COHOMOLOGY OF THE HYPERELLIPTIC TORELLI GROUP

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Abstract. Let $\mathcal{SI}(S_g)$ denote the hyperelliptic Torelli group of a closed surface S_g of genus g. This is the subgroup of the mapping class group of S_g consisting of elements that act trivially on $H_1(S_g;\mathbb{Z})$ and that commute with some fixed hyperelliptic involution of S_g. We prove that the cohomological dimension of $\mathcal{SI}(S_g)$ is $g-1$ when $g \geq 1$. We also show that $H_{g-1}(\mathcal{SI}(S_g);\mathbb{Z})$ is infinitely generated when $g \geq 2$. In particular, $\mathcal{SI}(S_3)$ is not finitely presentable. Finally, we apply our main results to show that the kernel of the Burau representation of the braid group B_n at $t = -1$ has cohomological dimension equal to the integer part of $n/2$, and it has infinitely generated homology in this top dimension.

1. Introduction

Let S_g denote the closed, connected, orientable surface of genus g, and let s be some fixed hyperelliptic involution of S_g. The mapping class group $\mathrm{Mod}(S_g)$ is the group of isotopy classes of orientation-preserving homeomorphisms of S_g, and the *hyperelliptic Torelli group* $\mathcal{SI}(S_g)$ is the subgroup of $\mathrm{Mod}(S_g)$ consisting of elements that commute with the homotopy class of s and that act trivially on $H_1(S_g;\mathbb{Z})$. The group $\mathcal{SI}(S_g)$ arises, for example, as the fundamental group of the branch locus of the period mapping [12, Section 4]. Also, for small g, Ellenberg [9] gives a description of the $\mathrm{Sp}(2g,\mathbb{Z})$-module structure of the cohomology of the full Torelli group (see below) in terms of the cohomology of $\mathcal{SI}(S_g)$.

Cohomological dimension. The *cohomological dimension* $\mathrm{cd}(G)$ of a group G is the supremum over all n so that there exists a G-module M with $H^n(G;M) \neq 0$. If a group G has torsion, then $\mathrm{cd}(G) = \infty$. On the other hand, if G contains a torsion-free subgroup H of finite index, then

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we can define the *virtual cohomological dimension* $\mathrm{vcd}(G) = \mathrm{cd}(H)$. It is a theorem of Serre that $\mathrm{vcd}(G)$ is well defined [25, Théorème 1].

Main Theorem 1. *For $g \geq 1$, we have* $\mathrm{cd}(\mathcal{SI}(S_g)) = g - 1$.

Dimensions of Torelli groups. Let $\mathcal{I}(S_g)$ denote the *Torelli group* of S_g, that is, the subgroup of $\mathrm{Mod}(S_g)$ consisting of elements that act trivially on $H_1(S_g;\mathbb{Z})$. Let $\mathcal{K}(S_g)$ denote the subgroup of $\mathcal{I}(S_g)$ generated by Dehn twists about separating simple closed curves. It is a fact that $\mathcal{SI}(S_g)$ is a subgroup of $\mathcal{K}(S_g)$; this follows immediately from the naturality property of Johnson's homomorphism τ [18, Lemma 2D] and Johnson's theorem that $\mathcal{K}(S_g) = \ker(\tau)$ [19, Theorem 6].

Since

$$\mathrm{Mod}(S_g) \geq \mathcal{I}(S_g) \geq \mathcal{K}(S_g) \geq \mathcal{SI}(S_g),$$

it follows from Fact 4.1 below that the dimensions of these groups also form a decreasing sequence. For $g \geq 2$, we in fact have the following:

$$\begin{aligned}
\mathrm{vcd}(\mathrm{Mod}(S_g)) &= 4g-5\\
\mathrm{cd}(\mathcal{I}(S_g)) &= 3g-5\\
\mathrm{cd}(\mathcal{K}(S_g)) &= 2g-3\\
\mathrm{cd}(\mathcal{SI}(S_g)) &= g-1.
\end{aligned}$$

The first equality is due to Harer [13, Theorem 4.1]. An alternate proof was given by Ivanov [16, Theorem 6.6]. The lower bound of $4g-5$ was also given by Mess [21, Proposition 1], and the upper bound follows from work of Culler–Vogtmann [8]. The inequality $\mathrm{cd}(\mathcal{I}(S_g)) \geq 3g-5$ was proven by Mess [21, Proposition 1], and the inequality $\mathrm{cd}(\mathcal{I}(S_g)) \leq 3g-5$ was proven by Bestvina–Bux–Margalit [2, Theorem A]. The dimension $\mathrm{cd}(\mathcal{K}(S_g))$ was computed by Bestvina–Bux–Margalit [2, Theorem B].

