbb{C}^\times)$ such that $\dim H^1(X_n;\mathbb{C}_\rho) \geq d$. One reason for studying the characteristic varieties of X_n is that they give fine information about the topology of its abelian covers. A general theorem of Arapura [1] implies that $V_d(X_n)$ is a union of algebraic subtori, possibly with some components translated away from the identity $\mathbf{1}$ by finite-order elements. Following Cohen–Suciu [14], we denote by $\check{V}_d(X_n)$ the union of the components of $V_d(X_n)$ that contain $\mathbf{1}$ (these are the so-called central components).

Let $\rho_I : \mathcal{PZ}_n \to \mu_2$ denote the homomorphism giving the irreducible representation V_I of $\mathcal{PZ}_n$ (see Section 8) and denote the unique extension $\mathrm{PB}_n \to \mu_2 \subset \mathbb{C}^\times$ by ρ_I as well.

Theorem 2.14. *Let $n \geq 3$. For $d \geq 2$, the characteristic variety $V_d(X_n)$ contains no 2-torsion. The set of 2-torsion points on $V_1(X_n)$ is $\{\rho_{g(I_3)}\}_{g\in S_n} \cup \{\rho_{g(I_4)}\}_{g\in S_n}$, which is contained in $\check{V}_1(X_n)$.*

For $n \leq 4$, all known components of the characteristic varieties of X_n contain $\mathbf{1}$ but it is an open problem to determine whether this holds for general n. Arapura's theorem implies that any component of $V_d(X_n)$ not containing $\mathbf{1}$ must contain some point of finite order. Therefore, if one could show that every torsion point on $V_d(X_n)$ were contained in $\check{V}_d(X_n)$, it would follow that $\check{V}_d(X_n) = V_d(X_n)$. By Theorem 2.14, any translated components of $V_d(X_n)$, should they exist, would have to be translated by an element of order at least 3.

Outline of the paper. The remainder of the paper essentially has three parts. The first part, Sections 3–7, is devoted to the proof of Theorem 2.1. In Section 3 we define maps

$$\begin{aligned} \psi &: \mathrm{B}_n[4] \to \mathbb{Z}^{\binom{n}{2}}, \\ \psi_{i\infty} &: \mathrm{B}_n[4] \to \mathbb{Z}^{\binom{2n-1}{2}} \quad 1 \leq i \leq n, \text{ and} \\ \psi_{ij} &: \mathrm{B}_n[4] \to \mathbb{Z}^{\binom{2n-2}{2}} \quad 1 \leq i < j \leq n \end{aligned}$$

that we will use to detect nontrivial classes in $H_1(\mathrm{B}_n[4];\mathbb{Q})$. The map ψ is simply the restriction of the abelianization of PB_n. This map is clearly not sufficient for our purposes, since the dimension of $H_1(\mathrm{B}_n[4];\mathbb{Q})$ asserted by Theorem 2.1 is much larger than that of $H_1(\mathrm{PB}_n;\mathbb{Q}) = \binom{n}{2}$. The ψ_{ij} are designed to detect elements of $H_1(\mathrm{B}_n[4];\mathbb{Q})$ coming from the commutator subgroup of PB_n. Roughly, they are defined as follows: lift an element of $\mathrm{B}_n[4]$ to a double cover of $\mathbb{D}_n$ and apply the abelianization of the pure braid group of the cover. We show at the end of Section 3 that these maps do indeed detect commutators of pure braids.

With the ψ_{ij} in hand, we complete the proof of Theorem 2.1 for $n = 3$ in Section 4. That the given basis elements are independent is proved using the ψ_{ij} and the fact that the first Betti number is 6, which comes easily from the equality $\mathrm{B}_3[4] = \mathrm{PB}_3^2$ and the isomorphism $\mathrm{PB}_3 \cong F_2 \times \mathbb{Z}$.

The proof of Theorem 2.1 for $n \geq 4$ is carried out in the next three sections. In Section 5 we use the ψ_{ij} to show that the basis elements from Theorem 2.1 are linearly independent; this step is similar in spirit to the $n = 3$ case, and is complicated mainly by the large number of homology classes being considered. Then in Section 6 we give an infinite spanning set for $H_1(\mathrm{B}_n[4];\mathbb{Q})$ whose elements have a particularly simple form: they are the images of squares of Dehn twists about curves surrounding two marked points. There is an obvious spanning set for $H_1(\mathrm{B}_n[4];\mathbb{Q})$ coming from the generating set for $\mathrm{B}_n[4]$ given by Brendle and the second author, namely the squares of Dehn twists, and our spanning set is a subset of this one. Finally, in Section 7 we complete the proof of Theorem 2.1. This section is the technical heart of the proof. The idea is to whittle down the spanning set from Section 6 using a series of relations in $H_1(\mathrm{B}_n[4];\mathbb{Q})$. The relations are obtained using a combination of the squared lantern relation, a Jacobi identity, the Witt–Hall identity, and the standard Artin relations for PB_n.

The second part of the paper is dedicated to the proof of Theorem 2.5. As above, this first requires an investigation of the representation theory of $\mathcal{Z}_n$. In particular, the statement of representation stability requires a naming system for its irreducible representations. This is carried out in Section 9.2. The main difficulty stems from the fact that $\mathcal{Z}_n$ does not split as a semi-direct product over S_n. We then prove Theorem 2.5 in Section 9.3 by exhibiting the irreducible representations from the statement of the theorem as explicit submodules and by verifying the three parts of the definition of uniform representation stability. These submodules are the spans of the $\mathcal{Z}_n$-orbits of elements a_{ij}, x_3, and x_4. The main obstacle towards proving Theorem 2.5 is simply locating the elements x_3 and x_4 in the first place.

Finally, the third part of the paper gives the proofs of the various applications of our main results. First in Section 10 we quickly dispense with Theorem 2.3 as a consequence of Theorem 2.1. Then in Section 11 we Theorem 2.9 and then use this to prove Theorem 2.8. For Theorem 2.9, the basic idea is to compare the dimension of $H^1(\mathrm{SMod}_g[4];\mathbb{Q})$ (Corollary 2.6) to the Euler characteristic of $\mathrm{SMod}_g[4]$. The latter is an enormous negative number, signaling the presence of large amounts of cohomology in odd degrees. A careful comparison of the odd Betti numbers of $\mathrm{SMod}_g[4]$ with the dimensions of the odd graded pieces of the exterior algebra $\Lambda^* H^1(\mathrm{SMod}_g[4];\mathbb{Q})$ then gives the result. At the end of Section 11 we prove Proposition 2.7 and Theorem 2.2.

Next, in Section 12 we prove Theorem 2.11. The idea is to show that there is a surjective map $H_1(\mathcal{BI}_{2g+1};\mathbb{Q}) \to H_1(\mathrm{SMod}_g[4];\mathbb{Q})$ and that the direct sum of this map with the second Johnson homomorphism is surjective. The result is then obtained by adding together the dimensions of the targets of these two maps. Finally, in Section 13 we prove Theorem 2.14 as an application of Theorem 2.5.

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3. Abelian quotients from double covers

The goal of this section is to define and describe the homomorphisms

$$\begin{aligned} \psi &: \mathrm{B}_n[4] \to \mathbb{Z}^{\binom{n}{2}}, \\ \psi_{i\infty} &: \mathrm{B}_n[4] \to \mathbb{Z}^{\binom{2n-1}{2}} \quad 1 \le i \le n, \text{ and} \\ \psi_{ij} &: \mathrm{B}_n[4] \to \mathbb{Z}^{\binom{2n-2}{2}} \quad 1 \le i < j \le n \end{aligned}$$

discussed in Section 2. We will denote the induced maps on $H_1(\mathrm{B}_n[4];\mathbb{Q})$ by the same symbols. We will use these homomorphisms in Sections 4 and 5 to detect non-zero homology classes in $H_1(\mathrm{B}_n[4];\mathbb{Q})$.

The map ψ will simply be defined as the restriction of the abelianization of PB_n. As discussed in Section 2, the ψ_{ij} will be defined in terms of 2-fold covers of $\mathbb{D}_n$. In Section 3.1 we describe the 2-fold covers used. Then in Section 3.2 we define the ψ_{ij} and in Section 3.3 we compute the images under the ψ_{ij} of each square of a Dehn twist in $\mathrm{B}_n[4]$. Finally we in Section 3.4 we give a naturality (equivariance) formula for the ψ_{ij} and use this formula to compute several examples.

One of the examples we compute at the end of the section is $\psi_{1\infty}([T_{23}^2, T_{12}])$. In particular we show it is non-zero. Of course ψ evaluates to zero on the commutator subgroup of PB_n, and so this computation verifies that the ψ_{ij} are indeed giving more information than ψ.

3.1. Double covers of the disk.

Denote the set of marked points of $\mathbb{D}_n$ by P. There is a correspondence

$$H_1(\mathbb{D}_n, P \cup \partial\mathbb{D}_n;\mathbb{Z}/2) \longleftrightarrow \{\text{2-fold branched covers of } (\mathbb{D}_n, P)\}/\sim$$

Here, a branched cover over $(\mathbb{D}_n, P)$ is a branched cover over $\mathbb{D}_n$ where the set of branch points lies in P.

The above correspondence can be explained as a sequence of three correspondences as follows. First, it is a consequence of Lefschetz duality that $H_1(\mathbb{D}_n, P \cup \partial\mathbb{D}_n; \mathbb{Z}/2)$ is isomorphic to $H^1(\mathbb{D}_n^\circ; \mathbb{Z}/2)$, where $\mathbb{D}_n^\circ$ is the surface obtained from $\mathbb{D}_n$ by removing the n marked points. Second, by basic covering space theory, the latter is in bijective correspondence with the equivalence classes of 2-fold covers of $\mathbb{D}_n^\circ$. Third, 2-fold covers over $\mathbb{D}_n^\circ$ are in bijection with 2-fold branched covers of $\mathbb{D}_n$ with branch set in P; we pass from one to the other by adding/subtracting the points of P and its preimage. The stated correspondence follows.

As above, let $[n]$ denote the set $\{1, \ldots, n\}$ and let $[n]^{\underline{2}}$ denote the set of pairs of elements of $[n]$. Also let $[n]_\infty$ denote $[n] \cup \{\infty\}$ and let $[n]^{\underline{2}}_\infty$ be the set of pairs of elements of $[n]_\infty$. There is a natural bijection between $[n]$ and P, where i corresponds to the ith marked point. If we think of $\partial\mathbb{D}_n$ has having the label ∞ then there is a further bijection between $[n]_\infty$ and the set of connected components of $P \cup \partial\mathbb{D}_n$. There is a map

$$\{[n]^{\underline{2}}_\infty\} \to \{\text{2-fold branched covers of } (\mathbb{D}_n, P)\}/\sim$$

defined as follows. For an element of $[n]^{\underline{2}}_\infty$ we obtain a nontrivial element of $H_1(\mathbb{D}_n, P \cup \partial\mathbb{D}_n; \mathbb{Z}/2)$ by choosing an arc between the corresponding components of $P \cup \partial\mathbb{D}_n$ (the arc should be disjoint from $P \cup \partial\mathbb{D}_n$ on its interior). This homology class, hence the resulting equivalence class of covers, is independent of the choice of arc. We refer to any resulting cover of $\mathbb{D}_n$ as an (ij)-cover of $\mathbb{D}_n$. An $(i\infty)$-cover of $\mathbb{D}_n$ is a disk with $2n-1$ marked points and any other (ij)-cover is an annulus with $2n-2$ marked points.

As elements of $[n]^{\underline{2}}_\infty$ only give equivalence classes of branched covers over $\mathbb{D}_n$, it will be helpful to fix specific (ij)-covers once and for all, as follows. First we fix a copy of $\mathbb{D}_n$ once and for all, as the closed unit disk in the plane with the marked points along the x-axis.

Then for each i we let $\alpha_{i\infty}$ be the vertical arc in $\mathbb{D}_n$ connecting the ith marked point to the upper boundary of $\mathbb{D}_n$. And for each $\{i, j\} \in [n]^{\underline{2}}$ we let α_{ij} be the semi-circular arc that connects the ith and jth marked points and lies above the x-axis.

We construct the specific (ij)-covers by taking two copies of $\mathbb{D}_n$, cutting each along the corresponding α_{ij}-arc and then gluing the two cut disks together. Each cut disk corresponds to a fundamental domain for the deck group. We think of the α_{ij} as branch cuts.

In the $(i\infty)$-cover $\tilde{\mathbb{D}}_n$, each marked point of $\mathbb{D}_n$ has two pre-images except for the ith, which has one. We label the preimage in $\tilde{\mathbb{D}}_n$ of the ith marked point with i. For each $j \neq i$ we label the preimages of the jth marked point in the first and second fundamental domains of $\tilde{\mathbb{D}}_n$ with j and j', respectively. For an (ij)-cover with $j \neq \infty$ the marked points in $\tilde{\mathbb{D}}_n$ are labeled similarly. We denote by $[n]'$ the set of symbols $\{1', \ldots, n'\}$. So the labels of the marked points of $\tilde{\mathbb{D}}_n$ lie in $[n] \cup [n]'$.

We remark that the cover $\tilde{\mathbb{D}}_n$, and hence the labeling of its marked points, is sensitive to the homotopy class of each α_{ij}, not just the corresponding class in $H_1(\mathbb{D}_n, P \cup \partial\mathbb{D}_n; \mathbb{Z}/2)$.

3.2. **Homomorphisms from double covers.** Our next goal is to define ψ and the ψ_{ij}. The ψ_{ij} will be defined as follows: given an element of $\mathrm{B}_n[4]$, lift it to the corresponding cover $\tilde{\mathbb{D}}_n$, and then take an abelian quotient of the pure mapping class group $\mathrm{PMod}(\tilde{\mathbb{D}}_n)$. In general, for a surface S with marked points, $\mathrm{PMod}(S)$ is the subgroup of $\mathrm{Mod}(S)$ given by the kernel of the action on the set of marked points.

We will carry out the plan described in the previous paragraph by explicitly describing the lifting maps and the abelian quotients. The latter will be aided by another homomorphism, called the capping homomorphism, which we define along the way.

Lifting. We begin with the lifting homomorphism. Consider an element of $[n]^{\underline{2}}_\infty$ and let $\tilde{\mathbb{D}}_n$ denote the corresponding branched cover. There is a homomorphism

$$\mathrm{Lift} : \mathrm{PB}_n \to \mathrm{Mod}(\tilde{\mathbb{D}}_n),$$

defined as follows. Let $f \in \mathrm{PB}_n \cong \mathrm{PMod}(\mathbb{D}_n)$ and let $\phi : \mathbb{D}_n \to \mathbb{D}_n$ be a representative homeomorphism fixing the boundary. Because f lies in PB_n it fixes the element of $H_1(\mathbb{D}_n, P \cup \partial\mathbb{D}_n; \mathbb{Z}/2)$ corresponding to $\tilde{\mathbb{D}}_n$. Hence ϕ lifts to a homeomorphism of $\tilde{\mathbb{D}}_n$. There is a unique lift that induces the identity map on $\partial\tilde{\mathbb{D}}_n$; let $\tilde{f}$ be the corresponding element of $\mathrm{Mod}(\tilde{\mathbb{D}}_n)$. Then Lift is defined by

$$\mathrm{Lift}(f) = \tilde{f}.$$

Lemma 3.1. *Let $n \geq 2$, let $\{i, j\} \subset [n]_\infty$, and let $\tilde{\mathbb{D}}_n$ denote the corresponding branched cover of $\mathbb{D}_n$. The lifting homomorphism* $\mathrm{Lift} : \mathrm{PB}_n \to \mathrm{Mod}(\tilde{\mathbb{D}}_n)$ *restricts to a homomorphism*

$$\mathrm{B}_n[4] \to \mathrm{PMod}(\tilde{\mathbb{D}}_n).$$

Proof. As mentioned in Section 2, it is a theorem of Brendle and the second author that $\mathrm{B}_n[4]$ is equal to the subgroup of PB_n generated by squares of Dehn twists. Therefore, it is enough to show that if c is a simple closed curve in $\mathbb{D}_n$ then $\mathrm{Lift}(T_c^2)$ lies in $\mathrm{PMod}(\tilde{\mathbb{D}}_n)$.

The preimage $\tilde{c}$ in $\tilde{\mathbb{D}}_n$ of a simple closed curve c in $\mathbb{D}_n$ is a 2-fold cover of c. In particular it has one or two components. In the first case, T_c^2 lifts to the Dehn twist about $\tilde{c}$. In the second case, T_c^2 lifts to the product of the squares of the Dehn twists about the components of $\tilde{c}$. In both cases, the lift lies in $\mathrm{PMod}(\tilde{\mathbb{D}}_n)$, as desired. □

Capping. We now proceed to the capping homomorphism. Let S be a surface with boundary. We choose one distinguished component of the boundary of S. Let $\hat{S}$ be the surface obtained by gluing a disk to this component. There is a homomorphism

$$\mathrm{Cap} : \mathrm{Mod}(S) \to \mathrm{Mod}(\hat{S}),$$

defined as follows: given an element f of $\mathrm{Mod}(S)$ we can represent it by a homeomorphism of S that fixes the boundary, and then extend this homeomorphism to $\hat{S}$ in such a way that the extension is the identity on the complement of S. The resulting mapping class is $\mathrm{Cap}(f)$.

The abelianization of the pure braid group. The abelianization of the pure braid group is:

$$\mathrm{PB}_n / [\mathrm{PB}_n, \mathrm{PB}_n] \cong H_1(\mathrm{PB}_n; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{2}}.$$

The abelianization map

$$\mathrm{PB}_n \to \mathbb{Z}^{\binom{n}{2}}$$

can be described as follows. There are $\binom{n}{2}$ forgetful homomorphisms

$$\mathrm{PB}_n \to \mathrm{PB}_2 \cong \mathbb{Z}$$

obtained by forgetting all but two of the marked points in $\mathbb{D}_n$, and the abelianization of PB_n is the direct sum of these homomorphisms.

We now give a slightly different, and more natural, description of the abelianization of PB_n. We will denote the element $\{i, j\}$ of $[n]^{\underline{2}}$ by (ij) (so $(ji) = (ij)$). Let $\mathbb{Z}\{(ij)\}_{i<j}$ denote the free abelian group on the set of elements of $[n]^{\underline{2}}$. We can write the abelianization of PB_n as

$$\mathrm{PB}_n \to \mathbb{Z}\{(ij)\}_{i<j},$$

where the (ij)-factor of $\mathbb{Z}\{(ij)\}_{i<j}$ corresponds to the map $\mathrm{PB}_n \to \mathrm{PB}_2$ where all marked points except the ith and the jth are forgotten. This notation will be especially useful when the labels of the marked points are not natural numbers, as is the case in our $(i\infty)$- and (ij)-covers.

For any subset $A = \{i_1, i_2, \cdots, i_k\}$ of $\{1, \ldots, n\}$ there is an associated element of the free abelian group $\mathbb{Z}\{(ij)\}_{i<j}$. This element may be denoted by (A) or $(i_1 i_2 \cdots i_k)$ and it is defined as

$$(i_1 i_2 \cdots i_k) = \sum_{p<q} (i_p i_q).$$

For example, $(123) = (12) + (13) + (23)$. In this notation it makes sense to interpret $(\emptyset)$ as the identity. This language makes it convenient to describe the image of a Dehn twist under the abelianization of PB_n: if c is a simple closed curve in $\mathbb{D}_n$ surrounding the marked points $\{i_1, \ldots, i_k\}$, then

$$T_c \mapsto (i_1 i_2 \cdots i_k).$$

The braid group B_n acts on PB_n, hence its abelianization, by conjugation. The action of a particular element $f \in \mathrm{B}_n$ depends only on its image in the quotient $\mathrm{B}_n / \mathrm{PB}_n$, which is isomorphic to the symmetric group on $[n]$. If the image of f is the permutation σ then the action f_* on $\mathbb{Z}\{(ij)\}_{i<j}$ is given by $f_*(ij) = (\sigma(i)\sigma(j))$.

The definition of ψ. As advertised, we define

$$\psi : \mathrm{B}_n[4] \to \mathbb{Z}\{(ij)\}_{i<j} \cong \mathbb{Z}^{\binom{n}{2}}$$

as simply the restriction of the abelianization of PB_n. From our description of the latter we immediately obtain a formula for the image of the square of an arbitrary Dehn twist under the map ψ: if c is a simple closed curve in $\mathbb{D}_n$ and A is the set of labels of marked points in the interior of c, then

$$\psi(T_c^2) = 2(A).$$

If $f \in \mathrm{B}_n$ maps to f_* in the symmetric group on $[n]$ then

$$\psi(f \cdot T_c^2) = 2(f_*(A)).$$

Our goal in the remainder of this section is to define the ψ_{ij} and obtain similar formulas for the image of a square of a Dehn twist.