In the case $g = 2$, the groups $\mathcal{I}(S_2)$, $\mathcal{K}(S_2)$, and $\mathcal{SI}(S_2)$ are all equal (combine [4, Theorem 8] with [24, Theorem 2][1]). This agrees with the fact that $3g-5$, $2g-3$, and $g-1$ are all equal when $g = 2$.

The hyperelliptic Johnson filtration. The *Johnson filtration* of $\mathrm{Mod}(S_g)$ is the sequence of groups $\mathcal{N}_k(S_g)$ defined by:

$$\mathcal{N}_k(S_g) = \ker(\mathrm{Mod}(S_g) \to \mathrm{Out}(\pi_1(S_g)/\pi_1^k(S_g))),$$

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[1] Powell states his result for $g \geq 3$, but his proof holds in the case $g = 2$.

where $\pi_1^k(S_g)$ is the kth term of the lower central series for $\pi_1(S_g)$. By definition, $\mathcal{N}_1(S_g) = \mathrm{Mod}(S_g)$ and $\mathcal{N}_2(S_g) = \mathcal{I}(S_g)$. It is a theorem of Johnson that $\mathcal{N}_3(S_g) = \mathcal{K}(S_g)$ [19]. An argument of Farb [10, Theorem 5.10] and the fact that $\mathcal{N}_k(S_g) \leq \mathcal{K}(S_g)$ for $k \geq 3$ gives

$$g - 1 \leq \mathrm{cd}(\mathcal{N}_k(S_g)) \leq 2g - 3$$

for $g \geq 2$ and $k \geq 3$ (see Fact 4.1 below).

We may also consider the groups $\mathcal{SN}_k(S_g) = \mathcal{N}_k(S_g) \cap \mathrm{SMod}(S_g)$. For $k \geq 1$, we have $\mathcal{SN}_k(S_g) \leq \mathcal{SI}(S_g)$, and so $\mathrm{cd}(\mathcal{SN}_k(S_g)) \leq g - 1$ for $g \geq 1$ and $k \geq 1$. On the other hand, we will prove in Proposition 4.14 below that $\mathcal{SN}_k(S_g)$ contains a subgroup isomorphic to $\mathbb{Z}^{g-1}$ for $g \geq 1$ and $k \geq 1$. Therefore, we have the following theorem.

Theorem 1.1. *For $g \geq 1$ and $k \geq 1$, we have*

$$\mathrm{cd}(\mathcal{SN}_k(S_g)) = g - 1.$$

Top-dimensional homology. Bestvina–Bux–Margalit proved that the top-dimensional homology of $\mathcal{I}(S_g)$ is infinitely generated [2, Theorem C]. We prove the analogous result for $\mathcal{SI}(S_g)$.

Main Theorem 2. *For $g \geq 2$, the group $H_{g-1}(\mathcal{SI}(S_g); \mathbb{Z})$ is infinitely generated.*

Since $\mathcal{I}(S_1)$ is trivial, Main Theorem 2 does not hold for $g = 1$. Mess proved that $\mathcal{SI}(S_2) = \mathcal{I}(S_2)$ is an infinite rank free group [22, Proposition 4], from which it immediately follows that $H_1(\mathcal{SI}(S_2); \mathbb{Z})$ is infinitely generated.

It is not known in general whether or not the groups $\mathcal{SI}(S_g)$ are finitely generated or finitely presented for $g \geq 3$. However, we have the following immediate consequence of Main Theorem 2.

Corollary 1.2. *The group $\mathcal{SI}(S_3)$ is not finitely presentable.*

The Burau representation. Let Bur_n denote the kernel of the reduced Burau representation at $t = -1$. In Section 5, we explain the precise connection between Bur_n and the hyperelliptic Torelli group. We obtain the following theorem.

Theorem 1.3. *For $n \geq 5$, we have*

$$\mathrm{cd}(\mathrm{Bur}_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Also, $H_{\lfloor \frac{n}{2} \rfloor}(\mathrm{Bur}_n; \mathbb{Z})$ is infinitely generated.