The definitions of the ψ_{ij}. We are finally ready to define the ψ_{ij}. First, for $i \in [n]$ we define $\psi_{i\infty}$ as the composition

$$\psi_{i\infty} : \mathrm{B}_n[4] \overset{\mathrm{Lift}}{\to} \mathrm{PMod}(\tilde{\mathbb{D}}_n) \to \mathbb{Z}\{(k\ell)\}_{\{k,\ell\} \subseteq L_{i\infty}} \cong \mathbb{Z}^{\binom{2n-1}{2}},$$

where $\tilde{\mathbb{D}}_n$ is the $(i\infty)$-cover of $\mathbb{D}_n$ and $L_{i\infty}$ is the set of labels of the marked points of $\tilde{\mathbb{D}}_n$. The existence of the first map is ensured by Lemma 3.1. The second map is the abelianization. The isomorphism at the end comes from the fact that the $(i\infty)$-cover is a disk with $2n-1$ marked points, with $L_{i\infty} = [n] \cup [n]' \setminus i'$.

For $\{i,j\} \in [n]^{\underline{2}}$ we denote by $\tilde{\mathbb{D}}_n$ the (ij)-cover of $\mathbb{D}_n$ and by L_{ij} the corresponding set of marked points. Then we define ψ_{ij} in the analogous way. The only difference is that we must apply the capping homomorphism (as above, we cap the component of the boundary lying in the second fundamental domain, where the marked points are labeled with primed numbers):

$$\psi_{ij} : \mathrm{B}_n[4] \overset{\mathrm{Lift}}{\to} \mathrm{PMod}(\tilde{\mathbb{D}}_n) \overset{\mathrm{Cap}}{\to} \mathrm{PB}_{2n-2} \to \mathbb{Z}\{(k\ell)\}_{\{k,\ell\} \subseteq L_{ij}} \cong \mathbb{Z}^{\binom{2n-2}{2}}.$$

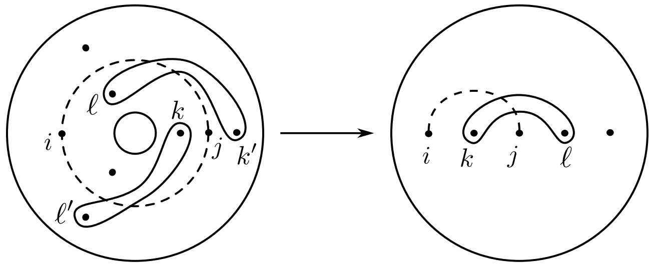

FIGURE 2. A representative case for the proof of Lemma 3.2

The induced maps. Since ψ and the ψ_{ij} are maps to abelian groups, they also induce maps

$$\begin{aligned} \psi &: H_1(\mathrm{B}_n[4];\mathbb{Q}) \to \mathbb{Q}^{\binom{n}{2}}, \\ \psi_{i\infty} &: H_1(\mathrm{B}_n[4];\mathbb{Q}) \to \mathbb{Q}^{\binom{2n-1}{2}} \quad 1 \le i \le n, \text{ and} \\ \psi_{ij} &: H_1(\mathrm{B}_n[4];\mathbb{Q}) \to \mathbb{Q}^{\binom{2n-2}{2}} \quad 1 \le i < j \le n \end{aligned}$$

As shown here, we refer to the induced maps by the same symbols as the original maps.

3.3. **Computations.** Our next goal is to describe the images under the ψ_{ij} of a square of a Dehn twist, first for squares of Artin generators and then for arbitrary squares of Dehn twists. The statement of the first lemma requires some notation. Let $\{i,j\}$ be an element of $[n]^{\underline{2}}_\infty$ with $i<j$ and let $\{k,\ell\}$ be an element of $[n]^{\underline{2}}$ with $k<\ell$. We say that $\{i,j\}$ and $\{k,\ell\}$ are *linked* if

$$i<k<j<\ell \quad \text{or} \quad k<i<\ell<j$$

and we say that they are *unlinked* if

$$i<k<\ell<j \quad \text{or} \quad k<i<j<\ell.$$

Also, we write $[n \setminus k,\ell]$ for $[n] \setminus \{k,\ell\}$.

Lemma 3.2. *Let $k,\ell \in [n]$ with $k<\ell$ and let $i,j \in [n]_\infty$ with $i<j$. Then*

$$\psi_{ij}(T_{k\ell}^2) = \begin{cases} (k\ell\ell') & \{i,j\}\cap\{k,\ell\} = \{k\} \\ (kk'\ell) & \{i,j\}\cap\{k,\ell\} = \{\ell\} \\ 2(k\ell)+2(k'\ell') & \{i,j\},\{k,\ell\} \textit{ unlinked} \\ 2(k\ell')+2(k'\ell) & \{i,j\},\{k,\ell\} \textit{ linked} \\ 2\left(\{k,\ell\}\cup[n\setminus k,\ell]'\right)+2([n\setminus k,\ell]') & \{i,j\}=\{k,\ell\} \end{cases}$$

Proof. Let c be the curve in $\mathbb{D}_n$ corresponding to the Artin generator $T_{k\ell}$. As discussed in the proof of Lemma 3.1, the preimage in $\tilde{\mathbb{D}}_n$ is either a single curve $\tilde{c}$ or a pair of curves $\tilde{c}_1, \tilde{c}_2$. In the first case $\mathrm{Lift}(T_c^2)$ is equal to $T_{\tilde{c}}$ and in the second case it is equal to $T_{\tilde{c}_1}^2 T_{\tilde{c}_1}^2$.

By the way the ψ_{ij} are defined, and because we already have a formula for the image of a Dehn twist in the abelianization the pure braid group, it remains to determine the preimage of c in each case. In fact, the only relevant feature of the preimage of c is the set of marked points that it surrounds.

There are nine cases, as the fifth case of the lemma only makes sense for (ij)-covers with $i,j \neq \infty$. A representative picture for the case where $j \neq \infty$ and $\{i,j\}$ and $\{k,\ell\}$ are linked is shown in Figure 2. The curve c is the boundary of a regular neighborhood of the arc $\alpha = \alpha_{k\ell}$. Therefore the preimage of c is the boundary of a regular neighborhood of the preimage of α. The path lift of α starting at the marked point k crosses the preimage of α_{ij} and ends at ℓ'; this path lift is hence a connected component of the preimage of α. Similarly, the other component of the preimage of α is an arc connecting the marked points k' and ℓ. So the preimage of c is a pair of curves, one surrounding the marked points k and ℓ' and one surrounding the marked points k' and ℓ. It follows that $\psi_{ij}(T_{k\ell}^2) = 2(k\ell') + 2(k'\ell)$, as in the statement of the lemma. The other cases are handled similarly. □

Closed formulas. We also have a closed formula for an arbitrary $\psi_{ij}(T_c^2)$. The formula has three parts, depending on how many of $\{i, j\}$ lie in the interior of c. Since we will not require these general formulas in the sequel, we do not supply the proofs.

Let A be the set of labels of the marked points lying in the interior of c. In the case where $A \cap \{i, j\} = \{i\}$ the formula is:

$$\psi_{ij}(T_c^2) = \left(A \cup (A - \{i\})'\right).$$

We now suppose that $A \cap \{i, j\} = \emptyset$. In this case there is a natural partition of A into two subsets: two elements of A are in the same subset if an arc that lies in the interior of c and connects the corresponding marked points intersects α_{ij} in an even number of points. If we denote the two subsets of A by A_1 and A_2, the formula is

$$\psi_{ij}(T_c^2) = 2(A_1 \cup A_2') + 2(A_1' \cup A_2).$$

In the third and final case, where $\{i, j\} \subseteq A$, we need to define two subsets B and C of $[n] \setminus A$. An element of $[n] \setminus A$ lies in B if and only if an arc that lies in the exterior of c and connects that marked point to $\partial \mathbb{D}_n$ crosses α_{ij} in an even number of points. Then C is the complement of B in $[n] \setminus A$. We have in this case

$$\psi_{ij}(T_c^2) = 2(A \cup C \cup B' \cup (A \setminus \{i, j\})') + 2(C \cup B').$$

It is straightforward to check that in the case where $T_c = T_{k\ell}$ our formulas here agree with Lemma 3.2. For instance, in the third case we have $A = \{k, \ell\}$, $B = [n] \setminus \{k, \ell\}$, and $C = \emptyset$. Thus our formula gives that $\psi_{k\ell}(T_{k\ell}^2)$ is

$$2(A \cup C \cup B' \cup (A \setminus \{i, j\})') + 2(C \cup B') = 2(\{k, \ell\} \cup [n \setminus k, \ell]') + 2([n \setminus k, \ell]'),$$

as per Lemma 3.2.

3.4. **Naturality.** Our final task in this section to give a formula for the image under ψ_{ij} of

$$T_{k\ell} \cdot f = T_{k\ell} f T_{k\ell}^{-1}$$

in terms of the image of f, where $f \in \mathrm{B}_n[4]$ and $T_{k\ell}$ is an Artin generator for PB_n. Since we already have a formula for each $\psi_{ij}(T_{k\ell}^2)$ (Lemma 3.2), this will give a formula for the image under ψ_{ij} of an arbitrary $T_{k\ell} \cdot T_{pq}^2$.

Let $\{i, j\} \in [n]^2_\infty$ with $i < j$ and $k, \ell \in [n]$ with $k < \ell$. We define a permutation $\iota_{k\ell}^{ij}$ of the set $[n] \cup [n]'$ as follows:

$$\iota_{k\ell}^{ij} = \begin{cases} (\ell\ \ell') & \{i, j\} \cap \{k, \ell\} = \{k\} \\ (k\ k') & \{i, j\} \cap \{k, \ell\} = \{\ell\} \\ id & \text{otherwise.} \end{cases}$$

In this formula we are using cycle notation for the symmetric group on $[n] \cup [n]'$, so $(k\ k')$ and $(\ell\ \ell')$ are transpositions.

Let $\{i, j\} \in [n]^2_\infty$, let $\tilde{\mathbb{D}}_n$ be the corresponding branched cover of $\mathbb{D}_n$, and let L_{ij} be the set of labels of the marked points of $\tilde{\mathbb{D}}_n$, as in Section 3.2. We may regard $\iota^{ij}_{k\ell}$ as a permutation of L_{ij}, and as such it acts on the abelianization of $\mathrm{PMod}(\tilde{\mathbb{D}}_n)$. We abuse notation and write the corresponding automorphism of the abelianization as $\iota^{ij}_{k\ell}$.

Lemma 3.3. *Let $\{i, j\} \in [n]^2_\infty$ with $i < j$ and $\{k, \ell\} \in [n]^2$ with $k < \ell$. Let f be an element of $\mathrm{B}_n[4]$. Then*

$$\psi_{ij}(T_{k\ell} \cdot f) = \iota^{ij}_{k\ell}\left(\psi_{ij}(f)\right).$$

Proof. For concreteness, we suppose that $j \neq \infty$; the case where $j = \infty$ is essentially the same. As in Section 3.2, the set L_{ij} is $[n] \cup [n]' \setminus \{i', j'\}$. Let ψ denote the abelianization of $\mathrm{PMod}(\tilde{\mathbb{D}}_n) \cong \mathrm{PB}_{2n-2}$. We have that $\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})$ is an element of $\mathrm{Mod}(\tilde{\mathbb{D}}_n) \cong \mathrm{B}_{2n-2}$. As such it acts on $\mathrm{PMod}(\tilde{\mathbb{D}}_n)$ by conjugation. As in Section 3.2 we denote the induced action on the abelianization by $\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})_*$. We have:

$$\begin{aligned}
\psi_{ij}(T_{k\ell} \cdot f) &= \psi(\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell} \cdot f)) \\
&= \psi(\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell}) \cdot \mathrm{Cap} \circ \mathrm{Lift}(f)) \\
&= \mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})_* \psi(\mathrm{Cap} \circ \mathrm{Lift}(f)) \\
&= \mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})_* \psi_{ij}(f).
\end{aligned}$$

It remains to check that $\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})_*$ is equal to $\iota^{ij}_{k\ell}$.

If $\{i, j\} \cap \{k, \ell\}$ is not a singleton, then the simple closed curve c in $\mathbb{D}_n$ corresponding to $\{k, \ell\}$ has an even number of intersections with the arc α_{ij}. Thus the preimage of c in $\tilde{\mathbb{D}}_n$ is a pair of curves. As in the proof of Lemma 3.1, it follows that $\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})$ lies in PB_{2n-2}. This agrees with the fact that $\iota^{ij}_{k\ell}$ is trivial in this case.

Suppose that on the other hand $\{i, j\} \cap \{k, \ell\} = \{k\}$. In this case the simple closed curve c in $\mathbb{D}_n$ corresponding to $\{k, \ell\}$ intersects the arc α_{ij} in a single point. So the preimage of c in $\tilde{\mathbb{D}}_n$ is a single curve $\tilde{c}$ surrounding the points labeled k, ℓ, and k'. And $\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})$ interchanges the points labeled k and k' (the square of $\mathrm{Cap} \circ \mathrm{Lift}(T_{k\ell})$ is the Dehn twist about $\tilde{c}$). This again agrees with the definition of $\iota k\ell^{ij}$. The case $\{i, j\} \cap \{k, \ell\} = \{\ell\}$ is exactly the same, with the roles of k and ℓ interchanged. ☐

We now give three sample computations with Lemmas 3.2 and 3.3. First, for any $n \geq 3$ we have:

$$\begin{aligned}
\psi_{1\infty}(T_{12} \cdot T_{23}^2) &= \iota^{1\infty}_{12}\left(\psi_{1\infty}(T_{23}^2)\right) \\
&= \iota^{1\infty}_{12}\left(2(23) + 2(2'3')\right) \\
&= 2(23') + 2(2'3).
\end{aligned}$$

Second, for any $n \geq 3$ we have:

$$\begin{aligned}
\psi_{12}(T_{23} \cdot T_{12}^2) &= \iota^{12}_{23}\left(\psi_{12}(T_{12}^2)\right) \\
&= \iota^{12}_{23}\left(2(123'4' \cdots n') + 2(3'4' \cdots n')\right) \\
&= 2(1234' \cdots n') + 2(34' \cdots n').
\end{aligned}$$

Third, for any $n \geq 3$ we have:

$$\begin{aligned}\psi_{1\infty}([T_{23}^2, T_{12}]) &= \psi_{1\infty}(T_{23}^2(T_{12} \cdot T_{23}^{-2})) \\ &= \psi_{1\infty}(T_{23}^2)\iota_{12}^{1\infty}\left(\psi_{1\infty}(T_{23}^{-2})\right) \\ &= 2(23) + 2(2'3') - \iota_{12}^{1\infty}(2(23) + 2(2'3')) \\ &= 2(23) + 2(2'3') - 2(2'3) - 2(23').\end{aligned}$$

As in Section 2, we refer to the image of $T_{k\ell}^2$ in $H_1(\mathrm{B}_n[4];\mathbb{Q})$ by $\tau_{k\ell}$. Regarding the ψ_{ij} as maps defined on $H_1(\mathrm{B}_n[4];\mathbb{Q})$, we can reinterpret the above calculations as:

$$\psi_{1\infty}(T_{12}\tau_{23}) = 2(23') + 2(2'3), \quad \psi_{12}(T_{23}\tau_{12}) = 2(1234' \cdots n') + 2(34' \cdots n'), \quad \text{and}$$
$$\psi_{1\infty}((1 - T_{12})\tau_{23}) = 2(23) + 2(2'3') - 2(2'3) - 2(23').$$

4. Proof of Theorem 2.1 for three strands

In this section we prove Theorem 2.1 for the case of $B_3[4]$. This result is stated as Corollary 4.3 below. We begin by showing that $\dim H_1(B_3[4];\mathbb{Q}) = 6$ (Proposition 4.1). Then we show that a set $\mathcal{S}'$ closely related to the set $\mathcal{S}$ from Theorem 2.1 forms a basis (Proposition 4.2). We then use this to prove Corollary 4.3.

Below, we denote by $\mathrm{PMod}_{0,m}$ the pure mapping class group of a sphere with m marked points.

Proposition 4.1. *The dimension of* $H_1(B_3[4];\mathbb{Q})$ *is equal to 6.*

Proof. We may glue a disk with one marked point to the boundary of $\mathbb{D}_3$ in order to obtain a sphere with four marked points. There is a resulting capping homomorphism

$$\mathrm{PB}_3 \to \mathrm{PMod}_{0,4}\,.$$

This map is defined analogously to the capping homomorphism in Section 3.2. The kernel is the infinite cyclic group generated by T_∂, the Dehn twist about the boundary of $\mathbb{D}_3$ [17, Proposition 3.19].

Since $\mathrm{B}_3[4] = \mathrm{PB}_3^2$ and since $\langle T_\partial \rangle \cap \mathrm{B}_3[4] = \langle T_\partial^2 \rangle$ we may restrict the capping homomorphism to obtain a short exact sequence

$$1 \to \langle T_\partial^2 \rangle \to B_3[4] \to \mathrm{PMod}_{0,4}^2 \to 1$$

where $\mathrm{PMod}_{0,4}^2$ denotes the subgroup of $\mathrm{PMod}_{0,4}$ generated by all squares. This extension gives rise to an exact sequence in homology

$$\mathbb{Q}\langle T_\partial^2 \rangle \to H_1(B_3[4];\mathbb{Q}) \to H_1(\mathrm{PMod}_{0,4}^2;\mathbb{Q}) \to 0$$

We analyze the terms on the right and left in turn. We claim that the term on the right is isomorphic to $\mathbb{Q}^5$. By the Birman exact sequence [17, Theorem 4.6] and the fact that $\mathrm{PMod}_{0,3}$ is trivial [17, Proposition 2.3] we have that $\mathrm{PMod}_{0,4}$ is a free group of rank 2. The index of $\mathrm{PMod}_{0,4}^2$ in $\mathrm{PMod}_{0,4}$ is 4 (it is the kernel of the mod 2 abelianization). By the Nielsen–Schreier formula the former is a free group of rank 5 and the claim follows.

We next claim that the first map is injective. The sequence of inclusions $\langle T_\partial^2 \rangle \to \mathrm{B}_3[4] \to \mathrm{PB}_3$ induces maps on homology $\mathbb{Q}\langle T_\partial^2 \rangle \to H_1(\mathrm{B}_3[4];\mathbb{Q}) \to H_1(\mathrm{PB}_3;\mathbb{Q})$. It follows from the discussion in Section 3.2 that the composition is nontrivial, and the claim follows.

It follows from the two claims that $\dim H_1(B_3[4];\mathbb{Q}) = 1 + 5 = 6$. □

Consider the set $\mathcal{S}' = \mathcal{S}_1 \cup \mathcal{S}_2'$ where

$$\mathcal{S}_1 = \{\tau_{12},\ \tau_{13},\ \tau_{23}\} \quad \text{and} \quad \mathcal{S}_2' = \{(1 - T_{13})\tau_{12},\ (1 - T_{23})\tau_{13},\ (1 - T_{12})\tau_{23}\}$$

Proposition 4.2. *The set $\mathcal{S}'$ is a basis for $H_1(B_3[4];\mathbb{Q})$.*

Proof. The first step is to compute the values of ψ on the elements of $\mathcal{S}'$. We find that

$$\psi(\tau_{12}) = 2(12) \qquad \psi((1-T_{13})\tau_{12}) = 0$$
$$\psi(\tau_{13}) = 2(13) \qquad \psi((1-T_{23})\tau_{13}) = 0$$
$$\psi(\tau_{23}) = 2(23) \qquad \psi((1-T_{12})\tau_{23}) = 0.$$

Thus $\{\tau_{12},\tau_{13},\tau_{23}\}$ is linearly independent and the span of $\mathcal{S}_2'$ is contained in the kernel of ψ. Thus it suffices to show that $\mathcal{S}_2'$ is linearly independent. Let δ_{ij} denote $(ij)+(i'j')-(ij')-(i'j)$. The images of the elements of $\mathcal{S}_2'$ under the $\psi_{i\infty}$ are as shown in the following table (the top-right entry was computed as an example in Section 3.4):

	$(1-T_{13})\tau_{12}$	$(1-T_{23})\tau_{13}$	$(1-T_{12})\tau_{23}$
$\psi_{1\infty}$	0	0	$2\delta_{23}$
$\psi_{2\infty}$	0	$-2\delta_{13}$	0
$\psi_{3\infty}$	$2\delta_{12}$	0	0

Since each element of $\mathcal{S}_2'$ has nontrivial image under exactly one $\psi_{i\infty}$ it follows that $\mathcal{S}_2'$ is linearly independent, and the proposition follows. □

As in Section 2, let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$\mathcal{S}_2 = \{T_{13}\tau_{12},\ T_{23}\tau_{13},\ T_{12}\tau_{23}\}.$$

The set $\mathcal{S}$ lies in the span of $\mathcal{S}'$ and vice versa. Since the two sets have the same cardinality, we have the following corollary of Proposition 4.2, which is the $n = 3$ case of Theorem 2.1.

Corollary 4.3. *The set $\mathcal{S}$ is a basis for $H_1(B_3[4];\mathbb{Q})$.*

5. A linearly independent set

Let $\mathcal{S}$ be the subset of $H_1(\mathrm{B}_n[4];\mathbb{Q})$ from Theorem 2.1. The goal of this section is to prove the "lower bound" for Theorem 2.1, namely, that $\mathcal{S}$ is linearly independent (Proposition 5.1 below). The proof will make use of the homomorphisms introduced in Section 3.

As in Section 4 we will prove that $\mathcal{S}$ is linearly independent by showing that a slightly different set $\mathcal{S}'$ is linearly independent. Specifically, let $\mathcal{S}' = \mathcal{S}_1 \cup \mathcal{S}_2' \cup \mathcal{S}_3'$, where

$$\mathcal{S}_1 = \{\tau_{ij} \mid 1 \le i < j \le n\},$$
$$\mathcal{S}_2' = \{(1-T_{jk})\tau_{ij}, (1-T_{jk})\tau_{ik}, (1-T_{ij})\tau_{jk} \mid 1 \le i < j < k \le n\}, \text{ and}$$
$$\mathcal{S}_3' = \{(1-T_{i\ell})(1-T_{jk})\tau_{ij}, (1-T_{ij})(1-T_{k\ell})\tau_{ik}, (1-T_{ik})(1-T_{j\ell})\tau_{i\ell} \mid 1 \le i < j < k < \ell \le n\}.$$

For a group G and $g,h \in G$ we denote by $[g,h]$ the commutator $ghg^{-1}h^{-1}$. The elements $(1-T_{k\ell})\tau_{ij}$ are the images of the commutators $[T_{k\ell}, T_{ij}^2]$ in $H_1(\mathrm{B}_n[4];\mathbb{Q})$. Similarly, the $(1-T_{pq})(1-T_{k\ell})\tau_{ij}$ are the images of $[T_{pq},[T_{k\ell},T_{ij}^2]]$.

Proposition 5.1. *For all $n \ge 4$ the sets $\mathcal{S}$ and $\mathcal{S}'$ are linearly independent.*

In the proof we use a homomorphism F_n defined as follows. Since $\mathrm{B}_n[4]$ is the subgroup of PB_n generated by all squares, there are well-defined maps $\mathrm{B}_n[4] \to \mathrm{B}_{n-k}[4]$ for all $0 \le k < n$ obtained by forgetting k of the marked points in $\mathbb{D}_n$. There are thus induced maps $H_1(\mathrm{B}_n[4];\mathbb{Q}) \to H_1(\mathrm{B}_{n-k}[4];\mathbb{Q})$. By introducing formal variables $\varepsilon_{ijk\ell}$, ε_{ijk} and ε_{ij} we can combine the maps with $2 \leq n-k \leq 4$ into a single homomorphism F_n. When $n=4$ it will also be convenient to use a function $\bar{F}_4$ that is defined in the same way as F_4 except without the terms corresponding to quadruples:

$$\bar{F}_4 : H_1(B_n[4];\mathbb{Q}) \to \left(\bigoplus_{i<j<k} H_1(B_3[4];\mathbb{Q}) \otimes \varepsilon_{ijk} \right) \bigoplus \left(\bigoplus_{i<j} H_1(B_2[4];\mathbb{Q}) \otimes \varepsilon_{ij} \right),$$

where ε_{ijk} corresponds to the map $\mathrm{B}_n[4] \to \mathrm{B}_3[4]$ obtained by forgetting the marked point not labeled by i, j, or k and ε_{ijk} corresponds to the map $\mathrm{B}_n[4] \to \mathrm{B}_2[4]$ obtained by forgetting the marked point not labeled by i or j.

Proof of Proposition 5.1. We proceed in two steps, first dealing with the case $n=4$ and then the general case. In both cases it suffices to show that $\mathcal{S}'$ is independent, since (as in the proof of Corollary 4.3) each element of $\mathcal{S}$ lies in the span of $\mathcal{S}'$ and vice versa.

We now proceed with the proof for the case $n=4$. We first claim that $\mathcal{S}_1$ is linearly independent. For each $i<j$ we have

$$\bar{F}_4(\tau_{ij}) = \tau_{ij} \otimes \varepsilon_{ij} + \sum \tau_{ij} \otimes \varepsilon_{abc},$$

where the sum is over all triples (a,b,c) with $a<b<c$ such that $i,j \in \{a,b,c\}$. Since each ε_{ij} only appears in the image of τ_{ij}, and since $H_1(\mathrm{B}_2[4];\mathbb{Q})$ is non-zero, the claim follows.

We next claim that $\mathcal{S}_1 \cup \mathcal{S}_2'$ is linearly independent. We begin by computing the images of the elements of $\mathcal{S}_2'$ under $\bar{F}_4$. For fixed $i<j<k$, we have

$$\begin{aligned}
\bar{F}_4((1-T_{jk})\tau_{ij}) &= (1-T_{jk})\tau_{ij} \otimes \varepsilon_{ijk}, \\
\bar{F}_4((1-T_{jk})\tau_{ik}) &= (1-T_{jk})\tau_{ik} \otimes \varepsilon_{ijk}, \text{ and} \\
\bar{F}_4((1-T_{ij})\tau_{jk}) &= (1-T_{ij})\tau_{jk} \otimes \varepsilon_{ijk}.
\end{aligned}$$

First of all, since there are no ε_{ij} terms here, it is enough to check that $\mathcal{S}_2'$ is independent. Second, since exactly three of the twelve elements of $\mathcal{S}_2'$ have an ε_{ijk} term in their images, it is enough to check that these three elements are linearly independent. By replacing i, j, and k with 1, 2, and 3, we see that this is equivalent to the statement that the set $\mathcal{S}_2'$ from the proof of Proposition 4.2 is independent. Applying that proposition, the claim is proven.

The set $\mathcal{S}_3'$ lies in the kernel of $\bar{F}_4$, and so to prove the $n=4$ case it suffices to show that $\mathcal{S}_3'$ is linearly independent. To do this we compute the images under ψ_{12}, ψ_{13}, and ψ_{14}:

	$(1-T_{14})(1-T_{23})\tau_{12}$	$(1-T_{12})(1-T_{34})\tau_{13}$	$(1-T_{13})(1-T_{24})\tau_{14}$
ψ_{12}	$4\delta_{34}$	0	0
ψ_{13}	0	$4\delta_{24}$	0
ψ_{14}	0	0	$4\delta_{23}$

As in the proof of Proposition 4.2 the symbol δ_{ij} denotes the element $(ij)+(i'j')-(ij')-(i'j)$.

It is clear from the table that the image of $\mathcal{S}_3'$ is independent, and so it remains to verify the entries of the table. The calculations for the three rows are similar. We show details for only the first one. For this we have

$$(1-T_{14})(1-T_{23})\tau_{12} = \tau_{12} - T_{14}\tau_{12} - T_{23}\tau_{12} + T_{14}T_{23}\tau_{12}$$

By Lemma 3.2, the image of τ_{12} under ψ_{12} is $2(123'4') + 2(3'4')$. By Lemma 3.3, the images of $T_{14}\tau_{12}$, $T_{23}\tau_{12}$, and $T_{14}T_{23}\tau_{12}$ are $2(123'4')+2(3'4')$, $2(123'4)+2(3'4)$, and $2(1234)+2(34)$. The calculation in the table follows. This completes the proof of the $n=4$ case.

We now proceed to the general case. The only elements of $\mathcal{S}'$ detected by the components of F_n corresponding to pairs of marked points are the elements of $\mathcal{S}_1$. The set $\mathcal{S}_1$ is linearly independent for the same reason it is in the proof of the $n = 4$ case, namely, the fact that $H_1(\mathrm{B}_2[4]; \mathbb{Q})$ is nontrivial. It then remains to show that $\mathcal{S}_2' \cup \mathcal{S}_3'$ is linearly independent. The only elements of the latter detected by the components of F_n corresponding to triples of marked points are the elements of $\mathcal{S}_2'$. By applying Proposition 4.2 to each choice of triple, we conclude that $\mathcal{S}_2'$ is linearly independent. It then remains to check that $\mathcal{S}_3'$ is linearly independent. This follows by evaluating F_n on $\mathcal{S}_3'$ and applying the $n = 4$ case of the proposition. This completes the proof. □

6. A spanning set in terms of Artin generators

The main goal of this section is to prove the following proposition. In the statement $\{T_{k\ell}\}$ is the set of Artin generators for PB_n.

Proposition 6.1. *For all $n \geq 3$, the $\mathbb{Q}[\overline{PB}_n]$-module $H_1(B_n[4]; \mathbb{Q})$ is generated by $\{\tau_{ij}\}$. Equivalently, the vector space $H_1(B_n[4]; \mathbb{Q})$ is spanned by*

$$\mathcal{T} = \{(1 - T_1) \cdots (1 - T_m)\tau_{ij} \mid m \geq 0,\ T_1, \ldots, T_m \in \{T_{k\ell}\}\}.$$

We now explain why the two statements of the proposition are equivalent. Clearly any element of $\mathcal{T}$ lies in the module spanned by the τ_{ij}. For the other direction, suppose we have an element of $H_1(B_n[4]; \mathbb{Q})$ of the form $T\tau_{ij}$ with $T \in \mathrm{PB}_n$. We first write this as $T_1 \cdots T_m \tau_{ij}$ with each $T_i \in \{T_{k\ell}\}$ (note that no inverses are needed because the actions of T_i and T_i^{-1} are the same). Then we may inductively apply the formula $T\tau_{ij} = -(1 - T)\tau_{ij} + \tau_{ij}$ in order to express the original element as a linear combination of elements of $\mathcal{T}$.

The boundary twist. Let T_∂ denote the element of PB_n corresponding to the Dehn twist about $\partial\mathbb{D}_n$, and let τ_∂ denote the image of T_∂^2 in $H_1(B_n[4]; \mathbb{Q})$. We first express τ_∂ as a linear combination of elements $T\tau_{ij}$ as in the statement of Proposition 6.1. Then we introduce the squared lantern relation and use it to prove the proposition.

Lemma 6.2. *For all $n \geq 2$ we have*

$$\tau_\partial = 2^{-\binom{n}{2}}(1 + T_{12}) \cdots (1 + T_{n-1,n}) \sum_{i<j} \tau_{ij}.$$

In particular τ_∂ lies in the span of the τ_{ij}.

Proof. The steps of the proof are:

(1) Every $\mathcal{Z}_n$-invariant element of $H_1(\mathrm{B}_n[4]; \mathbb{Q})$ is a multiple of τ_∂.
(2) The following element of $H_1(\mathrm{B}_n[4]; \mathbb{Q})$ is $\mathcal{Z}_n$-invariant:

$$x = (1 + T_{12})(1 + T_{13}) \cdots (1 + T_{n-1,n}) \sum_{i<j} \tau_{ij}.$$

(3) $x = 2^{\binom{n}{2}} \tau_\partial$.

For the first step let $H_1(\mathrm{B}_n[4]; \mathbb{Q})^{\mathcal{Z}_n}$ and $H_1(\mathrm{B}_n[4]; \mathbb{Q})_{\mathcal{Z}_n}$ denote the spaces of $\mathcal{Z}_n$-invariants and $\mathcal{Z}_n$-coinvariants of $H_1(\mathrm{B}_n[4]; \mathbb{Q})$, respectively. Since $\mathcal{Z}_n$ is finite there are isomorphisms

$$H_1(B_n[4]; \mathbb{Q})^{\mathcal{Z}_n} \cong H_1(\mathrm{B}_n[4]; \mathbb{Q})_{\mathcal{Z}_n} \cong H_1(\mathrm{B}_n; \mathbb{Q}) \cong \mathbb{Q},$$

where the last isomorphism is induced by the signed word length homomorphism $\mathrm{B}_n \to \mathbb{Z}$. We conclude that, up to scale, there is a unique $\mathcal{Z}_n$-invariant element of $H_1(B_n[4]; \mathbb{Q})$. The image of T_∂^2 under the signed word length homomorphism is $4\binom{n}{2}$. This is because T_∂ can be written as a product of T_{ij} where each T_{ij} appears exactly once and because the signed word length of each T_{ij} is 2. Thus the image of τ_∂ of in $H_1(\mathrm{B}_n;\mathbb{Q}) \cong \mathbb{Q}$ is equal to $4\binom{n}{2} \neq 0$, so any $\mathcal{Z}_n$-invariant element of $H_1(B_n[4];\mathbb{Q})$ is a multiple of τ_∂.

We now proceed to the second step. Let $\sigma \in \mathrm{B}_n$ and let σ_* be the induced permutation of $[n]^{\underline{2}}$. We will use two facts. The first fact is that the action of σ on the image of $\{T_{ij}\}$ in $\mathcal{PZ}_n$ is a permutation. The second fact is that $\sigma\tau_{ij}$ is equal to $T\tau_{\sigma_*(ij)}$ for some $T \in \mathcal{PZ}_n$. Both statements hold because $\sigma \cdot T_{ij}^k$ is conjugate in PB_n to $T_{\sigma_*(ij)}^k$.

By the previous paragraph, the action of σ on the ij-term of x is given by

$$\sigma \cdot (1+T_{12}) \cdots (1+T_{n-1,n})\tau_{ij} = (1+T_{12}) \cdots (1+T_{n-1,n})T\tau_{\sigma_*(ij)}.$$

Here we have used the fact that $\mathbb{Q}[\mathcal{PZ}_n]$ is commutative and so the permutation of $\{T_{ij}\}$ induced by σ is irrelevant.

To complete the proof of the second step, it is then enough to show for $T \in \mathcal{PZ}_n$ that

$$(1+T_{12}) \cdots (1+T_{n-1,n})T = (1+T_{12}) \cdots (1+T_{n-1,n})$$

in $\mathbb{Q}[\mathcal{PZ}_n]$. If $T = T_{ij}$, then this follows from the equality

$$(1+T_{ij})T_{ij} = T_{ij} + T_{ij}^2 = T_{ij} + 1 = 1 + T_{ij}$$

in $\mathbb{Q}[\mathcal{PZ}_n]$ and the commutativity of the latter. If T is a product of more than one T_{ij} then we apply this equality inductively. This completes the proof of the second step.

We now proceed to the third step. As above, the image of τ_∂ in $H_1(\mathrm{B}_n;\mathbb{Q}) \cong \mathbb{Q}$ is equal to $4\binom{n}{2}$. We similarly compute the image of x to be $4\binom{n}{2}2^{\binom{n}{2}}$. Thus $x = 2^{\binom{n}{2}}\tau_\partial$, as desired. The lemma follows. ☐

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. We will use the theorem of Brendle and the second author that $\mathrm{B}_n[4]$ is equal to the subgroup of PB_n generated by squares of Dehn twists. Because of this theorem it is enough to show that for any curve c the image of T_c^2 in $\mathrm{B}_n[4]$ lies in the $\mathbb{Q}[\mathcal{PZ}_n]$-submodule of $H_1(B_n[4];\mathbb{Q})$ generated by $\{\tau_{ij}\}$. We first prove this in the special case where c is c_k, the round circle in $\mathbb{D}_n$ surrounding the first k marked points.

Fix some $k \leq n$. The standard inclusion $\mathrm{B}_k \to \mathrm{B}_n$ induces an inclusion $f : \mathrm{B}_k[4] \to \mathrm{B}_n[4]$. The latter further induces a map $f_* : H_1(\mathrm{B}_k[4];\mathbb{Q}) \to H_1(\mathrm{B}_n[4];\mathbb{Q})$. The map f_* sends each τ_{ij} in $H_1(\mathrm{B}_k[4];\mathbb{Q})$ to τ_{ij} in $H_1(\mathrm{B}_n[4];\mathbb{Q})$.

By Lemma 6.2, the element $\tau_\partial \in H_1(\mathrm{B}_k[4];\mathbb{Q})$ lies in the submodule of $H_1(\mathrm{B}_k[4];\mathbb{Q})$ generated by the τ_{ij}. By the previous paragraph, it follows that $f_*(\tau_\partial) \in H_1(\mathrm{B}_n[4];\mathbb{Q})$ lies in the submodule of $H_1(\mathrm{B}_n[4];\mathbb{Q})$ generated by the τ_{ij}. But since $f(T_\partial^2) = T_{c_k}^2$ we have that $f_*(\tau_\partial)$ is the class of $T_{c_k}^2$, and so this completes the proof of the special case.

Let c be an arbitrary curve in $\mathbb{D}_n$. Say that c surrounds k marked points of $\mathbb{D}_n$. There is a braid $\sigma \in \mathrm{B}_n$ with $\sigma(c_k) = c$ (this is a special case of the change of coordinates principle [17, Section 1.3]). Thus the image of T_c^2 in $H_1(\mathrm{B}_n[4];\mathbb{Q})$ is obtained by applying the action of σ to the image of T_{c_k}. But the $\mathbb{Q}[\mathcal{PZ}_n]$-submodule of $H_1(B_n[4];\mathbb{Q})$ generated by $\{\tau_{ij}\}$ is invariant under the action of B_n (as in the proof of Lemma 6.2, each $\sigma\tau_{ij}$ is equal to $T\tau_{k\ell}$ for some k, ℓ and $T \in \mathrm{PB}_n$), so the proposition follows. ☐

7. Basis and dimension

The goal of this section is to complete the proof of Theorem 2.1, which states that the set $\mathcal{S}$ from Section 2 is a basis for $H_1(\mathrm{B}_4[4];\mathbb{Q})$.

Proposition 5.1 states that the subset $\mathcal{S}'$ of $H_1(\mathrm{B}_n[4];\mathbb{Q})$ is linearly independent. The cardinality of $\mathcal{S}'$ is

$$3\binom{n}{4}+3\binom{n}{3}+\binom{n}{2}.$$

Proposition 6.1 gives a spanning set $\mathcal{T}$ for $H_1(\mathrm{B}_n[4];\mathbb{Q})$ that contains $\mathcal{S}'$. So our task in this section is to show that the elements of $\mathcal{T}\setminus\mathcal{S}'$ lie in the span of $\mathcal{S}'$. In fact, we will see that the elements of $\mathcal{T}\setminus\mathcal{S}'$ are all multiples of elements of $\mathcal{S}'$.

Before proving Theorem 2.1 we state and prove Lemma 7.1, which will allow us to eliminate many of the elements from the spanning set $\mathcal{T}$ for $H_1(\mathrm{B}_n[4];\mathbb{Q})$ given in Proposition 6.1. Next we state and prove Lemma 7.2, which gives an expression for the class of $[T_{ij},T_{jk}]$ in $H_1(\mathrm{B}_n[4];\mathbb{Q})$ in terms of the τ_{ij}. We then state and prove Lemma 7.3, a pair of algebraic identities used to prove the subsequent Lemma 7.4, which gives certain equalities between elements of $\mathcal{T}$. We then finally proceed to the proof of the theorem.

Lemma 7.1. *Let $n\geq 2$. The following statements hold in $H_1(\mathrm{B}_n[4];\mathbb{Q})$.*
(1) For $i<j<k$ we have

$$T_{ik}\tau_{ij}=T_{jk}\tau_{ij},\qquad T_{ij}\tau_{ik}=T_{jk}\tau_{ik},\qquad \text{and}\qquad T_{ij}\tau_{jk}=T_{ik}\tau_{jk}.$$

(2) For pairwise distinct i,j,k,ℓ we have

$$T_{ij}\tau_{k\ell}=\tau_{k\ell}.$$

Proof. We begin with the first statement. For $i<j<k$, we have the following standard Artin relations in PB_n:

$$[T_{ik}T_{jk},T_{ij}]=[T_{jk}T_{ij},T_{ik}]=[T_{ij}T_{ik},T_{jk}]=1.$$

This implies that for $i<j<k$ we have

$$[T_{ik}T_{jk},T_{ij}^2]=[T_{jk}T_{ij},T_{ik}^2]=[T_{ij}T_{ik},T_{jk}^2]=1.$$

These relations are also expressible as

$$(T_{ik}T_{jk})\cdot T_{ij}^2=T_{ij}^2\qquad (T_{ij}T_{jk})\cdot T_{ik}^2=T_{ik}^2\qquad (T_{ij}T_{ik})\cdot T_{jk}^2=T_{jk}^2.$$

Since an element of PB_n and its inverse have the same action on $H_1(\mathrm{B}_n[4];\mathbb{Q})$ the above relations take the following form in $H_1(\mathrm{B}_n[4];\mathbb{Q})$:

$$T_{ik}\tau_{ij}=T_{jk}\tau_{ij}\qquad T_{ij}\tau_{ik}=T_{jk}\tau_{ik}\qquad T_{ij}\tau_{jk}=T_{ik}\tau_{jk}.$$

This completes the proof of the first statement.

We now proceed to the second statement. Let i, j, k, and ℓ be distinct. Let us also assume that $i<j$ and $k<\ell$. There are six possible configurations for $\{i,j,k,\ell\}$, two of which are linked and four of which are unlinked (see Section 3.3 for the definition of linked). In the unlinked cases the result follows from the fact that Dehn twists about disjoint curves commute. Thus it remains only to consider the linked cases $i<k<j<\ell$ and $k<i<\ell<j$. The two cases are essentially the same, so we deal only with the first.

If $i<k<j<\ell$ then in PB_n we have the standard Artin relation $(T_{j\ell}T_{ij}T_{j\ell}^{-1})\cdot T_{k\ell}=T_{k\ell}$ which (as above) gives the relation

$$(T_{j\ell}T_{ij}T_{j\ell}^{-1})\cdot T_{k\ell}^2=T_{k\ell}^2.$$

In $H_1(\mathrm{B}_n[4];\mathbb{Q})$ this takes the form $T_{ij}\tau_{k\ell} = \tau_{k\ell}$, as desired. □

In the next lemma we use $\bar{T}$ to denote the image of $T \in \mathrm{B}_n[4]$ in $H_1(\mathrm{B}_n[4];\mathbb{Q})$.