Our approaches to proving our main theorems are modeled on the arguments of the paper by Bestvina–Bux–Margalit [2]. On the other hand, some of the details are more subtle in the present situation, and we place most of our emphasis on these points.

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2. The complex of symmetric cycles

Our main theorems will be proven by analyzing the action of $\mathcal{SI}(S_g)$ on a contractible complex $\mathcal{SB}_x(S_g)$, which we define in this section. This complex is a symmetric version of the complex of minimizing cycles introduced by Bestvina–Bux–Margalit [2].

Fix some nonzero $x \in H_1(S_g;\mathbb{Z})$. The complex $\mathcal{SB}_x(S_g)$ will be defined as a certain set of isotopy classes of 1-cycles in S_g representing x. The complex does depend on the choice of x (there are finitely many isomorphism types of complexes for the infinitely many choices of x), but the main feature of $\mathcal{SB}_x(S_g)$, its contractibility, will not depend on x.

A *1-cycle* in S_g is a finite formal sum

$$\sum k_i c_i$$

where $k_i \in \mathbb{R}$, and each c_i is an oriented simple closed curve in S_g; the set $\{c_i : k_i \neq 0\}$ is called the *support.* We say that the 1-cycle is *simple* if the curves of the support are pairwise disjoint, and we say that it is *positive* if each k_i is positive.

Let $\mathcal{S}$ denote the set of isotopy classes of oriented simple closed curves in S_g. We may regard the isotopy class of a simple, positive 1-cycle in S_g as an element of $\mathbb{R}^{\mathcal{S}}_{\geq 0}$, the space of functions $\mathcal{S} \to \mathbb{R}_{\geq 0}$.

For an oriented simple closed curve (or 1-cycle) c, we denote by $\overleftarrow{c}$ the *reverse* of c, that is, the curve (or 1-cycle) obtained by reversing the orientation of c. A 1-cycle σ is *skew-symmetric* (with respect to the hyperelliptic involution s) if $s(\sigma) = \overleftarrow{\sigma}$.

A *skew-symmetric pair of curves* in S_g is a pair of disjoint, oriented simple closed curves in S_g interchanged and reversed by s, that is, a pair of disjoint, oriented curves of the form $\{c, s(\overleftarrow{c})\}$. Both curves in a skew-symmetric pair must be nonseparating. This follows, for example, from the fact that s acts by $-I$ on $H_1(S_g;\mathbb{Z})$.

A *skew-symmetric multicurve* in S_g is a nonempty collection of skew-symmetric pairs of curves in S_g that are homotopically nontrivial, pairwise disjoint, and pairwise non-homotopic. Note that a skew-symmetric multicurve has no connected components that are preserved by s. Also, two simple closed curves lying in a given skew-symmetric multicurve can be isotopic only if they lie in the same skew-symmetric pair.

A *basic skew-symmetric cycle* is a positive, skew-symmetric 1-cycle

$$\sum_{i=1}^{n} \frac{k_i}{2}(c_i + s(\overleftarrow{c_i}))$$

where the support $\{c_i, s(\overleftarrow{c_i})\}$ is a skew-symmetric multicurve, and where the $[c_i]$ form a linearly independent subset of $H_1(S_g;\mathbb{R})$.

Let $\mathcal{SM}$ denote the set of isotopy classes of skew-symmetric multicurves in S_g that are unions of supports of basic skew-symmetric cycles representing x.

Let $M = \{c_1, s(\overleftarrow{c_1}), \ldots, c_m, s(\overleftarrow{c_m})\}$ be a skew-symmetric multicurve whose isotopy class $[M]$ lies in $\mathcal{SM}$. The set

$$P_M = \left\{ (k_1, \ldots, k_m) \in \mathbb{R}^m_{\geq 0} : \sum_{i=1}^{m} \frac{k_i}{2}(c_i + s(\overleftarrow{c_i})) \text{ is a skew-symmetric} \right.$$
$$\left. \text{1-cycle representing } x \right\}$$

is a convex polytope in $\mathbb{R}^m_{\geq 0}$. Indeed, it is the convex hull of the points corresponding to basic skew-symmetric cycles representing x. The faces of P_M correspond exactly to skew-symmetric multicurves $M' \subseteq M$ with $[M'] \in \mathcal{SM}$.

The cell complex $\mathcal{SB}_x(S_g)$ is defined as follows: the set of cells is

$$\{P_M : [M] \in \mathcal{SM}\}.$$

We identify two cells if they are equal in $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ and endow the quotient with the weak topology. We refer to $\mathcal{SB}_x(S_g)$ as the *complex of symmetric cycles.*

Theorem 2.1. *Let $g \geq 1$, and let $x \in H_1(S_g;\mathbb{Z})$ be any primitive element. The complex $\mathcal{SB}_x(S_g)$ is contractible.*

Bestvina–Bux–Margalit studied a complex $\mathcal{B}_x(S_g)$ on which $\mathcal{SB}_x(S_g)$ is modeled. Theorem 2.1 can be proven in the same way as the contractibility of $\mathcal{B}_x(S_g)$; see [2, Theorem E] and [14, Proposition 7]. The only thing to check is that their functions Drain and Surger preserve skew-symmetry. But this is easy to verify. Thus, we do not repeat the proof.

Dimensions of cells. The quotient map $S_g \to S_g/\langle s\rangle$ is a branched cover of S_g over a sphere $S_{0,2g+2}$ with $2g+2$ *cone points* of order two, namely, the images of the $2g+2$ fixed points of s.

For our purposes, the cone points are simply marked points; we only use this terminology to distinguish these $2g+2$ points from other marked points. When we discuss simple closed curves (and homotopies of curves) in $S_{0,2g+2}$, we treat cone points (and all marked points) as if they are punctures. So, for instance, curves are not allowed to pass through cone points.

The image of any skew-symmetric multicurve M under the quotient $S_g \to S_{0,2g+2}$ is an unoriented multicurve $\overline{M}$ in $S_{0,2g+2}$, that is, a collection of essential, pairwise disjoint, pairwise non-homotopic simple closed curves in $S_{0,2g+2}$. Let $Z = Z(M)$ denote the number of components of $S_{0,2g+2} - \overline{M}$ that do not contain any of the $2g+2$ cone points, and let $P = P(M)$ denote the number of components that do contain cone points.

Proposition 2.2. *For any* $[M] \in \mathcal{SM}$, *we have*

$$\dim(P_M) = Z.$$

Proof. Let W denote the space of all 1-cycles that represent x and that are supported in M. This is a plane in $\mathbb{R}^{|M|}$, whose dimension is one fewer than the number of complementary components of M in S_g; see [2, Lemma 2.1] and [14, Proposition 5]. The number of such components is precisely $P + 2Z$.

Let $\sigma \in W$. For each skew-symmetric pair $\{c_i, s(\overleftarrow{c_i})\}$ in M, there is a line in W through σ of the form

$$\{\sigma + tc_i - t(s(\overleftarrow{c_i})) \mid t \in \mathbb{R}\};$$

indeed, $[tc_i - t(s(\overleftarrow{c_i}))] = 0$ in $H_1(S_g;\mathbb{Z})$. These lines are linearly independent in $\mathbb{R}^{|M|}$ (their direction vectors are nonzero in different coordinates) and so they determine an $|\overline{M}|$-dimensional plane U inside W.

The cell P_M is contained in the intersection of W with the subspace of $\mathbb{R}^{|M|}$ cut out by the conditions that, for each skew-symmetric pair $\{c_i, s(\overleftarrow{c_i})\}$, the coefficients of c_i and $s(\overleftarrow{c_i})$ are equal (specifically, P_M is the intersection of this plane with the positive orthant in $\mathbb{R}^{|M|}$). These $|\overline{M}|$ conditions specify a unique point in U. Thus, the dimension of P_M is $|\overline{M}|$ less than the dimension of W:

$$\dim(P_M) = P + 2Z - 1 - |\overline{M}| = P + 2Z - 1 - (P + Z - 1) = Z,$$

as desired. □

3. The Birman–Hilden theorem

Let $\mathrm{SHomeo}^+(S_g)$ denote the group of orientation-preserving homeomorphisms of S_g that commute with the hyperelliptic involution s. We define the *hyperelliptic mapping class group* $\mathrm{SMod}(S_g)$ to be the group of isotopy classes of elements of $\mathrm{SHomeo}^+(S_g)$. We do not, a priori, require the isotopies to be s-equivariant. Thus, $\mathrm{SMod}(S_g)$ is a subgroup of $\mathrm{Mod}(S_g)$.