Lemma 7.2. *For $n \geq 3$ and $i < j < k$, the following holds in $H_1(\mathrm{B}_n[4];\mathbb{Q})$:*

$$\overline{[T_{ij}, T_{jk}]} = \frac{1}{2}\left((1 - T_{ik})\tau_{ij} + (1 - T_{ij})\tau_{ik} - (1 - T_{ij})\tau_{jk}\right).$$

Proof. We first prove the lemma for the $n = 3$ case and then use this to obtain the general case.

Brendle and the second author proved a relation in $\mathrm{B}_3[4]$ called the squared lantern relation [8, Proposition 4.2]. In terms of the Artin generators for PB_n this relation can be written as

$$[T_{12}, T_{12} \cdot T_{13}] = T_{12}^2 T_{23}^2 T_{13}^2 T_\partial^{-2}$$

Conjugating both sides by T_{12}^{-1} and using the fact that T_∂ is central yields the relation

$$[T_{12}, T_{13}] = T_{12}^2 \left(T_{12}^{-1} \cdot T_{23}^2\right)\left(T_{12}^{-1} \cdot T_{13}^2\right) T_\partial^{-2}$$

(this commutator is conjugate to the one in the statement). Thus the following identity holds in $H_1(\mathrm{B}_3[4];\mathbb{Q})$:

$$\overline{[T_{12}, T_{13}]} = \tau_{12} + T_{12}\tau_{13} + T_{12}\tau_{23} - \tau_\partial.$$

Next we claim that

$$\tau_\partial = \frac{1}{2}((1 + T_{13})\tau_{12} + (1 + T_{12})\tau_{23} + (1 + T_{12})\tau_{13}).$$

By Lemma 6.2 we have that

$$\tau_\partial = \frac{1}{8}(1 + T_{12})(1 + T_{13})(1 + T_{23})(\tau_{12} + \tau_{13} + \tau_{23}).$$

Lemma 7.1 implies that $T_{12}\tau_{23} = T_{13}\tau_{23}$ and therefore that

$$(1 + T_{12})(1 + T_{13})(1 + T_{23})\tau_{23} = 2(1 + T_{12})^2\tau_{23} = 4(1 + T_{12})\tau_{23}.$$

Similar calculations show that

$$(1 + T_{12})(1 + T_{13})(1 + T_{23})\tau_{12} = 4(1 + T_{13})\tau_{12}$$

and

$$(1 + T_{12})(1 + T_{13})(1 + T_{23})\tau_{13} = 4(1 + T_{12})\tau_{13}.$$

We therefore have that

$$\begin{aligned}\tau_\partial &= \frac{1}{8}\left(4(1 + T_{13})\tau_{12} + 4(1 + T_{12})\tau_{13} + (1 + T_{12})\tau_{23}\right)\\ &= \frac{1}{2}((1 + T_{13})\tau_{12} + (1 + T_{12})\tau_{23} + (1 + T_{12})\tau_{13}),\end{aligned}$$

whence the claim.

Combining the claim and the above expression for $\overline{[T_{12}, T_{13}]}$ we obtain

$$\begin{aligned}\overline{[T_{12}, T_{13}]} &= \tau_{12} + T_{12}\tau_{13} + T_{12}\tau_{23} - \frac{1}{2}\left((1 + T_{13})\tau_{12} + (1 + T_{12})\tau_{23} + (1 + T_{12})\tau_{13}\right)\\ &= \frac{1}{2}((1 - T_{13})\tau_{12} - (1 - T_{12})\tau_{13} - (1 - T_{12})\tau_{23}).\end{aligned}$$

If we apply the braid generator σ_1 to the first and last expressions above we obtain

$$\begin{aligned}\overline{[T_{12},T_{23}]} &= \frac{1}{2}((1-T_{23})\tau_{12} - (1-T_{12})\tau_{23} - (1-T_{12})T_{12}\tau_{13})\\ &= \frac{1}{2}((1-T_{23})\tau_{12} - (1-T_{12})\tau_{23} + (1-T_{12})\tau_{13})\\ &= \frac{1}{2}((1-T_{13})\tau_{12} - (1-T_{12})\tau_{23} + (1-T_{12})\tau_{13})\end{aligned}$$

The second equality is obtained by multiplying the terms $(1-T_{12})$ and T_{12} and the third equality is obtained from the equality $(1-T_{23})\tau_{12} = (1-T_{13})\tau_{12}$ from Lemma 7.1.

For $n \geq 3$ and $i < j < k$ there exists an embedding $f : \mathbb{D}_3 \hookrightarrow \mathbb{D}_n$ such that the images of T_{12}, T_{13}, and T_{23} under the induced map $f_* : \mathrm{PB}_3 \to \mathrm{PB}_n$ are T_{ij}, T_{ik}, and T_{jk}, respectively. Applying f_* to the expression for $\overline{[T_{12},T_{23}]}$ in the $n=3$ case yields the lemma. □

For a group G, the commutator subgroup $[G,G]$ is a subgroup of G^2, the subgroup of G generated by squares of elements of G. For $g \in G^2$ we denote by $\bar{g}$ the image in $H_1(G^2;\mathbb{Q})$. The group G acts G^2 by conjugation. This induces an action of G on $H_1(G^2;\mathbb{Q})$, which descends to an action of $\bar{G} = G/G^2$: for $g \in G^2$ and $h \in G$ we have $h\bar{g} = \overline{hgh^{-1}}$.

Lemma 7.3. *Let G be a group, and let $x, y, z \in G$. We have the following identities in $H_1(G^2;\mathbb{Q})$, thought of as a $\mathbb{Q}[\bar{G}]$-module.*

$$\begin{aligned}\textit{Witt–Hall:}\quad & \overline{[x,yz]} = \overline{[x,y]} + y\overline{[x,z]}\\ \textit{Jacobi:}\quad & (1-x)\overline{[y,z]} - (1-y)\overline{[x,z]} + (1-z)\overline{[x,y]} = 0\end{aligned}$$

Proof. The Witt–Hall identity for groups is the equality

$$[x,yz] = [x,y]\left(y[x,z]y^{-1}\right)$$

in G, which can can checked by simply expanding both sides. Since $[G,G] \leqslant G^2$ we obtain from this the Witt–Hall identity in the statement.

We now proceed to the Jacobi identity. The strategy is to express $[x,[y,z]]$ in two ways and to set the resulting expressions equal to each other. On one hand, since

$$[x,[y,z]] = (x[y,z]x^{-1})[y,z]^{-1}$$

we have

$$\overline{[x,[y,z]]} = (x-1)\overline{[y,z]}.$$

On the other hand, writing

$$[x,[y,z]] = [x,(yz)(zy)^{-1}]$$

we obtain

$$\begin{aligned}\overline{[x,[y,z]]} &= \overline{[x,yz]} + yz\overline{[x,(zy)^{-1}]}\\ &= \overline{[x,y]} + y\overline{[x,z]} + yz\overline{[x,(zy)^{-1}]}\\ &= \overline{[x,y]} + y\overline{[x,z]} + yz\left((zy)^{-1}\overline{[zy,x]}\right)\\ &= \overline{[x,y]} + y\overline{[x,z]} - \overline{[x,zy]}\\ &= \overline{[x,y]} + y\overline{[x,z]} - \left(\overline{[x,z]} + z\overline{[x,y]}\right)\\ &= -(1-y)\overline{[x,z]} + (1-z)\overline{[x,y]},\end{aligned}$$

where the first, second, and fifth equalities use the Witt–Hall identity, the third equality uses the relation $[a,b^{-1}] = b^{-1}[b,a]b$, and the fourth equality uses the fact that G^2, hence $[G,G]$ acts trivially on $H_1(G^2;\mathbb{Q})$. The Jacobi identity follows. $\square$

Lemma 7.4. *For all $p<q<r<s$ we have*

$$\begin{aligned}(1-T_{ps})(1-T_{qr})\tau_{pq} &= (1-T_{ps})(1-T_{qr})\tau_{rs}\\ (1-T_{pq})(1-T_{rs})\tau_{pr} &= -(1-T_{pq})(1-T_{rs})\tau_{qs}\\ (1-T_{pr})(1-T_{qs})\tau_{ps} &= (1-T_{pr})(1-T_{qs})\tau_{qr}\end{aligned}$$

Proof. We first prove the lemma in the case $n=4$ and then use this to obtain the general case. When n is 4, we have that p, q, r, and s are 1, 2, 3, and 4. Thus the statement of the lemma reduces to the following three specific equalities:

$$\begin{aligned}(1-T_{14})(1-T_{23})\tau_{12} &= (1-T_{14})(1-T_{23})\tau_{34}\\ (1-T_{12})(1-T_{34})\tau_{13} &= -(1-T_{12})(1-T_{34})\tau_{24}\\ (1-T_{13})(1-T_{24})\tau_{14} &= (1-T_{13})(1-T_{24})\tau_{23}\end{aligned}$$

We will prove the second equality using the Jacobi identity (Lemma 7.3) and then use the B_n-action to derive the other two. Specifically, by inserting $x=T_{12}$, $y=T_{23}$, $z=T_{34}$ into the Jacobi identity and using the fact that $[T_{12},T_{34}]=1$ in PB_4, we obtain the following equality in $H_1(\mathrm{B}_n[4];\mathbb{Q})$:

$$(1-T_{12})\overline{[T_{23},T_{34}]} = -(1-T_{34})\overline{[T_{12},T_{23}]}.$$

Applying Lemma 7.2 twice with (i,j,k) equal to $(1,2,3)$ and $(2,3,4)$, and inserting the results into both sides of the above equation and simplifying, we obtain the second equality.

Acting on both sides of the second equality with σ_2 and σ_3, respectively, yields the first and third equalities. This completes the proof of the lemma in the case $n=4$.

We now address the general case. Let $f:\{1,2,3,4\}\to\{p,q,r,s\}$ be the unique increasing map. There is an embedding $\mathbb{D}_4\to\mathbb{D}_n$ so that the induced homomorphism $\mathrm{B}_4\to\mathrm{B}_n$ maps T_{ij} to $T_{f(i)f(j)}$. It follows that the induced homomorphism on homology maps τ_{ij} to $\tau_{f(i)f(j)}$. The images of the equalities from the $n=4$ case are the desired equalities. $\square$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Proposition 5.1 the set $\mathcal{S}$ is linearly independent. To prove the theorem, we must show that $\mathcal{S}$ spans $H_1(\mathrm{B}_n[4];\mathbb{Q})$. Since $\mathcal{S}$ has cardinality $3\binom{n}{4}+3\binom{n}{3}+\binom{n}{2}$, it suffices to show that there is some spanning set of this size.

The second statement of Proposition 6.1 states that $H_1(\mathrm{B}_n[4];\mathbb{Q})$ is spanned as $\mathbb{Q}$-vector space by the set

$$\mathcal{T} = \{(1-T_1)\cdots(1-T_m)\tau_{ij} \mid m\geq 0, T_1,\ldots,T_m\in\{T_{k\ell}\}\}.$$

Let $\mathcal{T}_0=\mathcal{T}$. The goal is to successively eliminate elements of $\mathcal{T}_0$ until we obtain a set $\mathcal{T}_3$ with exactly $3\binom{n}{4}+3\binom{n}{3}+\binom{n}{2}$ elements. The elements of $\mathcal{T}_3$ will in fact all be scalar multiples of the elements of the set $\mathcal{S}'$ from Proposition 5.1.

We first claim that $H_1(\mathrm{B}_n[4];\mathbb{Q})$ is spanned by the subset $\mathcal{T}_1$ of $\mathcal{T}_0$ containing all elements of the form

$$\prod_{k\neq i,j}(1-T_{ik})^{\varepsilon_k}\,\tau_{ij}$$

where each ε_k lies in $\{0,1\}$. Each element of $\mathcal{T}_1$ is an element of $\mathcal{T}_0$ where the corresponding m is at most $n-2$ (but not all such elements of $\mathcal{T}_0$ lie in $\mathcal{T}_1$).

To prove the claim we consider an element

$$x = (1 - T_1) \cdots (1 - T_m)\tau_{ij}$$

of $\mathcal{T}_0$ and consider a single term $(1 - T_p)$ of the product. Suppose that T_p is the Artin generator $T_{k\ell}$. If x is non-zero then the product $(1 - T_p)\tau_{ij}$ must be non-zero. By the second statement of Lemma 7.1 the intersection of $\{k, \ell\}$ with $\{i, j\}$ must contain exactly one element. Using the first statement of Lemma 7.1 we may assume without loss of generality that the intersection is $\{i\}$. This leaves exactly $n-2$ possibilities for $\{k, \ell\}$, namely, the sets $\{i, \ell\}$ with $\ell \neq i, j$. We also have that $(1 - T_p)^2 = 2(1 - T_p)$ in $\mathbb{Q}[\mathcal{PZ}_n]$, and so may further assume that each term in the product appears at most once. The claim now follows.

We next claim that $H_1(\mathrm{B}_n[4]; \mathbb{Q})$ is spanned by the subset $\mathcal{T}_2$ of $\mathcal{T}_1$ consisting of elements where there are only two or fewer factors of the form $(1 - T_{ik})$. In other words, $\mathcal{T}_2$ consists of elements of the following form

$$\tau_{ij}, \qquad (1 - T_{ik_1})\tau_{ij}, \qquad (1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij}$$

where each k_1 and k_2 lies outside $\{i, j\}$ and in each product $k_1 \neq k_2$. To prove the claim, it suffices to show that an element of $\mathcal{T}_1$ of the form

$$(1 - T_{ik_3})(1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij}$$

lies in the span of $\mathcal{T}_1$, where k_3 does not lie in $\{k_1, k_2\}$. Since the latter is equal to

$$(1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij} - T_{ik_3}(1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij}$$

and the first of these terms already lies in $\mathcal{T}_2$ it suffices to show that the second term

$$T_{ik_3}(1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij}$$

lies in the span of $\mathcal{T}_2$. The basic strategy is to use Lemmas 7.1 and 7.4 to convert the latter into an element of the form

$$\pm T_{ik_3}(1 - T_{\star\star})(1 - T_{\star\star})\tau_{k_1k_2}.$$

By Lemma 7.1 and the fact that i, k_1, k_2, and k_3 are all distinct we have that $T_{ik_3}\tau_{k_1k_2} = \tau_{k_1k_2}$. If both of the $T_{\star\star}$ terms are of the form $T_{k_1\star}$ then the given element lies in $\mathcal{T}_2$ (up to sign). If either of the $T_{\star\star}$ terms are of the form $T_{k_2\star}$, then we may apply Lemma 7.1 to replace it with $T_{k_1\star}$, leading to the previous case. If either $T_{\star\star}$ term is not of the form $T_{k_1\star}$ or $T_{k_2\star}$, then the corresponding product $(1 - T_{\star\star})\tau_{k_1k_2}$ equals 0. In all cases, the given element lies in the span of $\mathcal{T}_2$.

In order to convert $T_{ik_3}(1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij}$ into the desired form, we proceed in two steps. The first step is to replace either T_{ik_1} or T_{ik_2} with a different Artin generator, so that the result is one of the six types of elements listed in the statement of Lemma 7.4. Here is how we do this. The disjoint sets $\{i, j\}$ and $\{k_1, k_2\}$ are either linked or unlinked. If they are unlinked then we can replace T_{ik_1} with T_{jk_1} by Lemma 7.1. If they are linked then we can replace T_{ik_2} with T_{jk_2} by the same lemma.

Since we have converted the given element $T_{ik_3}(1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij}$ into one of the six forms in the statement of Lemma 7.4, we can apply the corresponding equality from Lemma 7.4 and we obtain an element of the desired form. The claim is now proved.

Our final claim is that $H_1(\mathrm{B}_n[4]; \mathbb{Q})$ is spanned by the subset $\mathcal{T}_3$ of $\mathcal{T}_2$ consisting of all of the τ_{ij}, all of the terms of the elements of the form $(1 - T_{ik_1})\tau_{ij}$, and among the elements of the form $(1 - T_{ik_2})(1 - T_{ik_1})\tau_{ij}$, only those that satisfy

$$i = \min\{i, j, k_1, k_2\}.$$

This claim follows from Lemma 7.4. Indeed, of the six types of elements in the statement of that lemma, there are three types that do not satisfy the condition $i = \min\{i, j, k_1, k_2\}$, and in each case the element on the other side of the equality does satisfy the condition.

To complete the proof, it remains to check that the cardinality of $\mathcal{T}_3$ is $3\binom{n}{4}+3\binom{n}{3}+\binom{n}{2}$. The number of τ_{ij} with $i<j$ is $\binom{n}{2}$, the number of $(1-T_{ik_1})\tau_{ij}$ with $i<j$ and $k \notin \{i,j\}$ is $3\binom{n}{3}$, and the number of $(1-T_{ik_2})(1-T_{ik_1})\tau_{ij}$ with $i<j$, with $k_1<k_2$ and with $k_1,k_2 \notin \{i,j\}$ is $3\binom{n}{4}$. Adding these three terms together gives the desired result. □

8. Representation theory of $\mathcal{Z}_n$

In this section we prove Theorem 2.4, which states that the $V_n(\rho,\lambda)$ are irreducible representations of $\mathcal{Z}_n$ and moreover that every irreducible representation of $\mathcal{Z}_n$ is isomorphic to exactly one $V_n(\rho,\lambda)$.

In Section 8.1 we define the map $\mathcal{Z}_n^I \to \mathcal{Z}_m^I \times S_{n-m}$ used in the definition of the $V_n(\rho,\lambda)$ and prove that it is surjective (Lemma 8.1). Then in Section 8.2 we give a complete criterion for a representation of $\mathcal{Z}_n$ to be irreducible (Proposition 8.3) and use this to show that the $V_n(\rho,\lambda)$ are irreducible. Finally in Section 8.3 we complete the proof of Theorem 2.4.

8.1. **Projection maps.** Our definition of the $V_n(\rho,\lambda)$ was predicated on the existence of a map $\mathcal{Z}_n^I \to \mathcal{Z}_m^I \times \mathcal{Z}_{n-m}$. In this section we prove Lemma 8.1, which gives such a map.

Let I be an element of $\mathbb{I}_m$ and let $n \geq m$. By the definition of $\mathbb{I}_m$ the union of the elements of I is $[m]$. As in Section 2, we may regard I as a subset of $[n]^{\underline{2}}$. There are forgetful maps $f_1 : \mathrm{B}_n^I \to \mathrm{B}_m^I$ and $f_2 : \mathrm{B}_n^I \to \mathrm{B}_{n-m}$ obtained by forgetting the last $n-m$ strands and the first m strands, respectively. Since the f_i take squares of pure braids to squares of pure braids, and since $\mathrm{B}_n[4] = \mathrm{PB}_n^2$, there are induced maps

$$F_1 : \mathcal{Z}_n^I \to \mathcal{Z}_m^I \quad \text{and} \quad F_2 : \mathcal{Z}_n^I \to \mathcal{Z}_{n-m}\,.$$

Let P the composition of $F_1 \times F_2$ with the natural surjection $\mathrm{B}_m^I \times \mathrm{B}_{n-m} \to \mathrm{B}_m^I \times S_{n-m}$. Let $K_{n,m}$ be the subgroup of $\mathcal{PZ}_n$ generated by the images of the T_{ij} with $j > m$.

In the proof of the lemma, we will need the following isomorphism:

$$\overline{\mathrm{PB}}_n \cong \bigoplus_{[n]^{\underline{2}}} \mathbb{Z}/2.$$

This isomorphism follows from the description of the abelianization of PB_n in Section 3.2 and the fact that $\mathrm{B}_n[4]$ is the kernel of the mod 2 abelianization of PB_n.

Lemma 8.1. *The map*

$$P : \mathcal{Z}_n^I \to \mathcal{Z}_m^I \times S_{n-m}$$

is surjective with kernel $K_{n,m}$.

Proof. We first show that P is surjective. Let $(g,\sigma) \in \mathcal{Z}_m^I \times S_{n-m}$. Let $\iota : \mathrm{B}_m^I \times \mathrm{B}_{n-m} \leqslant \mathrm{B}_n$ be the natural inclusion, induced by disjoint embeddings $\mathbb{D}_m \to \mathbb{D}_n$ and $\mathbb{D}_{n-m} \to \mathbb{D}_n$. Let $\tilde{\sigma}$ be a lift of σ to B_{n-m}. Then $P \circ \iota(g,\tilde{\sigma}) = (g,\sigma)$. Thus P is surjective.