There is a short exact sequence

$$1 \to \langle s \rangle \to \mathrm{SHomeo}^+(S_g) \to \mathrm{Homeo}^+(S_{0,2g+2}) \to 1.$$

This is useful because $S_{0,2g+2}$ is a simpler object than S_g. As such, one would hope for an analogous short exact sequence on the level of mapping class groups. Birman–Hilden [4, Theorem 7] proved that, for $g \geq 2$, there is indeed such a short exact sequence:

$$1 \to \langle [s] \rangle \to \mathrm{SMod}(S_g) \to \mathrm{Mod}(S_{0,2g+2}) \to 1.$$

This theorem amounts to the fact that, if an element of $\mathrm{SHomeo}^+(S_g)$ is isotopic to the identity, then it is isotopic to the identity within $\mathrm{SHomeo}^+(S_g)$.

We require a souped-up version. Let P be a set of $2p$ marked points in S_g and say that s interchanges the points of P in pairs. Let $\overline{P}$ denote the image of P in $S_{0,2g+2}$. Let $\mathrm{SMod}(S_g, P)$ be the group of isotopy classes of orientation-preserving homeomorphisms of S_g that commute with s and preserve the set P. Similarly, define $\mathrm{Mod}(S_{0,2g+2}, \overline{P})$ as the group of isotopy classes of orientation-preserving homeomorphisms of $S_{0,2g+2}$ that preserve the set of $2g+2$ cone points and preserve the set $\overline{P}$.

We have the following generalized short exact sequence, also due to Birman–Hilden [5, Theorem 1].

Theorem 3.1. *Let* $g \geq 0$*. If* $g = 1$*, assume that* $p > 0$*, and if* $g = 0$*, assume that* $p > 1$*. There is a short exact sequence:*

$$1 \to \langle [s] \rangle \to \mathrm{SMod}(S_g, P) \to \mathrm{Mod}(S_{0,2g+2}, \overline{P}) \to 1.$$

The conclusion of Theorem 3.1 does not hold as stated for the case where $g = 1$ and $p = 0$. Indeed, consider the element ϕ of $\mathrm{SHomeo}^+(T^2)$ that is rotation by π in one of the two circle factors. Let $\overline{\phi}$ denote the image of ϕ in $\mathrm{Homeo}^+(S_{0,4})$. The mapping class $[\phi]$ is trivial, but the mapping class $[\overline{\phi}]$ is nontrivial, as it induces a nontrivial permutation of the cone points of $S_{0,4}$. Thus, we do not have a natural well-defined map $\mathrm{SMod}(T^2) \to \mathrm{Mod}(S_{0,4})$.

We can, however, modify Theorem 3.1 in the case $g = 1$, $p = 0$. First of all, each element of $\mathrm{Mod}(T^2)$ has a (linear) representative that commutes with s, and so $\mathrm{SMod}(T^2) = \mathrm{Mod}(T^2)$. Second, there is a non-canonical isomorphism $\mathrm{Mod}(T^2) \to \mathrm{Mod}(T^2, p)$, where p is one of the fixed points of s. The reason for this is that each element of $\mathrm{Mod}(T^2)$ has a (linear) representative that fixes the image of the origin under the covering map $\mathbb{R}^2 \to T^2$.

Let $\overline{p}$ denote the image of p in $S_{0,4}$, and let $\mathrm{Mod}(S_{0,4}, \overline{p})$ denote the subgroup of $\mathrm{Mod}(S_{0,4})$ consisting of elements that fix the marked point $\overline{p}$. We have the following special case of the Birman–Hilden theorem.

Theorem 3.2. *There is a short exact sequence:*

$$1 \to \langle [s] \rangle \to \mathrm{SMod}(T^2) \to \mathrm{Mod}(S_{0,4}, \overline{p}) \to 1.$$

Note that, in the statement of Theorem 3.2, the group $\mathrm{Mod}(S_{0,4}, \overline{p})$ is a subgroup of $\mathrm{Mod}(S_{0,4})$, not $\mathrm{Mod}(S_{0,5})$, since $\overline{p}$ is already a cone point of $S_{0,4}$.

4. Cohomological dimension

In this section, we prove Main Theorem 1, which states that $\mathrm{cd}(\mathcal{SI}(S_g)) = g - 1$. We start by showing that $\mathrm{cd}(\mathcal{SI}(S_g)) \geq g - 1$ (Proposition 4.2).

We will use the following fact [7, Chapter VIII, Proposition 2.4].