It remains to determine the kernel of P. First, we observe that $K_{n,m}$ is contained in the kernel. Since the stated generating set for $K_{n,m}$ has $\binom{n}{2} - \binom{m}{2}$, elements, and since these elements are part of the standard basis for $\mathcal{PZ}_n$, it follows that $K_{n,m}$ has cardinality $2^{\binom{n}{2}-\binom{m}{2}}$. Computing the cardinalities of $\mathcal{Z}_n$, $\mathcal{Z}_m^I$, and $\mathcal{Z}_{n-m}$, we see that the kernel of P must have cardinality $2^{\binom{n}{2}-\binom{m}{2}}$. The result follows. □

8.2. **I-Isotypic representations and a criterion for irreducibility.** We will give in this section a characterization of the irreducible representations of $\mathcal{Z}_n$, Proposition 8.3 below. As a consequence, we deduce in Corollary 8.4 that the $V_n(\rho, \lambda)$ are irreducible.

Our characterization uses the notion of an I-isotypic representation, and so we begin with this idea. It follows from the above description of $\overline{\mathrm{PB}}_n$ that

$$H^1(\overline{\mathrm{PB}}_n; \mu_2) \cong \prod_{[n]^{\underline{2}}} \mu_2,$$

and so elements of $H^1(\mathcal{PZ}_n; \mu_2)$ are labeled by subsets of $[n]^{\underline{2}}$ (recall $\mu_2 = \{\pm 1\}$). We may identify $H^1(\mathcal{PZ}_n; \mu_2)$ with $\mathrm{Hom}(\mathcal{PZ}_n, \mu_2)$, and we denote the homomorphism corresponding to $I \subseteq [n]^{\underline{2}}$ by ρ_I. We denote the corresponding 1-dimensional representation of $\mathcal{PZ}_n$ over $\mathbb{C}$ by V_I.

Let Γ be a subgroup of $\mathcal{Z}_n$ that contains $\mathcal{PZ}_n$; for instance $\Gamma = \mathcal{Z}_n^I$ for some I. Let V be a representation of Γ over $\mathbb{C}$ and let $I \subseteq [n]^{\underline{2}}$. We will say that a subspace W of V is I-isotypic if it is a $\mathcal{PZ}_n$-submodule of V and there is a $\mathcal{PZ}_n$-module isomorphism $W \cong V_I^{\oplus m}$ for some $m \geq 1$.

Lemma 8.2. *Let Γ be a subgroup of $\mathcal{Z}_n$ that contains $\mathcal{PZ}_n$, and let V be a representation of Γ over $\mathbb{C}$. If $W \subset V$ is I-isotypic, then for all $\sigma \in \Gamma$ we have that σW is $\sigma(I)$-isotypic.*

Proof. Let $v \in W$, let $\sigma \in \Gamma$, and let T_{ij} denote the image of an Artin generator for PB_n in $\mathcal{PZ}_n$. It suffices to show that $T_{ij}(\sigma v)$ is equal to $\rho_{\sigma(I)}(T_{ij})\,(\sigma v)$. We indeed have:

$$\begin{aligned} T_{ij}\,(\sigma v) &= \sigma\left(\sigma^{-1} T_{ij} \sigma\right) v = \sigma T_{\sigma^{-1}\{i,j\}} v \\ &= \sigma \rho_I(T_{\sigma^{-1}\{i,j\}})\,(v) = \rho_I(T_{\sigma^{-1}\{i,j\}})\,(\sigma v) = \rho_{\sigma(I)}(T_{ij})\,(\sigma v)\,, \end{aligned}$$

as desired. □

Proposition 8.3. *Let W be a representation of $\mathcal{Z}_n$ over $\mathbb{C}$. Then W is irreducible if and only if there exists an $I \subseteq [n]^{\underline{2}}$ and an irreducible, I-isotypic $\mathcal{Z}_n^I$-submodule $W_I \subset W$ so that we have a $\mathcal{Z}_n$-module isomorphism*

$$W \cong \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W_I.$$

Proof. First assume W is irreducible. Let $\mathrm{Res}_{\mathcal{PZ}_n}^{\mathcal{Z}_n} W = \bigoplus_I W_I$ be the decomposition into isotypic subspaces. By Lemma 8.2 we have that gW_I is $g(I)$-isotypic for each $g \in \mathcal{Z}_n$. Thus $gW_I = W_{g(I)}$ and the $\mathcal{Z}_n$-action permutes the W_I. Since W is irreducible, the induced action on the set of indices I is transitive. Hence for any choice of I there is an isomorphism of $\mathcal{Z}_n$-modules $W \cong \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W_I$. Since W is irreducible, W_I is an irreducible $\mathcal{Z}_n^I$-module.

For the other direction, assume that W is a $\mathcal{Z}_n$-module of the form $\mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W_I$ for some irreducible, I-isotypic $\mathcal{Z}_n^I$-module W_I. Let W' be the irreducible $\mathcal{Z}_n$-submodule of W that contains W_I. For any $g \in \mathcal{Z}_n$ we have $gW_I \subseteq W'$. Since gW_I is $g(I)$-isotypic (Lemma 8.2), W' contains the direct sum $\bigoplus_{g \in \mathcal{Z}_n / \mathcal{Z}_n^I} gW_I$. This direct sum is isomorphic to the $\mathcal{Z}_n$-module $\mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W_I$, which we assumed to be isomorphic to W. Thus $W' = W$, as desired. □

Corollary 8.4. *Each $\mathcal{Z}_n$-representation $V_n(\rho, \lambda)$ is irreducible.*

Proof. Fix some $V_n(\rho, \lambda)$. From the definition there is an $m \leq n$, a full subset I of $[m]^{\underline{2}}$, an irreducible $\mathcal{Z}_m^I$-representation $V_m(\rho)$, and an irreducible S_{n-m}-representation $V(\lambda)$ so that

$$V_n(\rho, \lambda) = \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n}\left(V_m(\rho) \boxtimes V_{n-m}(\lambda)\right).$$

Since we are working over an algebraically closed field of characteristic 0, and since $V_m(\rho)$ and $V_{n-m}(\lambda)$ are irreducible, and since the action of $\mathcal{Z}_n^I$ on $V_m(\rho) \boxtimes V_{n-m}(\lambda)$ factors through the surjective map P from Lemma 8.1, the tensor product $V_m(\rho) \boxtimes V_{n-m}(\lambda)$ is an irreducible $\mathcal{Z}_n^I$-representation. Since ρ is I-isotypic by assumption, and since the image of $\mathcal{PZ}_n$ under P lies in $\mathcal{Z}_m^I \leqslant \mathcal{Z}_m^I \times S_{n-m}$ it follows that $V_m(\rho) \boxtimes V_{n-m}(\lambda)$ is I-isotypic. The corollary is thus an immediate consequence of Proposition 8.3. □

8.3. **Classification of representations.** We are almost ready to prove Theorem 2.5, our classification of irreducible representations of $\mathcal{Z}_n$. What remains is to distinguish between different representations of the form $V_n(\rho, \lambda)$. The following technical lemma provides the required tools for this.

Lemma 8.5. *Let $n \geq 2$ and let $I, J \subseteq [n]^2$.*

(1) Let U be an I-isotypic $\mathcal{Z}_n^I$ module. There is a J-isotypic $\mathcal{Z}_n^J$-module W with $\mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} U \cong \mathrm{Ind}_{\mathcal{Z}_n^J}^{\mathcal{Z}_n} W$ if and only if I and J lie in the same $\mathcal{Z}_n$-orbit.

(2) If U, W are irreducible I-isotypic $\mathcal{Z}_n^I$-modules, $\mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} U \cong \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W$ if and only if $U \cong W$.

Proof. We begin with the first statement. For the reverse implication, we first observe that if I and J lie in the same $\mathcal{Z}_n$-orbit, which is to say that they lie in the same S_n-orbit, then B_n^I and B_n^J are conjugate in B_n. It follows that $\mathcal{Z}_n^I$ and $\mathcal{Z}_n^J$ are conjugate in $\mathcal{Z}_n$. The desired conclusion is then given by the first part of the first exercise in Section III.5 of Brown's book [9].

For the forward implication, suppose that $\mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} U \cong \mathrm{Ind}_{\mathcal{Z}_n^J}^{\mathcal{Z}_n} W$. Since U and W are I- and J-isotypic, respectively, it follows that $\mathrm{Res}_{\mathcal{PZ}_n}^{\mathcal{Z}_n} \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} U$ and $\mathrm{Res}_{\mathcal{PZ}_n}^{\mathcal{Z}_n} \mathrm{Ind}_{\mathcal{Z}_n^J}^{\mathcal{Z}_n} W$ are each direct sums of copies of representations of the form $V_{g(I)}$ and $V_{g(J)}$ for $g \in \mathcal{Z}_n$ (possibly with multiplicities), respectively. We conclude that there is a g so that $V_{g(I)}$ is isomorphic to V_J as $\mathcal{PZ}_n$-modules. But this implies that $g(I) = J$, as desired.

We proceed to the second statement. The reverse implication is trivial. For the forward implication suppose that U and W are non-isomorphic I-isotypic irreducible $\mathcal{Z}_n^I$-modules. It suffices to prove that

$$\mathrm{Hom}_{\mathcal{Z}_n}\left(\mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} U, \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W\right) = 0.$$

By Frobenius reciprocity we have an isomorphism

$$\mathrm{Hom}_{\mathcal{Z}_n}\left(\mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} U, \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W\right) \cong \mathrm{Hom}_{\mathcal{Z}_n^I}\left(U, \mathrm{Res}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W\right).$$

Let $E \subset \mathcal{Z}_n$ be a set of representatives for the set of double cosets $\mathcal{Z}_n^I \setminus \mathcal{Z}_n \,/\, \mathcal{Z}_n^I$. Using the formula $g\, \mathcal{Z}_n^I\, g^{-1} = \mathcal{Z}_n^{g(I)}$ for $g \in \mathcal{Z}_n$ we have an isomorphism of $\mathcal{Z}_n^I$-modules

$$\mathrm{Res}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W = \bigoplus_{g \in E} \mathrm{Ind}_{\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}}^{\mathcal{Z}_n^I} \mathrm{Res}_{\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}}^{\mathcal{Z}_n^{g(I)}} gW$$

(see [9, p.69 Proposition 5.6(b)]), and so

$$\mathrm{Hom}_{\mathcal{Z}_n^I}\left(U, \mathrm{Res}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} \mathrm{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W\right) \cong \bigoplus_{g \in E} \mathrm{Hom}_{\mathcal{Z}_n^I}\left(U, \mathrm{Ind}_{\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}}^{\mathcal{Z}_n^I} \mathrm{Res}_{\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}}^{\mathcal{Z}_n^{g(I)}} gW\right).$$

Applying Frobenius reciprocity once more, we see that the right-hand side is isomorphic to

$$\bigoplus_{g \in E} \mathrm{Hom}_{\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}}\left(\mathrm{Res}_{\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}}^{\mathcal{Z}_n^I} U, \mathrm{Res}_{\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}}^{\mathcal{Z}_n^{g(I)}} gW\right).$$

Since any $\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}$-module map between U and gW restricts to a $\mathcal{PZ}_n$-module map, the fact that U is I-isotypic and gW is $g(I)$-isotypic implies that there can be no non-trivial $\mathcal{Z}_n^I \cap \mathcal{Z}_n^{g(I)}$-module maps between them unless $g(I) = I$. Since g ranges over a set of representatives for the set of double cosets $\mathcal{Z}_n^I \setminus \mathcal{Z}_n \,/\, \mathcal{Z}_n^I$, the only g for which this condition is satisfied is $g = id$. Thus the only nontrivial summand in the above direct sum is $\operatorname{Hom}_{\mathcal{Z}_n^I}(U, W)$, which vanishes by Schur's lemma, because U and W are non-isomorphic irreducible $\mathcal{Z}_n^I$-modules. ☐

We are finally ready to prove Theorem 2.4, which states that every $V_n(\rho, \lambda)$ is an irreducible $\mathcal{Z}_n$-representation and conversely that every irreducible $\mathcal{Z}_n$-representation is isomorphic to exactly one $V_n(\rho, \lambda)$.

Proof of Theorem 2.4. Corollary 8.4 already gives that the $V_n(\rho, \lambda)$ are irreducible. Let V be an arbitrary irreducible $\mathcal{Z}_n$-representation. We would like to show that V is isomorphic to some $V_n(\rho, \lambda)$ as a $\mathcal{Z}_n$-module. By Proposition 8.3, there is an $I \subseteq [n]^{\underline{2}}$ and an irreducible I-isotypic $\mathcal{Z}_n^I$-representation W_I such that $V = \operatorname{Ind}_{\mathcal{Z}_n^I}^{\mathcal{Z}_n} W_I$. By the first statement of Lemma 8.5 we may assume that the union of the elements of I is $[m]$ for some m. Let $K_{n,m}$ be the kernel of the map P, as in Lemma 8.1. Since the generators for $K_{n,m}$ act trivially on W_I, the $\mathcal{Z}_n^I$-action on W_I descends to an action of the quotient $\mathcal{Z}_m^I \times S_{n-m}$. Since W_I is an irreducible representation of $\mathcal{Z}_n^I$, it is an irreducible representation of the quotient $\mathcal{Z}_m^I \times S_{n-m}$. Since we are working over an algebraically closed field of characteristic 0 an irreducible representation of a direct product of groups decomposes as an external tensor product of irreducible representations of the two factors [31]. In particular, there are irreducible representations U_1 and U_2 of B_m^I and S_{n-m} such that $W_I \cong U_1 \boxtimes U_2$ as $\mathcal{Z}_m^I \times S_{n-m}$-modules. Since W_I is I-isotypic and since $\mathcal{PZ}_n$ acts trivially on U_2 it follows that U_1 is I-isotypic.

To complete the proof of the theorem, it remains to prove the uniqueness statement. Suppose that $V_n(\rho, \lambda)$ and $V_n(\rho', \lambda')$ are isomorphic as $\mathcal{Z}_n$-modules. By the second statement of Lemma 8.5, the $\mathcal{Z}_n$-modules $V_m(\rho) \boxtimes V_{n-m}(\lambda)$ and $V_{m'}(\rho') \boxtimes V_{n-m'}(\lambda')$ from which $V_n(\rho, \lambda)$ and $V_n(\rho', \lambda')$ are induced must be isomorphic. It follows from the first statement of Lemma 8.5 that $m = m'$. Since the tensor products are isomorphic, it follows that the individual factors are as well (as we are working over $\mathbb{C}$). ☐

8.4. **A non-splitting.** The following proposition ties up a loose end from Section 2.

Proposition 8.6. *The following extension is not split:*

$$1 \to \mathcal{PZ}_n \to \mathcal{Z}_n \to S_n \to 1.$$

Proof. The surjection $\mathcal{Z}_n \to S_n$ induces a surjection

$$H_1(\bar{\mathrm{B}}_n; \mathbb{Z}) \to H_1(S_n; \mathbb{Z}) \cong \mathbb{Z}/2$$

We claim that $H_1(\bar{\mathrm{B}}_n; \mathbb{Z}) \cong \mathbb{Z}/4$. Since there is no split surjection $\mathbb{Z}/4 \to \mathbb{Z}/2$, the proposition follows from this.

Since $\mathcal{Z}_n$ is the quotient of B_n by $\mathrm{B}_n[4]$ there is an exact sequence

$$H_1(\mathrm{B}_n[4]; \mathbb{Z}) \to H_1(\mathrm{B}_n; \mathbb{Z}) \to H_1(\mathcal{Z}_n; \mathbb{Z}) \to 0.$$

We have $H_1(\mathrm{B}_n; \mathbb{Z}) \cong \mathbb{Z}$. The image of $H_1(\mathrm{PB}_n; \mathbb{Z})$ in $H_1(\mathrm{B}_n; \mathbb{Z})$ is $2\mathbb{Z}$, since each Artin generator evaluates to 2 under the length homomorphism on B_n. Since $\mathrm{B}_n[4]$ is PB_n^2 the image of $H_1(\mathrm{B}_n[4]; \mathbb{Z})$ in $H_1(\mathrm{B}_n; \mathbb{Z})$ is $4\mathbb{Z}$. The claim follows. ☐

9. Representation stability

In this section we prove Theorem 2.5, which gives the decomposition of $H_1(\mathrm{B}_n[4];\mathbb{C})$ into irreducible $\mathcal{Z}_n$-representations, and also states that the $H_1(\mathrm{B}_n[4];\mathbb{C})$ satisfy uniform representation stability.

In Section 9.1, we define the representations $V_3(\rho_3)$ and $V_4(\rho_4)$ of $\bar{\mathrm{B}}_3^{I_3}$ and $\bar{\mathrm{B}}_4^{I_4}$ that are used in the representations $V_n(\rho_3,0)$ and $V_n(\rho_4,0)$ from the statement of Theorem 2.5. Then we prove the isomorphisms from Theorem 2.5 in Section 9.2 by exhibiting the given $V_n(\rho,\lambda)$ as $\mathcal{Z}_n$-submodules of $H_1(\mathrm{B}_n[4];\mathbb{C})$. These submodules are the spans of the orbits of the elements

$$x_3 = (1-T_{13})\prod_{4\le j\le n}(1+T_{1j})(1+T_{2j})\tau_{12} \qquad x_4 = (1-T_{14})(1-T_{23})\tau_{12}.$$

(note that x_3 is only defined for $n \geq 3$ and x_4 only for $n \geq 4$). Finally, in Section 9.3 we complete the proof of Theorem 2.5 by showing that $H_1(\mathrm{B}_n[4];\mathbb{C})$ satisfies the definition of uniform representation stability.

In this section we denote the span of $x \in H_1(\mathrm{B}_n[4];\mathbb{C})$ by $\langle x\rangle$.

9.1. **Representations of $\mathcal{Z}_n$.** The representations $V_3(\rho_3)$ and $V_4(\rho_4)$ of $\mathcal{Z}_n^{I_3}$ and $\mathcal{Z}_n^{I_4}$ will both be 1-dimensional representations obtained from homomorphisms $\rho_k : \mathcal{Z}_k^{I_k} \to \mu_2$. We first define maps $\omega_k : \mathrm{B}_n^{I_k} \to \mathbb{Z}$ (Lemma 9.1) and then obtain the ρ_k from the mod 2 reductions of the ω_k.

In order to define the ω_k we take a different point of view on braids, as follows. Let $C_n(\mathbb{R}^2)$ be the space of configurations of n distinct, indistinguishable points in $\mathbb{R}^2$. Choose a base point for $C_n(\mathbb{R}^2)$ where the n points lie on a horizontal line. There is a natural isomorphism $\pi_1(C_n(\mathbb{R}^2)) \cong \mathrm{B}_n$. We label the points in the base point of $C_n(\mathbb{R}^2)$ by $[n]$ from left to right. A loop in $C_n(\mathbb{R}^2)$ induces a permutation of $[n]$, and this is the usual homomorphism $\mathrm{B}_n \to S_n$. If we represent a braid by a loop in $C_n(\mathbb{R}^2)$, then the ith strand of this braid representative is the path traced out by the point labeled i (the terminology is explained by considering a spacetime diagram of the loop). Let

$$\xi_{ij} : \mathrm{B}_n \to \tfrac{1}{2}\mathbb{Z}$$

be the function that counts the total winding number of the ith strand with the jth strand. This is well defined because of our choice of base point for $C_n(\mathbb{R}^2)$.

With this in hand, we define a function $\omega_3 : \mathrm{B}_n^{I_3} \to \mathbb{Z}$ by the formula

$$\omega_3 = \xi_{13} + \xi_{23}.$$

We similarly we define $\omega_4 : \mathrm{B}_n^{I_4} \to \mathbb{Z}$ by

$$\omega_4 = \xi_{13} + \xi_{14} + \xi_{23} + \xi_{24}.$$

The subgroup $\mathrm{B}_n^{I_3}$ can alternatively be described as the subgroup of B_n preserving the subsets $\{1,2\}$ and $\{3\}$ of $[n]$. Similarly, $\mathrm{B}_n^{I_4}$ can be described as the subgroup preserving the pair of sets $\{\{1,2\},\{3,4\}\}$.

A priori the functions ω_3 and ω_4 are not well defined, since the natural codomain is $\frac{1}{2}\mathbb{Z}$ in both cases.

Lemma 9.1. *For $k \in \{3,4\}$, the function ω_k is a well-defined homomorphism.*

Proof. We begin by showing that ω_3 and ω_4 are well defined. For any braid in $\mathrm{B}_n^{I_3}$, the 1st and 2nd strands both start and end to the left of the 3rd strand. It follows that both ξ_{13} and ξ_{23} map $\mathrm{B}_n^{I_3}$ to $\mathbb{Z}$. Thus, ω_3 is a well-defined function to $\mathbb{Z}$.

For any braid in $\mathrm{B}_n^{I_4}$ that preserves $\{1,2\}$ the numbers ξ_{13}, ξ_{14}, ξ_{23}, and ξ_{24} are all integers, similar to the ω_3 case. Also, for any braid in $\mathrm{B}_n^{I_4}$ that interchanges $\{1,2\}$ and $\{3,4\}$, none of ξ_{13}, ξ_{14}, ξ_{23}, and ξ_{24} are integers, and so again ω_4 is well defined.

To complete the proof it remains to show that ω_3 and ω_4 are homomorphisms. We observe that $\mathrm{B}_n^{I_3}$ can alternatively be described as the subgroup of B_n preserving the subsets $\{1,2\}$ and $\{3\}$. Similarly, $\mathrm{B}_n^{I_4}$ is the subgroup preserving the pair of sets $\{\{1,2\},\{3,4\}\}$.