Fact 4.1. *If* H *is a subgroup of a group* G*, then* $\mathrm{cd}(H) \leq \mathrm{cd}(G)$*.*

Proposition 4.2. *For $g \geq 1$, we have* $\mathrm{cd}(\mathcal{SI}(S_g)) \geq g-1$.

Proof. We can find a collection of $g-1$ mutually disjoint, essential, homotopically distinct, separating simple closed curves in S_g that are fixed by s. The Dehn twists about these curves generate a subgroup of $\mathcal{SI}(S_g)$ that is isomorphic to $\mathbb{Z}^{g-1}$ [11, Lemma 3.17]. It is a basic fact that $\mathrm{cd}(\mathbb{Z}^n) = n$ for any $n \geq 0$; see [7, Section VIII.2]. Applying Fact 4.1, we deduce the desired lower bound. $\square$

We now aim to show that $\mathrm{cd}(\mathcal{SI}(S_g)) \leq g-1$ (Proposition 4.13). Our basic tool is the following fact [7, Section VIII.2, Exercise 4].

Proposition 4.3. *Suppose that a group G acts on a contractible cell complex X. We have*

$$\mathrm{cd}(G) \leq \sup_{\tau}\{\mathrm{cd}(\mathrm{Stab}_G(\tau)) + \dim(\tau)\}$$

where the supremum is taken over all cells τ of X.

Of course, we will apply Proposition 4.3 to the case of the action of $\mathcal{SI}(S_g)$ on the complex of symmetric cycles $\mathcal{SB}_x(S_g)$.

4.1. The Birman exact sequence and dimension.

Let $S_{g,n}$ denote a closed, connected, orientable surface of genus g with $n > 0$ marked points. The group $\mathrm{Mod}(S_{g,n})$ is the group of isotopy classes of orientation-preserving homeomorphisms of S_g that preserve the set of n marked points.

Assume $2g+n > 3$. Denote the nth marked point of $S_{g,n}$ by p, and let $\mathrm{Mod}(S_{g,n}, p)$ denote the subgroup of $\mathrm{Mod}(S_{g,n})$ preserving p. There is a natural map $\mathrm{Mod}(S_{g,n}, p) \to \mathrm{Mod}(S_{g,n-1})$ obtained by forgetting that p is marked. The Birman exact sequence [3, Section 1] identifies the kernel:

$$1 \to \pi_1(S_{g,n-1}, p) \to \mathrm{Mod}(S_{g,n}, p) \to \mathrm{Mod}(S_{g,n-1}) \to 1.$$

Let $\mathrm{PMod}(S_{g,n})$ denote the subgroup of $\mathrm{Mod}(S_{g,n})$ consisting of elements that induce the trivial permutation of the marked points. We also have the restriction:

$$1 \to \pi_1(S_{g,n-1}, p) \to \mathrm{PMod}(S_{g,n}) \to \mathrm{PMod}(S_{g,n-1}) \to 1.$$

We would like to use the Birman exact sequence to gain information about the cohomology of $\mathrm{Mod}(S_{g,n})$ and its subgroups. The key is the following fact [7, Chapter VIII, Proposition 2.4].

Fact 4.4. *Suppose we have a short exact sequence of groups*

$$1 \to K \to G \to Q \to 1.$$

Then $\operatorname{cd}(G) \leq \operatorname{cd}(K) + \operatorname{cd}(Q)$.

Proposition 4.5. *For* $n \geq 3$ *we have*

$$\operatorname{cd}(\operatorname{PMod}(S_{0,n})) \leq n - 3.$$

Proof. The group $\operatorname{PMod}(S_{0,3})$ is trivial [11, Proposition 2.3], and hence it has cohomological dimension 0. Since $\pi_1(S_{0,n})$ is a free group, it has cohomological dimension 1. The proposition then follows by applying the Birman exact sequence and Fact 4.4 inductively. □

In the case of $g \geq 1$, we will require the following more delicate bound on cohomological dimension. As above, $\operatorname{PMod}(S_g, P)$ is the group of isotopy classes of homeomorphisms of S_g fixing each point in P.

Proposition 4.6. *Let* $g \geq 1$*. Suppose* P *is a set of* p *pairs of marked points in* S_g*, where the points in each pair are identified by* s*. Let* H *be some subgroup of* $\operatorname{SMod}(S_g)$ *with* $[s] \notin H$*. Let* $F : \operatorname{SMod}(S_g, P) \cap \operatorname{PMod}(S_g, P) \to \operatorname{SMod}(S_g)$ *be the forgetful map, and let* G *be a subgroup of* $F^{-1}(H)$*. Then* $\operatorname{cd}(G) \leq \operatorname{cd}(H) + p$.

Proof. Let $\overline{P}$ denote the image of P in $S_{0,2g+2}$. Since $[s] \notin \operatorname{PMod}(S_g, P)$, the Birman–Hilden theorem (Theorems 3.1 and 3.2) implies that the groups $F^{-1}(H)$ and H are identified isomorphically with their images in $\operatorname{Mod}(S_{0,2g+2}, \overline{P})$ and $\operatorname{Mod}(S_{0,2g+2})$, respectively. Applying the Birman exact sequence inductively, and using Fact 4.4 and the fact that $\operatorname{cd}(\pi_1(S_{0,n})) = \operatorname{cd}(F_{n-1}) = 1$, we obtain $\operatorname{cd}(F^{-1}(H)) \leq \operatorname{cd}(H) + p$. By Fact 4.1, we have $\operatorname{cd}(G) \leq \operatorname{cd}(F^{-1}(H))$, and the proposition follows. □

4.2. **Dimensions of cell stabilizers.** In this section, we fix some $g \geq 2$ and we fix some skew-symmetric multicurve M with $[M] \in \mathcal{SM}$. The stabilizer of $[M]$ in $\mathcal{SI}(S_g)$ is exactly the stabilizer of the cell $P_M \subseteq \mathcal{SB}_x(S_g)$ in $\mathcal{SI}(S_g)$.

As above, we denote the image of M in $S_g/\langle s\rangle \cong S_{0,2g+2}$ by $\overline{M}$. Say that $S_{0,2g+2} - \overline{M}$ has P connected components that contain some of the $2g+2$ cone points and Z components that do not contain any cone points. Denote these subsurfaces by $\overline{R}_1, \ldots, \overline{R}_P$ and $\overline{R}_{P+1}, \ldots, \overline{R}_{P+Z}$, respectively.

Say that $\overline{R}_i$ contains k_i cone points and that the preimage R_i of $\overline{R}_i$ in S_g has genus g_i. Note that $k_i > 1$, for otherwise the boundary of R_i would be null-homotopic. Denote the number of components of $\overline{M}$ in the boundary of $\overline{R}_i$ by p_i, so each $\overline{R}_i$ is homeomorphic to a sphere with k_i cone points and p_i punctures. For our purposes, punctures play the same role as marked points.

Lemma 4.7. *Let $1 \leq i \leq P$. Then R_i is homeomorphic to $S_{g_i,2p_i}$, where $g_i = (k_i - 2)/2$.*

Proof. By the Riemann–Hurwitz formula [11, Section 7.2.2], the orbifold Euler characteristic of $\overline{R}_i$ is

$$\chi(\overline{R}_i) = 2 - p_i - k_i/2.$$

Since orbifold Euler characteristic is multiplicative under orbifold covering maps, we have

$$\chi(R_i) = 4 - 2p_i - k_i.$$

Now, to each curve of $\overline{M}$, there corresponds exactly two curves of M. Therefore, R_i has $2p_i$ punctures. Also, when $k_i > 0$, the cover R_i has one connected component. Plugging the last two facts into the general formula $\chi(S_{g,n}) = 2 - 2g - n$, we obtain a second formula for the Euler characteristic of R_i:

$$\chi(R_i) = 2 - 2g_i - 2p_i.$$

Combining our two formulas for $\chi(R_i)$, we find that $g_i = (k_i - 2)/2$. □

Lemma 4.8. *We have*

$$\sum_{i=1}^{P} g_i = g - P + 1.$$

Proof. Combining Lemma 4.7 with the fact that $\sum k_i = 2g + 2$, we have

$$\sum_{i=1}^{P} g_i = \sum_{i=1}^{P} \frac{k_i - 2}{2} = \left(\frac{1}{2}\sum_{i=1}^{P} k_i\right) - P = \frac{2g+2}{2} - P = g - P + 1.$$

□

Lemma 4.9. *We have*

$$|\overline{M}| + 1 = P + Z.$$

Proof. The quantity on the right hand side is the total number of components of $S_{0,2g+2} - \overline{M}$. Since $S_{0,2g+2}$ is a sphere, the number of complementary components is $|\overline{M}| + 1$. ☐

Let $G(M)$ be the free abelian group generated by the Dehn twists in the curves of M.