We begin with ω_3. Let $g \in \mathrm{B}_n^{I_3}$. We color the 1st and 2nd strands red and the 3rd strand blue. Then $\omega_3(g)$ is the sum of the winding numbers of red strands with blue strands. If g and h are two elements of $\mathrm{B}_n^{I_3}$ then the colorings of the strands in $\mathbb{R}^2$ for g and h agree with the coloring of gh (defined in the same way as the one for g). It follows that ω_3 is a homomorphism.

The case of ω_4 is similar. In this case, given $g \in \mathrm{B}_n^{I_4}$ we color the 1st and 2nd strands red and we color the 3rd and 4th strands blue. Then $\omega_4(g)$ again is the sum of the winding numbers of red strands with blue strands. Suppose now that h is another element of $\mathrm{B}_n^{I_4}$. If g preserves the set $\{1,2\}$ then we color the strands of h in the same way that we colored the strands of g. Otherwise, if g interchanges $\{1,2\}$ and $\{3,4\}$ then we color h in the opposite way: the 1st and 2nd strands are blue and the 3rd and 4th strands are red. For either coloring of h, the number $\omega_4(h)$ counts the sum of the winding numbers of red strands with blue strands. The chosen colorings of g and h agree with the coloring on gh. It follows that ω_4 is a homomorphism. □

The homomorphisms ω_3 and ω_4 induce homomorphisms $\mathrm{B}_n^{I_3} \to \mu_2$ and $\mathrm{B}_n^{I_4} \to \mu_2$. The pure braid group PB_n, hence $\mathrm{B}_n[4]$, is contained in each $\mathrm{B}_n^{I_k}$. Since $\mathrm{B}_n[4]$ is equal to PB_n^2 the image of $\mathrm{B}_n[4]$ under each map is trivial. It follows that ω_3 and ω_4 induce homomorphisms $\bar{\mathrm{B}}_n^{I_3} \to \mu_2$ and $\bar{\mathrm{B}}_n^{I_4} \to \mu_2$. Further restricting to $n = 3$ and $n = 4$ gives the desired homomorphisms

$$\rho_3 : \bar{\mathrm{B}}_3^{I_3} \to \mu_2 \quad \text{and} \quad \rho_4 : \bar{\mathrm{B}}_4^{I_4} \to \mu_2.$$

These homomorphisms give rise to the representations $V_3(\rho_3)$ and $V_4(\rho_4)$ from Section 2.

Lemma 9.2. *Let $k \in \{3,4\}$. The representation $V_k(\rho_k)$ is I_k-isotypic.*

Proof. Since each ρ_k defines a 1-dimensional representation, it is enough to check that the restriction of ρ_k to $\mathcal{PZ}_k$ is equal to ρ_{I_k}. For any I the homomorphism ρ_I can be written as

$$\rho_I = \sum_{\{i,j\} \in I} \frac{1}{2}\xi_{ij} \mod 2.$$

The lemma now follows from this and the expressions of the ρ_{I_k} in terms of the ξ_{ij}. □

9.2. **The irreducible decomposition.** We are now in a position to prove the first part of Theorem 2.5, which we state separately as Proposition 9.5 below. We require two lemmas.

In the statement of the first lemma, σ_{13} denotes the half-twist in B_n whose square is T_{13}. In terms of the standard generators for B_n, we can write σ_{13} as $(\sigma_3\sigma_2)\sigma_1(\sigma_3\sigma_2)^{-1}$. Also, when an element is not defined we simply drop it from the proposed generating sets (for instance T_{14} is not an element of $\mathrm{B}_3^{I_3}$).

Lemma 9.3. *For $n \geq 3$ the group $\mathrm{B}_n^{I_3}$ is generated by the set*

$$\{T_{13}, T_{14}, T_{34}, \sigma_1\} \cup \{\sigma_i \mid i \geq 4\}.$$

For $n \geq 4$ the group $\mathrm{B}_n^{I_4}$ is generated by the set

$$\{T_{23}, T_{45}, \sigma_1, \sigma_2\sigma_{13}^{-1}\} \cup \{\sigma_i \mid i \geq 5\}.$$

Proof. We begin with the case of $\mathrm{B}_n^{I_3}$. To simplify the exposition we assume $n \geq 4$; the case $n = 3$ is obtained by ignoring the elements T_{14} and T_{34}. The stabilizer of I_3 in S_n is the image of $S_2 \times S_1 \times S_{n-3}$ under the standard inclusion. The σ_i in the proposed generating set map to the standard generators for this subgroup. Thus it suffices to check that every Artin generator $T_{ij} \in \mathrm{PB}_n$ lies in the group generated by the proposed generators. Using the fact that T_{13}, T_{14}, and T_{34} lie in the generating set and inductively applying the formulas $\sigma_j T_{ij} \sigma_j^{-1} = T_{i,j+1}$ and $\sigma_i^{-1} T_{ij} \sigma_i = T_{i-1,j}$ shows that each T_{ij} is a product of the proposed generators, as desired.

We now treat $\mathrm{B}_n^{I_4}$. Again, to simplify the exposition we assume $n \geq 5$; the case $n = 4$ is obtained by ignoring the element T_{45}. The stabilizer of I_4 in S_n is isomorphic to $(S_2 \times S_2 \times S_{n-4}) \rtimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ factor is any element of order 2 that interchanges $\{1,2\}$ with $\{3,4\}$. The element $\sigma_2\sigma_{13}^{-1}$ maps to $(14)(23)$, giving the $\mathbb{Z}/2$ factor. The element σ_1 maps to the generator of the first S_2 factor. Since the $\mathbb{Z}/2$ factor interchanges the S_2 factors, the generator for the other S_2 factor also is in the image. The σ_i with $i \geq 5$ map to the standard generators for the S_{n-4} factor. So again the lemma reduces to the problem of exhibiting each T_{ij} as a product of generators. This is achieved in the same way as in the previous case. □

Lemma 9.4. *Let $k \in \{3, 4\}$ and let $n \geq k$. The subspace $\langle x_k \rangle$ of $H_1(\mathrm{B}_n[4]; \mathbb{C})$ is a $\mathcal{Z}_n^{I_k}$-module isomorphic to $V_k(\rho_k) \boxtimes V_{n-k}(0)$.*

Proof. We begin by observing that x_3 and x_4 are nonzero in $H_1(\mathrm{B}_n[4]; \mathbb{C})$. The element x_4 is certainly nonzero, as it is one of the basis elements of $H_1(\mathrm{B}_n[4]; \mathbb{C})$ from Theorem 2.1. To see that the element x_3 is nonzero, we apply the forgetful map $\mathrm{PB}_n \to \mathrm{PB}_3$ that forgets the last $n-3$ strands. Via this map, $(1 + T_{1j})$ and $(1 + T_{2j})$ both map to 2 in $\mathbb{Q}[\mathcal{PZ}_3]$, and $(1 - T_{13})\tau_{12}$ maps to $(1 - T_{13})\tau_{12}$ in $H_1(\mathrm{B}_3[4]; \mathbb{C})$. Since the latter is one of the basis elements for $H_1(\mathrm{B}_3[4]; \mathbb{C})$ from Theorem 2.1 it follows that the image of x_3, hence x_3 itself, is nonzero.

As in Section 2, the action of $\mathcal{Z}_n^{I_k}$ on $V_k(\rho_k) \boxtimes V_{n-k}(0)$ factors through the surjection to $P : \mathcal{Z}_n^{I_k} \to \mathcal{Z}_k^{I_k} \times S_{n-k}$ from Lemma 8.1. Let P_1 denote the composition of P with projection to the first factor.

For each k, Lemma 9.3 gives a set of generators for $\mathrm{B}_n^{I_k}$. We will show that the image g of each generator in $\mathcal{Z}_n^{I_k}$ preserves $\langle x_k \rangle$ and moreover that $g x_k = \rho_k \circ P_1(g) x_k$. Since the representation $V_k(\rho_k) \boxtimes V_{n-k}(0)$ is determined by $\rho_k \circ P_1$ the lemma follows from this. In the argument we refer to an element of $\mathrm{B}_n^{I_k}$ and its image in $\mathcal{Z}_n^{I_k}$ by the same symbol.

We begin with the case $k = 3$. Again, to simplify the exposition, we assume $n \geq 4$. For T_{13} we have $T_{13}(1 - T_{13}) = -(1 - T_{13})$ and so $T_{13} x_3 = -x_3 = \rho_3 \circ P_1(T_{13}) x_3$, as desired. For T_{14} we have $T_{14}(1 + T_{14}) = 1 + T_{14}$ and so again $T_{14} x_3 = x_3 = \rho_3 \circ P_1(T_{14}) x_3$. Next we have $T_{34}\tau_{12} = \tau_{12}$ and so $T_{34} x_3 = x_3 = \rho_3 \circ P_1(T_{34}) x_3$.

For σ_1 we use the following relations in B_n: $\sigma_1 T_{1j} \sigma_1^{-1} = T_{2j}$ for $j \geq 3$, $\sigma_1 T_{2j} \sigma_1^{-1} = T_{12} T_{1j} T_{12}^{-1}$ for $j \geq 3$, and σ_1 commutes with T_{12}^2. Since $\mathcal{PZ}_n$ is abelian we have $T_{12} T_{1j} T_{12}^{-1} =$

T_{1j} in $\mathcal{Z}_n^{I_3}$. Using these facts and Lemma 7.1 in turn we obtain that

$$\begin{aligned}\sigma_1 \cdot (1-T_{13}) \prod_{4\le j\le n}(1+T_{1j})(1+T_{2j})\tau_{12} &= (1-T_{23}) \prod_{4\le j\le n}(1+T_{1j})(1+T_{2j})\tau_{12} \\ &= (1-T_{13}) \prod_{4\le j\le n}(1+T_{1j})(1+T_{2j})\tau_{12},\end{aligned}$$

which is to say that $\sigma_1 x_3 = x_3 = \rho_3 \circ P_1(\sigma_3)x_3$.

Finally we must deal with the σ_j with $j \geq 4$. As elements of B_n, each of these commutes with T_{12}^2 and T_{13}. Similarly, for $i \in \{1,2\}$ and $k \notin \{1,2,j-1,j\}$ we have that σ_j commutes with T_{ij}. Also, for $i \in \{1,2\}$ we have that $\sigma_j T_{ij} \sigma_j^{-1} = T_{i,j+1}$ and $\sigma_j T_{i,j+1}\sigma_j^{-1} = T_{j,j+1}T_{ij}T_{j,j+1}^{-1}$ in B_n. As before we have the the latter is equal to T_{ij} in $\mathcal{Z}_n^{I_3}$. We deduce that $\sigma_j x_3 = x_3 = \rho_3 \circ P_1(\sigma_j)x_3$ for $j \geq 4$.

We now treat the case $k = 4$. To simplify the exposition, we assume $n \geq 5$. For T_{45}, we have that $T_{45}\tau_{12} = \tau_{12}$ and so $T_{45}x_4 = x_4 = \rho_4 \circ P_1(T_{45})x_4$. For T_{23} we have $T_{23}(1-T_{23}) = -(1-T_{23})$ and so $T_{23}x_4 = -x_4 = \rho_4 \circ P_1(T_{23})x_4$. Since σ_i commutes with T_{14}, T_{23}, and T_{12}^2 for $i \geq 5$ we also have $\sigma_i x_4 = x_4 = \rho_4 \circ P_1(\sigma_i)x_4$ for $i \geq 5$. Using the relations $\sigma_1 T_{14}\sigma_1^{-1} = T_{24}$ and $\sigma_1 T_{23}^2 \sigma_1^{-1} = T_{12}T_{13}^2T_{12}^{-1}$ in B_n we have

$$\sigma_1 \cdot (1-T_{14})(1-T_{23})\tau_{12} = (1-T_{24})(1-T_{13})\tau_{12} = (1-T_{14})(1-T_{23})\tau_{12}$$

and so $\sigma_1 x_4 = x_4 = \rho_4 \circ P_1(\sigma_1)x_4$. Finally, we will show that $\sigma_2\sigma_{13}^{-1}x_4 = x_4 = \rho_4 \circ P_1(\sigma_2\sigma_{13}^{-1})x_4$. The braid $\sigma_2\sigma_{13}^{-1}$ commutes with T_{14} and T_{23}, and $(\sigma_2\sigma_{13}^{-1})T_{12}^2(\sigma_2\sigma_{13}^{-1})^{-1} = T_{34}^2$ in B_n. Applying these facts and the first equality of Lemma 7.4 in turn we have:

$$\sigma_2\sigma_{13}^{-1} \cdot (1-T_{14})(1-T_{23})\tau_{12} = (1-T_{14})(1-T_{23})\tau_{34} = (1-T_{14})(1-T_{23})\tau_{12},$$

as desired. □

Proposition 9.5. *There are $\mathcal{Z}_n$-equivariant isomorphisms*

$$H_1(\mathrm{B}_n[4];\mathbb{C}) \cong \begin{cases} V_2(1,(0)) & n=2 \\ V_3(1,(0)) \oplus V_3(1,(1)) \oplus V_3(\rho_3,(0)) & n=3 \\ V_n(1,(0)) \oplus V_n(1,(1)) \oplus V_n(1,(2)) \oplus V_n(\rho_3,(0)) \oplus V_n(\rho_4,(0)) & n \geq 4. \end{cases}$$

Proof of Proposition 9.5. As usual, to simplify the exposition, we assume $n \geq 4$. The other cases are obtained by ignoring the appropriate terms.

The first step is to show that we have the following isomorphisms of $\mathcal{Z}_n$-modules:

$$H_1(\mathrm{B}_n[4];\mathbb{C}) \cong \begin{cases} \mathrm{Ind}_{\mathcal{Z}_2^{I_2}}^{\mathcal{Z}_2} \mathbb{C} & n=2 \\ \mathrm{Ind}_{\mathcal{Z}_3^{I_2}}^{\mathcal{Z}_3} \mathbb{C} \oplus \mathrm{Ind}_{\mathcal{Z}_3^{I_3}}^{\mathcal{Z}_3} \langle x_3 \rangle & n=3 \\ \mathrm{Ind}_{\mathcal{Z}_n^{I_2}}^{\mathcal{Z}_n} \mathbb{C} \oplus \mathrm{Ind}_{\mathcal{Z}_n^{I_3}}^{\mathcal{Z}_n} \langle x_3 \rangle \oplus \mathrm{Ind}_{\mathcal{Z}_n^{I_4}}^{\mathcal{Z}_n} \langle x_4 \rangle & n \geq 4, \end{cases}$$

where $\mathbb{C}$ is the trivial $\mathcal{Z}_2^{I_2}$ module. The second step is to identify the summands of this decomposition with the summands in the statement of the theorem.

We begin with the first step. The index of $\mathcal{Z}_n^{I_k}$ in B_n is $\binom{n}{2}$, $3\binom{n}{3}$, and $3\binom{n}{4}$ for k equal to 2, 3, and 4, respectively. Thus the dimension of the purported decomposition equals the dimension of $H_1(\mathrm{B}_n[4];\mathbb{C})$ as given in Theorem 2.1. Since representations of finite groups are completely reducible in characteristic zero, it thus suffices to show that the representations

$$\operatorname{Ind}_{\mathcal{Z}_n^{I_2}}^{\mathcal{Z}_n} \mathbb{C}, \quad \operatorname{Ind}_{\mathcal{Z}_n^{I_3}}^{\mathcal{Z}_n} \langle x_3\rangle, \quad \text{and} \quad \operatorname{Ind}_{\mathcal{Z}_n^{I_4}}^{\mathcal{Z}_n} \langle x_4\rangle$$

appear as submodules of $H_1(\mathrm{B}_n[4];\mathbb{C})$ and that they pairwise intersect in the zero vector. We deal with each summand in turn.

We start with the first submodule $\operatorname{Ind}_{\mathcal{Z}_n^{I_2}}^{\mathcal{Z}_n} \mathbb{C}$. For $i<j$ we define

$$\alpha_{ij} = \prod_{r<s}(1+T_{rs})\tau_{ij}.$$

The α_{ij} are linearly independent since α_{ij} is detected exactly by the forgetful map $\mathrm{B}_n[4] \to \mathrm{B}_2[4]$ that forgets all but the ith and jth marked points in $\mathbb{D}_n$. Since $g\alpha_{ij} = \alpha_{g(i)g(j)}$ for all $g \in \mathcal{Z}_n$ it follows that $H_1(\mathrm{B}_n[4];\mathbb{C})$ contains the $\mathcal{Z}_n$-module

$$\bigoplus_{[g]\in \mathcal{Z}_n\,/\,\mathcal{Z}_n^{I_2}} g\langle\alpha_{12}\rangle = \operatorname{Ind}_{\mathcal{Z}_n^{I_2}}^{\mathcal{Z}_n}\langle\alpha_{12}\rangle = \operatorname{Ind}_{\mathcal{Z}_n^{I_2}}^{\mathcal{Z}_n}\mathbb{C},$$

as desired.

Next, we identify the submodules $\operatorname{Ind}_{\mathcal{Z}_n^{I_k}}^{\mathcal{Z}_n}\langle x_k\rangle$. Fix $k\in\{3,4\}$. We consider the subspaces $g\langle x_k\rangle$ of $H_1(\mathrm{B}_n[4];\mathbb{C})$ for $[g]\in \mathcal{Z}_n\,/\,\mathcal{Z}_n^{I_k}$. Since $\langle x_k\rangle$ is a $\mathcal{Z}_n^{I_k}$-module (Lemma 9.4), these subspaces do not depend on the choice of $g\in[g]$. It follows from the fact that $\langle x_k\rangle$ is I_k-isotypic (Lemma 9.2) and Lemma 8.2 that the $g\langle x_k\rangle$ are mutually non-isomorphic, and so $H_1(\mathrm{B}_n[4];\mathbb{C})$ contains the direct sum. We may write this direct sum as

$$\bigoplus_{[g]\in \mathcal{Z}_n\,/\,\mathcal{Z}_n^{I_k}} g\langle x_k\rangle = \operatorname{Ind}_{\mathcal{Z}_n^{I_k}}^{\mathcal{Z}_n}\langle x_k\rangle,$$

which is the desired submodule.

Since I_2, I_3, and I_4 lie in different $\mathcal{Z}_n$-orbits, it follows from Lemma 8.2 that the three summands we have found have trivial intersection pairwise. This completes the first step.

We now proceed to the second step. By Lemma 9.4, the second and third summands from the first step agree with the summands $V_n(\rho_3,(0))$ and $V_n(\rho_4,(0))$ from the statement of the proposition. It therefore remains to show that we have an isomorphism of $\mathcal{Z}_n$-modules

$$\operatorname{Ind}_{\mathcal{Z}_n^{I_2}}^{\mathcal{Z}_n}\mathbb{C} \cong V_n(1,(0)) \oplus V_n(1,(1)) \oplus V_n(1,(2)).$$

Any representation of $\mathcal{Z}_n$ where $\mathcal{PZ}_n$ acts trivially can be naturally identified with a representation of S_n. Applying this identification to the right-hand side of the above isomorphism yields the S_n-representation $V_n(0)\oplus V_n(1)\oplus V_n(2)$. Recall that on the left-hand side $\mathbb{C}$ is the trivial $\mathcal{Z}_n^{I_2}$-representation and so again $\mathcal{PZ}_n$ acts trivially. Applying the same identification to the left-hand side yields the S_n-representation $\operatorname{Ind}_{S_2\times S_{n-2}}^{S_n}\mathbb{C}$. It follows from the branching rule that the latter is isomorphic to $V_n(0)\oplus V_n(1)\oplus V_n(2)$, as desired. □

9.3. **Uniform representation stability.** In this section we prove the second statement of Theorem 2.5, namely, that the sequence $\{H_1(\mathrm{B}_n[4];\mathbb{Q})\}$ of $\mathcal{Z}_n$-modules is uniformly representation stable (Proposition 9.8 below).

Lemma 9.6. *The standard embedding* $\mathrm{B}_n \to \mathrm{B}_{n+1}$ *induces injective maps* $\mathrm{B}_n[4]\to\mathrm{B}_{n+1}[4]$ *and* $\mathcal{Z}_n\to\mathcal{Z}_{n+1}$.

Proof. As mentioned in Section 2, Brendle and the second author proved that the group $\mathrm{B}_n[4]$ is equal to PB_n^2. It follows that the image of $\mathrm{B}_n[4]$ under $\mathrm{B}_n \to \mathrm{B}_{n+1}$ is contained in $\mathrm{B}_{n+1}[4]$. Thus the standard embedding $\mathrm{B}_n \to \mathrm{B}_{n+1}$ induces a well-defined maps $\mathrm{B}_n[4] \to \mathrm{B}_{n+1}[4]$ and $\mathcal{Z}_n \to \mathcal{Z}_{n+1}$. The first map is injective since it is the restriction of an injective map. The injectivity of the second map is equivalent to the statement that the preimage of $\mathrm{B}_{n+1}[4]$ under $\mathrm{B}_n \to \mathrm{B}_{n+1}$ is contained in $\mathrm{B}_n[4]$. This follows from the fact that PB_n^2, hence $\mathrm{B}_n[4]$, is the kernel of the mod 2 abelianization of PB_n and the fact that the following square commutes, where the horizontal arrows are the mod 2 abelianizations, and the vertical maps are the inclusions:

$$\begin{array}{ccc} \mathrm{PB}_{n+1} & \longrightarrow & (\mathbb{Z}/2)^{\binom{n+1}{2}} \\ \uparrow & & \uparrow \\ \mathrm{PB}_n & \longrightarrow & (\mathbb{Z}/2)^{\binom{n}{2}} \end{array}$$

This completes the proof. □

Lemma 9.7. *For each $k \geq 0$ and $n \geq 0$, the vector spaces $H_k(\mathrm{B}_n[4];\mathbb{Q})$ form a consistent sequence of $\mathcal{Z}_n$-representations with respect to the maps induced by the standard inclusions.*

Proof. For each $g \in \mathrm{B}_n$ there is a commutative diagram of groups

$$\begin{array}{ccc} \mathrm{B}_n[4] & \longrightarrow & \mathrm{B}_{n+1}[4] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{B}_n[4] & \longrightarrow & \mathrm{B}_{n+1}[4] \end{array}$$

where g acts by conjugation. The lemma then follows by applying H_k to all four groups, and using the fact that a group acts trivially on its homology groups. □

Proposition 9.8. *The sequence $\{H_1(\mathrm{B}_n[4];\mathbb{Q})\}$ of $\mathcal{Z}_n$-modules is uniformly representation stable.*

Proof. We check the three parts of the definition of uniform representation stability in turn. The standard inclusion map $\mathrm{B}_n[4] \to \mathrm{B}_{n+1}[4]$ from Lemma 9.6 is a right inverse to the surjective map $\mathrm{B}_{n+1}[4] \to \mathrm{B}_n[4]$ obtained by forgetting the last strand. It follows that the induced maps $\varphi_n : H_1(\mathrm{B}_n[4];\mathbb{C}) \to H_1(\mathrm{B}_{n+1}[4];\mathbb{C})$ are injective. It follows from the first statement of Proposition 6.1 and the fact that every τ_{ij} lies in the same $\mathcal{Z}_{n+1}$-orbit as τ_{12} that the $\mathcal{Z}_{n+1}$-span of $\varphi_n(H_1(\mathrm{B}_n[4];\mathbb{C}))$ is equal to $H_1(\mathrm{B}_{n+1}[4];\mathbb{C})$. Finally, the condition on the multiplicities of the irreducible components follows immediately from Proposition 9.5. □

10. A non-generating set

In this short section we use Lemma 6.2 to prove Theorem 2.3, which states that if $\mathcal{BI}_n \leqslant G \leqslant \mathrm{B}_n[4]$ then G is not generated by even powers of Dehn twists about curves surrounding two points.

Proof of Theorem 2.3. Brendle and the second author proved that the standard forgetful maps $\mathrm{B}_n[4] \to \mathrm{B}_3[4]$ induce a surjection $G \to \mathrm{B}_3[4]$; see [8, Corollary 4.4]. Under any such forgetful map, an even power of a Dehn twist about a curve surrounding two marked points either maps to the identity or to an even power of a Dehn twist about a curve surrounding two marked points. Thus it suffices to prove the result for the case $n = 3$.

By Lemma 6.2, we have in $H_1(\mathrm{B}_3[4];\mathbb{Q})$ that

$$\tau_\partial = \frac{1}{2}\left(\tau_{12} + \tau_{13} + \tau_{23} + T_{13}\tau_{12} + T_{12}\tau_{13} + T_{12}\tau_{23}\right).$$

On the other hand, if T_∂^2 could be written as a product of even powers of Dehn twists about curves surrounding two points, there would exist integers $c_1, \dots, c_6$ such that

$$\tau_\partial = c_1\tau_{12} + c_2\tau_{13} + c_3\tau_{23} + c_4T_{13}\tau_{12} + c_5T_{12}\tau_{13} + c_6T_{12}\tau_{23}$$

But this is impossible, since τ_{12}, τ_{13}, τ_{23}, $T_{13}\tau_{12}$, $T_{12}\tau_{13}$, and $T_{12}\tau_{23}$ are exactly the elements of our basis $\mathcal{S}$ for $H_1(\mathrm{B}_3[4];\mathbb{Q})$ from Corollary 4.3. □

11. Albanese cohomology

In this section, we will prove Theorems 2.8, and 2.9, which state that $H^*_{Alb}(\mathrm{B}_n[4];\mathbb{Q})$ is a proper subalgebra of $H^*(\mathrm{B}_n[4];\mathbb{Q})$ for all $n \geq 15$ and that $H^*_{Alb}(\mathrm{SMod}_g[4];\mathbb{Q})$ is a proper subalgebra of $H^*(\mathrm{SMod}_g[4];\mathbb{Q})$ for $g \geq 7$, respectively. We conclude the section with the proofs of Theorem 2.2 and Proposition 2.7. The former gives the Betti numbers of $\mathrm{B}_3[4]$ and $\mathrm{B}_4[4]$, while the latter gives a new (large) lower bound on the top Betti number of $\mathrm{Mod}_2[4]$.

11.1. **Interpretations of the level 4 braid group à la Brendle–Margalit.** In this section, it will be advantageous to recast the group $\mathrm{B}_n[4]$ in two different ways. Specifically, we will utilize the following two isomorphisms, which hold for $g \geq 1$:

$$\mathrm{B}_{2g+1}[4] \cong \mathrm{SMod}_g[4] \times \mathbb{Z}$$
$$\mathrm{B}_{2g+1}[4] \cong \mathrm{PMod}_{0,2g+2}^2 \times \mathbb{Z}.$$

As in Section 4, $\mathrm{PMod}_{0,n}$ denotes the pure mapping class group of a sphere with n marked points and $\mathrm{PMod}_{0,n}^2$ is the subgroup generated by all squares.

Neither of the above isomorphisms are stated explicitly by Brendle and the second author. However, both are easily obtained from their work, as we shall explain currently.

We begin with the first isomorphism. Brendle and the second author [7, Theorem 4.2] proved the analogous isomorphism $\mathcal{BI}_{2g+1} \cong \mathcal{SI}_g \times \mathbb{Z}$ (their theorem actually refers to the hyperelliptic Torelli group $\mathcal{SI}_g^1$ of a surface with boundary instead of $\mathcal{BI}_{2g+1}$, but as explained in their introduction the groups $\mathcal{SI}_g^1$ and $\mathcal{BI}_{2g+1}$ are naturally isomorphic). The proof of their isomorphism applies verbatim in our situation, except with the Torelli group replaced with the level 4 mapping class group.

The second isomorphism follows from the theorem of Brendle and the second author that $\mathrm{B}_{2g+1}[4] \cong \mathrm{PB}_{2g+1}^2$ and the fact that PB_n splits as a direct product as $\mathrm{PMod}_{0,n+1} \times \mathbb{Z}$; see [17, p. 252].

We can also combine the above two isomorphisms in order to obtain the isomorphism

$$\mathrm{SMod}_g[4] \cong \mathrm{PMod}_{0,2g+2}^2$$

for $g \geq 1$. Indeed, the group $\mathrm{B}_{2g+1}[4]$ has infinite cyclic center, and so the composition of the two isomorphisms above must identify the two given $\mathbb{Z}$-factors.

In this section we will use one other fact from the work of Brendle and the second author. They observed [8, Corollary 4.4] that each of the forgetful maps $\mathrm{PB}_n \to \mathrm{PB}_m$ induces a surjective homomorphism

$$\mathrm{B}_n[4] \to \mathrm{B}_m[4].$$

(cf. the proof of Theorem 2.3). This map is split. For instance if the forgetful map $\mathrm{PB}_n \to \mathrm{PB}_m$ is the one obtained by forgetting the last $n-m$ marked points of $\mathbb{D}_n$ then the splitting is the restriction of the standard inclusion $\mathrm{B}_m \to \mathrm{B}_n$. Both the surjectivity and the existence of the splitting follow directly from the isomorphism $\mathrm{B}_{2g+1}[4] \cong \mathrm{PB}^2_{2g+1}$.

11.2. **The proofs of Theorems 2.8 and 2.9.** Our next goal is to prove Theorems 2.8 and 2.9, which state that the Albanese cohomology algebras of $\mathrm{B}_n[4]$ and $\mathrm{SMod}_g[4]$ are proper subalgebras of $H^*(\mathrm{B}_n[4];\mathbb{Q})$ and $H^*(\mathrm{SMod}_g[4];\mathbb{Q})$ for $n \geq 15$ and $g \geq 7$, respectively. We give two lemmas that give the cohomological dimension and the Euler characteristic of $\mathrm{SMod}_g[4]$ before proceeding to the proofs of Theorem 2.9 and 2.8 (in that order).

For a group G we denote by $\operatorname{cd} G$ its cohomological dimension.

Lemma 11.1. *For $n \geq 3$ we have*

$$\operatorname{cd} \mathrm{PMod}^2_{0,n} = n - 3$$

and for $g \geq 1$ we have

$$\operatorname{cd} \mathrm{SMod}_g[4] = 2g - 1.$$

Proof. For $n \geq 3$ we have $\operatorname{cd} \mathrm{PMod}_{0,n} = n - 3$. The two statements now follow from the fact that $\mathrm{PMod}^2_{0,n}$ has finite index in $\mathrm{PMod}_{0,n}$ and the isomorphism $\mathrm{SMod}_g[4] \cong \mathrm{PMod}^2_{0,2g+2}$ from Section 11.1, respectively. □

Lemma 11.2. *For $g \geq 1$ we have*

$$\chi(\mathrm{SMod}_g[4]) = -2^{\binom{2g+1}{2}-1}(2g-1)!$$

Proof. As explained in Section 11.1, the group $\mathrm{SMod}_g[4]$ is isomorphic to $\mathrm{PMod}^2_{0,2g+2}$. We will compute the Euler characteristic of the latter.

We claim that the index of $\mathrm{PMod}^2_{0,n}$ in $\mathrm{PMod}_{0,n}$ is $2^{\binom{n-1}{2}-1}$. Since for any group G we have $G/G^2 \cong H_1(G;\mathbb{Z}/2)$, it follows that

$$\mathrm{PMod}_{0,n} / \mathrm{PMod}^2_{0,n} \cong H_1(\mathrm{PMod}_{0,n};\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\binom{n-1}{2}-1}$$

The last isomorphism follows from the splitting $\mathrm{PB}_{n-1} \cong \mathrm{PMod}_{0,n} \times \mathbb{Z}$ and the usual description of the abelianization of PB_{n-1}. The claim follows.

Harer and Zagier [23, p. 476] proved that

$$\chi(\mathrm{PMod}_{0,n}) = (-1)^{n-3}(n-3)!$$

For any group G and a subgroup G' of finite index we have $\chi(G') = [G : G']\chi(G')$. The lemma follows by combining this fact with the claim. □

Proof of Theorem 2.9. By Lemma 11.1 we have $\operatorname{cd} \mathrm{SMod}_g[4] = 2g - 1$. Therefore, in order to show that $H^*_{Alb}(\mathrm{B}_n[4];\mathbb{Q})$ is a proper subalgebra of $H^*(\mathrm{B}_n[4];\mathbb{Q})$ we must show that the image of the cup product map

$$\Lambda^i H^1(\mathrm{SMod}_g[4];\mathbb{Q}) \to H^i(\mathrm{SMod}_g[4];\mathbb{Q}).$$

fails to be surjective for some $2 \leq i \leq 2g - 1$.

Let b_i denote the ith Betti number of $\mathrm{SMod}_g[4]$ and let d_i denote the dimension of the image of the above cup product map. Our basic strategy is to show that there is some i between 2 and $2g - 1$ with $b_i > d_i$. To do this we will estimate the d_i from above and the b_i from below.

We first claim that

$$d_i \leq \binom{b_1}{2g-1}$$

for all $2 \leq i \leq 2g-1$. As $g \geq 7$ it follows from Corollary 2.6 that $2g-1 < b_1/2$, and so

$$d_i \leq \dim \Lambda^i H^1(\mathrm{SMod}_g[4];\mathbb{Q}) = \binom{b_1}{i} \leq \binom{b_1}{2g-1}$$

for $i \leq 2g-1$, as desired.

We next claim that

$$b_{2k-1} > \frac{1}{g-1}\left(2^{\binom{2g+1}{2}-1}(2g-1)! - b_1\right)$$

for some k with $2 \leq k \leq g$. Since (as above) the cohomological dimension of $\mathrm{SMod}_g[4]$ is $2g-1$ we have the following immediate consequence of Lemma 11.2:

$$b_1 + b_3 + \cdots + b_{2g-1} > 2^{\binom{2g+1}{2}-1}(2g-1)!$$

The claim follows.

Combining the two claims, it is now enough to show that

$$\frac{1}{g-1}\left(2^{\binom{2g+1}{2}-1}(2g-1)! - b_1\right) > \binom{b_1}{2g-1}$$

for $g \geq 7$.

For $g = 7$ we can verify the inequality numerically. Direct computation shows that the right-hand side is on the order of 10^{38} and that the left-hand side is on the order of 10^{40}.

We now treat the general case $g \geq 8$. We will perform four strengthenings of the desired inequality in order to obtain an inequality that we can prove with basic calculus. First, using the estimate $\binom{n}{k} \leq n^k/k!$ and the estimate $(2g-1)! > \left(\frac{2g-1}{e}\right)^{2g-1}\sqrt{2\pi(2g-1)}$ we obtain the stronger inequality

$$\frac{1}{g-1}\left(2^{\binom{2g+1}{2}-1}\left(\frac{2g-1}{e}\right)^{2g-1}\sqrt{2\pi(2g-1)} - b_1\right) > \frac{b_1^{2g-1}}{(2g-1)!}$$

Next, adding $b_1/(g-1)$ to both sides and using the fact that $b_1/(g-1) < b_1 \leq \binom{b_1}{2g-1}$, that $\binom{n}{k} \leq n^k$, and that $b_1^{2g-1} < b_1^{2g}$, we obtain the even stronger inequality

$$2^{\binom{2g+1}{2}-1}\left(\frac{2g-1}{e}\right)^{4g-2} 2\pi(2g-1) > 2b_1^{2g}.$$

It follows from Theorem 2.1 and the estimate $\binom{n}{k} \leq n^k/k!$ that $b_1 < \frac{(2g+2)^4}{6}$. Using this and dividing both sides of the last inequality by 2 we obtain the even stronger inequality

$$2^{\binom{2g+1}{2}-1}\left(\frac{2g-1}{e}\right)^{4g-2} \pi(2g-1) > \left(\frac{(2g+2)^4}{6}\right)^{2g}.$$

Since both sides of the last inequality are positive, we may take the logarithms of both sides in order to obtain the equivalent inequality

$$\begin{aligned}\left(2g^2+g-1\right)\ln 2 + &(4g-2)\left(\ln(2g-1)-1\right) + \ln(2g-1) + \ln \pi \\ &> 8g\ln(2g+2) - 2g\ln 6.\end{aligned}$$

Set

$$G(x) = \left(2x^2+x-1\right)\ln 2 + (4x-2)\left(\ln(2x-1)-1\right) + \ln(2x-1) + \ln \pi$$

and

$$H(x) = 8x\ln(2x+2) - 2x\ln 6.$$

The last inequality can be restated as $G(g) > H(g)$. By direct computation, the function $F(x) = G(x) - H(x)$ satisfies $F(8) > 0$ and $F'(8) > 0$. Furthermore, for $x \geq 8$ we have that

$$\begin{aligned} F''(x) &= 4\ln 2 + \frac{8}{2x-1} - \frac{8}{x+1} - \frac{4}{(2x-1)^2} - \frac{8}{(x+1)^2} \\ &> 4\ln 2 + 0 - \frac{8}{9} - \frac{4}{225} - \frac{8}{81} \\ &> 0 \end{aligned}$$

where we have used the fact that $x \geq 8$ in the first inequality. This implies that $F'(x)$ is increasing for $x \geq 8$, and therefore that $F(x)$ is increasing for all $x \geq 8$. The theorem follows. □

Proof of Theorem 2.8. We will now derive Theorem 2.8 from Theorem 2.9. Because of the isomorphism $\mathrm{B}_{2g+1}[4] \cong \mathrm{SMod}_g[4] \times \mathbb{Z}$ there is a split surjective homomorphism $\mathrm{B}_{2g+1}[4] \to \mathrm{SMod}_g[4]$ induced by projection onto the first factor. Any section $\sigma : \mathrm{SMod}_g[4] \to B_{2g+1}[4]$ induces a surjection

$$\sigma^* : H^*(B_{2g+1}[4]; \mathbb{Q}) \to H^*(\mathrm{SMod}_g[4]; \mathbb{Q})$$

Let $\sigma^*(1)$ denote the algebra homomorphism

$$\Lambda^* H^1(B_{2g+1}[4]; \mathbb{Q}) \to \Lambda^* H^1(\mathrm{SMod}_g[4]; \mathbb{Q})$$

induced by σ^* in degree 1. This map is surjective.

The relationships between σ^*, $\sigma^*(1)$, and the cup product are given by the following commutative diagram:

$$\begin{array}{ccc} \Lambda^* H^1(B_{2g+1}[4]; \mathbb{Q}) & \xrightarrow{\sigma_g^*(1)} & \Lambda^* H^1(\mathrm{SMod}_g[4]; \mathbb{Q}) \\ \big\downarrow \smile & & \big\downarrow \smile \\ H^*(B_{2g+1}[4]; \mathbb{Q}) & \xrightarrow{\sigma_g^*} & H^*(\mathrm{SMod}_g[4]; \mathbb{Q}). \end{array}$$

We complete the proof of Theorem 2.8 by first dealing with the case of n odd, followed by the case of n even.

By Theorem 2.9, the rightmost cup product in the above diagram fails to be surjective for $g \geq 7$. This implies that for all $g \geq 7$ the cup product $\Lambda^* H^1(\mathrm{B}_{2g+1}[4]; \mathbb{Q}) \to H^*(\mathrm{B}_{2g+1}[4]; \mathbb{Q})$ is not surjective. This proves Theorem 2.8 for n odd with $n \geq 15$.

It remains to deal with the case of n even. Let $\mathrm{B}_{2g+2}[4] \to \mathrm{B}_{2g+1}[4]$ be the map induced by forgetting the last marked point of $\mathbb{D}_n$ and let s be any section, for instance the one induced by the standard inclusion $\mathrm{B}_{2g+1} \to \mathrm{B}_{2g+2}$. Replacing $\mathrm{B}_{2g+1}[4]$, $\mathrm{SMod}_g[4]$, and σ in the diagram above with $\mathrm{B}_{2g+2}[4]$, $\mathrm{B}_{2g+1}[4]$, and s, respectively, and applying the odd n case of Theorem 2.8, we obtain that for $g \geq 7$ the cup product $\Lambda^* H^1(\mathrm{B}_{2g+2}[4]; \mathbb{Q}) \to H^*(\mathrm{B}_{2g+2}[4]; \mathbb{Q})$ is not surjective. This completes the proof. □

11.3. **Higher Betti numbers.** In this section we will prove Theorem 2.2, which gives the Betti numbers of $\mathrm{B}_n[4]$ for $n = 3, 4$ and Proposition 2.7, which gives a lower bound for the top Betti number of $\mathrm{Mod}_2[4]$. We begin with a lemma.

Lemma 11.3. *For all* $n \geq 1$ *we have* $\operatorname{cd} \mathrm{B}_n[4] = n - 1$ *and* $\chi(\mathrm{B}_n[4]) = 0$.

Proof. For $n \geq 1$ we have $\operatorname{cd} \mathrm{PB}_n = n - 1$; the lower bound comes from the existence of a free abelian subgroup of rank $n-1$ (generated by Dehn twists) and the upper bound comes from the decomposition of PB_n into an $(n-1)$-fold iterated semidirect product of free groups (via combing). Since $\mathrm{B}_n[4]$ has finite index in PB_n, the first statement follows.

It follows from Arnol'd's computation [2, Corollary 2] of the Poincaré polynomial of PB_n that $\chi(\mathrm{PB}_n) = 0$. Since $\mathrm{B}_n[4]$ has finite index in PB_n, we obtain the second statement. □

Proof of Theorem 2.2. In the proof we will denote the ith Betti number of a group G by $b_i(G)$ and we will abbreviate $b_i(\mathrm{B}_n[4])$ by b_i.

We begin with the case of $n = 3$. By the second statement of Lemma 11.3 we have

$$\chi(\mathrm{B}_3[4]) = b_0 - b_1 + b_2 = 0.$$

By the $n = 3$ case of Theorem 2.1 we have $b_1(\mathrm{B}_3[4]) = 6$. Since $b_0 = 1$, we find that $b_2(\mathrm{B}_3[4]) = 5$. By the first statement of Lemma 11.3, we have found all of the nontrivial Betti numbers of $\mathrm{B}_3[4]$.

Next we treat the case $n = 4$. As in Section 11.1 we have $\mathrm{B}_4[4] \cong \mathrm{PMod}_{0,5}^2 \times \mathbb{Z}$. From this the Künneth theorem gives

$$H^j(\mathrm{B}_4[4]; \mathbb{Q}) \cong H^j(\mathrm{PMod}_{0,5}^2; \mathbb{Q}) \oplus H^{j-1}(\mathrm{PMod}_{0,5}^2; \mathbb{Q})$$

for all $j \geq 1$. Thus for all $j \geq 1$ we have

$$b_j = b_j(\mathrm{PMod}_{0,5}^2) + b_{j-1}(\mathrm{PMod}_{0,5}^2).$$

It follows from Lemma 11.3 and the isomorphism $\mathrm{B}_{2g+1}[4] \cong \mathrm{PMod}_{0,2g+2}^2 \times \mathbb{Z}$ (Section 11.1) that $\mathrm{cd}\,\mathrm{PMod}_{0,5}^2 = 2$. Thus by Lemma 11.2

$$64 = \chi(\mathrm{PMod}_{0,5}^2) = 1 - b_1(\mathrm{PMod}_{0,5}^2) + b_2(\mathrm{PMod}_{0,5}^2).$$

Since $b_1(\mathrm{PMod}_{0,5}^2) = 20$, we obtain $b_2(\mathrm{PMod}_{0,5}^2) = 83$. Thus

$$b_2(\mathrm{B}_4[4]) = b_2(\mathrm{PMod}_{0,5}^2) + b_1(\mathrm{PMod}_{0,5}^2) = 83 + 20 = 103.$$

Finally, since $\chi(\mathrm{B}_4[4]) = 0$ we have

$$b_3(\mathrm{B}_4[4]) = 1 - b_1(\mathrm{B}_4[4]) + b_2(\mathrm{B}_4[4]) = 1 - 21 + 103 = 83.$$

Since $\mathrm{cd}\,\mathrm{B}_4[4] = 3$, we have found all of the non-trivial Betti numbers of $\mathrm{B}_4[4]$. □

Proof of Proposition 2.7. Throughout we use the equality $\mathrm{SMod}_2[4] = \mathrm{Mod}_2[4]$, which follows immediately from the equality $\mathrm{SMod}_2 = \mathrm{Mod}_2$; see [17, Section 9.4.2].

By Lemma 11.2 we have $\chi(\mathrm{Mod}_2[4]) = -3072$ and by Lemma 11.1 we have $\mathrm{cd}\,\mathrm{Mod}_2[4] = 3$. Thus

$$1 - b_1 + b_2 - b_3 = -3072.$$

By Corollary 2.6 we have $b_1 = 54$, whence $b_3 = 3019 + b_2$. It remains to bound b_2 from below.

Since $\mathrm{B}_5[4] \cong \mathrm{Mod}_2[4] \times \mathbb{Z}$, the Künneth theorem gives

$$H_2(\mathrm{Mod}_2[4]; \mathbb{Q}) \oplus H_1(\mathrm{Mod}_2[4]; \mathbb{Q}) \cong H_2(\mathrm{B}_5[4]; \mathbb{Q})$$

and therefore that

$$b_2 + b_1 = \dim H_2(\mathrm{B}_5[4]; \mathbb{Q})$$

Since the map $\mathrm{B}_5[4] \to \mathrm{B}_4[4]$ induced by forgetting the last marked point in $\mathbb{D}_5$ is split, the induced map

$$H_2(\mathrm{B}_5[4]; \mathbb{Q}) \to H_2(\mathrm{B}_4[4]; \mathbb{Q})$$

is surjective. By the $n = 4$ case of Theorem 2.2 we have

$$\dim H_2(\mathrm{B}_5[4]; \mathbb{Q}) \geq \dim H_2(\mathrm{B}_4[4]; \mathbb{Q}) = 103$$

It follows that $b_2 \geq 103 - 54 = 49$ and therefore that $b_3 \geq 3019 + 49 = 3068$. □

12. Hyperelliptic Torelli groups

In this section prove Theorem 2.11, which states that

$$\dim H_1(\mathcal{SI}_g;\mathbb{Q}) \geq \frac{1}{6}\left(20g^4+12g^3-5g^2+9g-6\right).$$

After recalling some facts about the second Johnson homomorphism τ_2, we proceed to the proof of the theorem. At the end of the section we prove Proposition 2.12.

In this section, $\mathrm{Sp}_g(\mathbb{Z})[m]$ denotes the level m congruence subgroup of $\mathrm{Sp}_g(\mathbb{Z})$, that is, the kernel of the mod m reduction map.

The second Johnson homomorphism. Let $\pi = \pi_1(\Sigma_g,*)$ and let $\pi^{(k)}$ denote the kth term of the lower central series of π. We define $\mathcal{L}_k = \pi^{(k)}/\pi^{(k+1)}\otimes\mathbb{Q}$. Let $\mathcal{K}_g$ denote the subgroup of $\mathcal{I}_g$ generated by Dehn twists about separating simple closed curves. The second Johnson homomorphism is a Mod_g-equivariant homomorphism

$$\tau_2 : \mathcal{K}_g \to \mathrm{Hom}(\mathcal{L}_1,\mathcal{L}_3);$$

see the papers by Hain and Morita [21, 26] for the definition. The image of τ_2 is a representation of $\mathrm{Sp}_g(\mathbb{Z})$. Work of Hain [19] implies that the image is isomorphic to the restriction to $\mathrm{Sp}_g(\mathbb{Z})$ of the irreducible $\mathrm{Sp}_g(\mathbb{Q})$-representation $V(2\lambda_2)$, where $\lambda_1,\lambda_2,\dots,\lambda_g$ is a system of fundamental weights for $\mathrm{Sp}_g(\mathbb{Q})$ (see also [26, p.377]).

The group $\mathcal{SI}_g$ is contained in $\mathcal{K}_g$; see of the paper by Brendle, Putman, and the second author [6, p. 268]. Thus we may restrict τ_2 to $\mathcal{SI}_g$ to obtain

$$j : \mathcal{SI}_g \to V(2\lambda_2).$$

The group $\mathcal{SI}_g$ is normal in $\mathrm{SMod}_g[2]$. Also, A'Campo proved that the symplectic representation $\mathrm{Mod}_g \to \mathrm{Sp}_g(\mathbb{Z})$ induces an isomorphism $\mathrm{SMod}_g[2]/\mathcal{SI}_g \cong \mathrm{Sp}_g(\mathbb{Z})[2]$. It follows that j is $\mathrm{SMod}_g[2]$-equivariant and that it induces an $\mathrm{Sp}_g(\mathbb{Z})[2]$-equivariant map

$$j_* : H_1(\mathcal{SI}_g;\mathbb{Q}) \to V(2\lambda_2).$$

Proof of Theorem 2.11. Let $i : \mathcal{SI}_g \to \mathrm{SMod}_g[4]$ denote the inclusion and consider the map

$$\Phi : H_1(\mathcal{SI}_g;\mathbb{Q}) \to H_1(\mathrm{SMod}_g[4];\mathbb{Q}) \oplus V(2\lambda_2)$$

defined by

$$\Phi(x) = (i_*(x), j_*(x)).$$

By Corollary 2.6 the dimension of the first summand is

$$3\binom{2g+1}{4} + 3\binom{2g+1}{3} + \binom{2g+1}{2} - 1.$$

The dimension of the second summand is also known (see [21, Lemma 8.5]):

$$\dim V(2\lambda_2) = \frac{g(g-1)(4g^2+4g-3)}{3}.$$

Since the sum of these two dimensions is the desired lower bound, it suffices to prove that Φ is surjective. To do this, we will first show that i_* and j_* are surjective.

First we show that i_* is surjective. It follows from the aforementioned theorem of A'Campo that the map $\mathrm{Mod}_g \to \mathrm{Sp}_g(\mathbb{Z})$ induces an isomorphism $\mathrm{SMod}_g[4]/\mathcal{SI}_g \cong \mathrm{Sp}_g(\mathbb{Z})[4]$. We therefore have an exact sequence

$$H_1(\mathcal{SI}_g;\mathbb{Q}) \to H_1(\mathrm{SMod}_g[4];\mathbb{Q}) \to H_1(\mathrm{Sp}_g(\mathbb{Z})[4];\mathbb{Q})$$

The homology group $H_1(\mathrm{Sp}_g(\mathbb{Z})[m];\mathbb{Q})$ is zero for $g \geq 2$ and $m \geq 0$ (see [27, p. 3]). Thus i_* is surjective.

We now show that the map j_* is surjective for each $g \geq 2$. By the Borel density theorem, any lattice $\Gamma \leq \mathrm{Sp}_g(\mathbb{R})$ is Zariski dense; see [27, p. 766]. It follows that the irreducible $\mathrm{Sp}_g(\mathbb{R})$-module $V(2\lambda_2) \otimes \mathbb{R}$ is irreducible as a Γ-module. Since tensoring a reducible representation with $\mathbb{R}$ results in a reducible representation, it follows that $V(2\lambda_2)$ is irreducible as a Γ-module. As j_* is $\mathrm{Sp}_g(\mathbb{Z})[2]$-equivariant, it suffices to show that j_* is non-zero.

Morita proved that if c is any (nontrivial) separating curve in Σ_g then $\tau_2(T_c)$ is non-zero [26, Proposition 1.1]. If c is any separating curve in Σ_g that is preserved by the hyperelliptic involution s, then T_c lies in $\mathcal{SI}_g$. It follows that j is non-zero, and hence that that j_* is non-zero, hence surjective.

Finally, we show that Φ is surjective. Let $(x,y) \in H_1(\mathrm{SMod}_g[4];\mathbb{Q}) \oplus V(2\lambda_2)$. We will show that (x,y) lies in the image of Φ. Since both i_* and j_* are surjective, we can choose $\widetilde{x},\widetilde{y} \in H_1(\mathcal{SI}_g;\mathbb{Q})$ such that

$$i_*(\widetilde{x}) = x \quad \text{and} \quad j_*(\widetilde{y}) = y.$$

Let $y_1 = -j_*(\widetilde{x})+y$. Since $V(2\lambda_2)$ is an irreducible $\mathrm{Sp}_g(\mathbb{Z})[4]$-representation, the corresponding space of coinvariants $V(2\lambda_2)_{\mathrm{Sp}_g(\mathbb{Z})[4]}$ is trivial. This is the same as saying that $V(2\lambda_2)$ is spanned by

$$\left\{(h-1)v_h \mid h \in \mathrm{Sp}_g(\mathbb{Z})[4],\ v_h \in H_1(\mathcal{SI}_g;\mathbb{Q})\right\}.$$

In particular, there is a finite set $\mathcal{H} \subset \mathrm{Sp}_g(\mathbb{Z})[4]$ such that

$$y_1 = \sum_{h\in\mathcal{H}} (h-1)v_h$$

where each v_h lies in $V(2\lambda_2)$.

For each $h \in \mathcal{H}$, let $\tilde{v}_h$ be an element of the j_*-preimage of v_h. Let

$$\widetilde{y}_1 = \sum_{h\in\mathcal{H}} (h-1)\widetilde{v}_h.$$

By construction, we have that $j_*(\widetilde{y}_1) = y_1 = -j_*(\widetilde{x}) + y$.

Since $\mathcal{SI}_g$ is normal in $\mathrm{SMod}_g[4]$, and since $\mathrm{SMod}_g[4]/\mathcal{SI}_g \cong \mathrm{Sp}_g(\mathbb{Z})[4]$, the map i_* is $\mathrm{Sp}_g(\mathbb{Z})[4]$-equivariant.Thus

$$i_*(\widetilde{y}_1) = \sum_{h\in\mathcal{H}} (h-1)i_*(\widetilde{v}_h) = \sum_{h\in\mathcal{H}} hi_*(\widetilde{v}_h) - i_*(\widetilde{v}_h).$$

Since $\mathcal{SI}_g$ is contained in $\mathrm{SMod}_g[4]$ there is a well-defined action of the quotient $\mathrm{Sp}_g(\mathbb{Z})[4]$ on $H_1(\mathrm{SMod}_g[4];\mathbb{Q})$. But the action of $\mathrm{SMod}_g[4]$ on $H_1(\mathrm{SMod}_g[4];\mathbb{Q})$ is trivial and so the action of $\mathrm{Sp}_g(\mathbb{Z})[4]$ is trivial. Thus our last expression for $i_*(\widetilde{y}_1)$ is zero. It follows that $\Phi(\widetilde{x} + \widetilde{y}_1) = (x,y)$, as desired. □

We now prove Proposition 2.12, which states that for n odd the first homology $H_1(\mathcal{BI}_n;\mathbb{Q})$ is infinite dimensional if the sequence $(\dim H_1(\mathrm{B}_n[m];\mathbb{Q}))_{m=1}^{\infty}$ unbounded.

Proof of Proposition 2.12. As in the statement, let $n = 2g+1$ be odd. For each m we have $\mathcal{BI}_{2g+1} \subset \mathrm{B}_{2g+1}[m]$. Indeed, $\mathcal{BI}_{2g+1}$ is equal to the intersection of all $\mathrm{B}_{2g+1}[m]$ with $m \geq 1$. It follows from the work of Brendle and the second author [8] that for $g \geq 1$ there is an isomorphism

$$\mathrm{B}_{2g+1}[2m]/\mathcal{BI}_{2g+1} \cong \mathrm{Sp}_g(\mathbb{Z})[2m].$$

The fact that $H_1(\mathrm{Sp}_g(\mathbb{Z})[2m];\mathbb{Q}) = 0$ for $g \geq 2$ (see, for example, [28]) implies that there is a surjection

$$H_1(\mathcal{BI}_{2g+1};\mathbb{Q}) \to H_1(\mathrm{B}_{2g+1}[2m];\mathbb{Q}).$$

From this, it follows that if the sequence $(\dim H_1(\mathrm{B}_n[2m];\mathbb{Q}))_{m=1}^{\infty}$ were unbounded then $H_1(\mathcal{BI}_{2g+1};\mathbb{Q})$ would be infinite dimensional. To complete the proof, it now suffices to observe that the transfer homomorphism gives a surjection $H_1(\mathrm{B}_{2g+1}[2m];\mathbb{Q}) \to H_1(\mathrm{B}_{2g+1}[m];\mathbb{Q})$. So if the sequence $(\dim H_1(\mathrm{B}_n[m];\mathbb{Q}))_{m=1}^{\infty}$ unbounded the sequence $(\dim H_1(\mathrm{B}_n[2m];\mathbb{Q}))_{m=1}^{\infty}$ would be unbounded as well. □

13. 2-torsion on the characteristic varieties of the braid arrangement

The goal of this section is to prove Theorem 2.14. We first introduce a general branching rule that gives the restriction of an irreducible $\mathcal{Z}_n$-representation of to $\mathcal{PZ}_n$ (Lemma 13.1). To prove the theorem we apply the lemma to our description of $H_1(\mathrm{B}_n[4];\mathbb{C})$ from Theorem 2.5 in order to explicitly compute all of the 2-torsion points that lie on the characteristic variety of the braid arrangement.

Lemma 13.1. *Let $n \geq 2$. Assume that ρ is an irreducible I-isotypic representation of $\mathcal{Z}_m^I$ for some full subset $I \subset [m]^{\underline{2}}$ where $m \leq n$. Then we have an isomorphism of $\mathcal{PZ}_n$-modules*

$$\mathrm{Res}^{\mathcal{Z}_n}_{PB_n} V_n(\rho,\lambda) \cong (\dim V_m(\rho))\,(\dim V(\lambda)) \bigoplus_{g\in \mathcal{Z}_n\,/\,\mathcal{Z}_n^I} V_{g(I)}$$

Proof. By the definition of the $V_n(\rho,\lambda)$, by the formula for the restriction of an induced representation [9, Proposition 5.6(b)], and by the fact that $\overline{PB}_n \subset \mathcal{Z}_n^J$ for every $J \subset [n]^{\underline{2}}$, we have

$$\mathrm{Res}^{\mathcal{Z}_n}_{PB_n} V_n(\rho,\lambda) = \mathrm{Res}^{\mathcal{Z}_n}_{PB_n} \mathrm{Ind}^{\mathcal{Z}_n}_{\mathcal{Z}_n^I} V_m(\rho) \boxtimes V(\lambda) \cong \bigoplus_{g\in \mathcal{Z}_n\,/\,\mathcal{Z}_n^I} \mathrm{Res}^{\mathcal{Z}_n^{g(I)}}_{\mathcal{PZ}_n} gV_m(\rho)\boxtimes V(\lambda).$$

To complete the proof, we observe that $V_{n-m}(\lambda)$ restricts to a direct sum of $\dim V_{n-m}(\lambda)$ copies of the trivial $\mathcal{PZ}_n$-module and that $V_m(\rho)$ restricts to the direct sum of $\dim V_m(\rho)$ copies of the representation V_I. Thus $V_m(\rho)\boxtimes V_{n-m}(\lambda)$ restricts to the direct sum of $(\dim V_m(\rho))\,(\dim V_{n-m}(\lambda))$ copies of V_I. Employing Lemma 8.2, we see that $gV_m(\rho)\boxtimes V_{n-m}(\lambda)$ restricts to the direct sum of $(\dim V_m(\rho))\,(\dim V_{n-m}(\lambda))$ copies of $V_{g(I)}$. The result follows. □

Proof of Theorem 2.14. A 2-torsion point of $V_d(X_n)$ is a homomorphism $\rho : \mathrm{PB}_n \to \mu_2$ with $\dim H^1(X_n;\mathbb{C}_\rho) \geq d$. As in Section 8, any such ρ is equal to some ρ_I. For any $I \subset [n]^{\underline{2}}$, we may identify the fiber of $\mathbb{C}_{\rho_I}$ with the $\mathcal{PZ}_n$-module V_I, viewed as a PB_n-module. The fact that $\mathcal{PZ}_n = \mathrm{PB}_n\,/\,\mathrm{B}_n[4]$ is a finite group implies that the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathcal{PZ}_n; H^q(\mathrm{B}_n[4];V_I)) \implies H^{p+q}(\mathrm{PB}_n;V_I)$$

degenerates at the E_2 page. This gives isomorphisms

$$H^1(X_n;\mathbb{C}_{\rho_I}) \cong H^1(\mathrm{PB}_n;V_I) \cong \left(H^1(\mathrm{B}_n[4];\mathbb{C})\otimes V_I\right)^{\mathcal{PZ}_n}.$$

We conclude that $\dim H^1(X_n;\mathbb{C}_{\rho_I})$ is equal to the multiplicity of V_I in $H^1(\mathrm{B}_n[4];\mathbb{C}) \cong H_1(\mathrm{B}_n[4];\mathbb{C})$, regarded as a $\mathcal{PZ}_n$-module. Combining Theorem 2.5 with Lemma 13.1 we see that

$$\mathrm{Res}^{\mathcal{Z}_n}_{\mathcal{PZ}_n} H_1(\mathrm{B}_n[4];\mathbb{C}) \cong \begin{cases} \mathbb{C}^3 \oplus \left(\bigoplus_{g\in\mathcal{Z}_3\,/\,\mathcal{Z}_3^{I_3}} V_{g(I_3)}\right) & n=3 \\ \mathbb{C}^{\binom{n}{2}} \oplus \left(\bigoplus_{g\in\mathcal{Z}_n\,/\,\mathcal{Z}_n^{I_3}} V_{g(I_3)} \oplus \bigoplus_{g\in\mathcal{Z}_n\,/\,\mathcal{Z}_3^{I_4}} V_{g(I_4)}\right) & n\geq 4 \end{cases}$$

Thus the multiplicity of any nontrivial V_I in $\mathrm{Res}^{\mathcal{Z}_n}_{\mathcal{PZ}_n} H_1(\mathrm{B}_n[4];\mathbb{C})$ is at most 1. That is, for $d \geq 2$ there are no 2-torsion points on $V_d(X_n)$. Further, this decomposition shows that the 2-torsion points on $V_1(X_n)$ are exactly those of the form $\rho_{g(I_3)}$ or $\rho_{g(I_4)}$ for $g \in S_n$. It remains only to show that these points lie on $\check{V}_1(X_n)$.

Cohen–Suciu [14] found explicit equations for all of the components of $\check{V}_1(X_n)$. For $i < j < k$ there is a component

$$V_{ijk} = \{\mathbf{t} \in (\mathbb{C}^\times)^{\binom{n}{2}} : t_{ij}t_{ik}t_{jk} = 1 \text{ and } t_{pq} = 1 \text{ if } |\{p,q\} \cap \{i,j,k\}| \leq 1\}$$

and for each 4-element set $I = \{i,j,k,\ell\}$ with $i < j < k < \ell$ there is a component

$$V_{ijk\ell} = \{\mathbf{t} \in (\mathbb{C}^\times)^{\binom{n}{2}} : t_{pq} = t_{rs} \text{ if } \{p,q\} \cup \{r,s\} = I, t_{pq} = 1 \text{ if } \{p,q\} \not\subset I\ , \prod t_{pq} = 1\}.$$

We directly verify that the 2-torsion points of the form $\rho_{g(I_3)}$ lie in $V_{g(123)}$ and those of the form $\rho_{g(I_4)}$ lie in $V_{g(1234)}$. This completes the proof of the theorem. □

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