Lemma 4.10. *The group $G(M) \cap \mathcal{SI}(S_g)$ is trivial.*

Proof. Because M contains no separating curves (see Section 2), the intersection $G(M) \cap \mathcal{K}(S_g)$ is trivial [2, Theorem A.1]. As in the introduction, $\mathcal{SI}(S_g) \leq \mathcal{K}(S_g)$. The lemma follows. ☐

Lemma 4.11. *Assume that Main Theorem 1 is true for all genera between 1 and $g-1$ inclusive. We have*

$$\mathrm{cd}(\mathrm{Stab}_{\mathcal{SI}(S_g)}(M)) \leq g - 1 - Z.$$

Proof. There is an exact sequence

$$1 \to G(M) \to \mathrm{Stab}_{\mathrm{Mod}(S_g)}(M) \to \mathrm{Mod}(S_g - M)$$

(see [11, Proposition 3.20]). Since $G(M) \cap \mathcal{SI}(S_g)$ is trivial (Lemma 4.10), $\mathrm{Stab}_{\mathcal{SI}(S_g)}(M)$ is isomorphic to its image G in $\mathrm{SMod}(S_g - M)$.

By a theorem of Ivanov, each element of G fixes each R_i and fixes each puncture of each R_i [17, Theorem 3]. Thus for each i there is a well-defined map $\mathrm{Stab}_{\mathcal{SI}(S_g)}(M) \to \mathrm{PMod}(R_i) \cap \mathrm{SMod}(R_i)$ (we define $\mathrm{SMod}(R_i)$ in the usual way); denote the image by G_i. The group G is contained in $\prod G_i$. By Fact 4.4, $\mathrm{cd}(G) \leq \sum \mathrm{cd}(G_i)$.

We claim that

(1) for $1 \leq i \leq P$, we have $\mathrm{cd}(G_i) \leq g_i - 1 + p_i$, and
(2) for $P + 1 \leq i \leq P + Z$, we have $\mathrm{cd}(G_i) \leq p_i - 3$.

We start with the first statement. If $k_i = 2$, then $g_i = 0$. Specifically, $\overline{R_i}$ is a sphere with p_i punctures and two cone points. By the Birman–Hilden theorem (Theorem 3.1), there is an injective homomorphism $\mathrm{PMod}(R_i) \cap \mathrm{SMod}(R_i) \to \mathrm{Mod}(\overline{R_i})$ (in the case $p_i = 1$, Theorem 3.1 does not apply; however, in this case, $\mathrm{PMod}(R_i) = 1$). Each element of the image fixes each of the p_i punctures. Because the image of $\mathcal{SI}(S_g)$ in $\mathrm{Mod}(S_{0,2g+2})$ lies in $\mathrm{PMod}(S_{0,2g+2})$ (see [1, Lemma 2]), the image of G_i in $\mathrm{Mod}(\overline{R_i})$ lies in $\mathrm{PMod}(\overline{R_i}) \cong \mathrm{PMod}(S_{0,p_i+2})$. By Proposition 4.5, Fact 4.1, and the fact that $g_i = 0$, we have $\operatorname{cd}(G_i) \leq \operatorname{cd}(\operatorname{PMod}(\overline{R}_i)) \leq (p_i + 2) - 3 = g_i - 1 + p_i$.

Now assume $k_i > 2$, i.e., $g_i > 0$. By filling in the punctures of R_i, we obtain a forgetful map $\operatorname{PMod}(R_i) \cap \operatorname{SMod}(R_i) \to \operatorname{SMod}(S_{g_i})$. The image of G_i under this map is a subgroup of $\mathcal{SI}(S_{g_i})$ [2, Lemma 5.10]. By assumption, we have $\operatorname{cd}(\mathcal{SI}(S_{g_i})) \leq g_i - 1$. By Proposition 4.6, we have $\operatorname{cd}(G_i) \leq g_i - 1 + p_i$.

We now address the second statement, which treats the case where $k_i = 0$. The surface $\overline{R}_i$ is homeomorphic to a sphere with p_i punctures. As in the previous cases, the group G_i is isomorphic to a subgroup of $\operatorname{PMod}(\overline{R}_i)$. By Fact 4.1 then, $\operatorname{cd}(G_i) \leq \operatorname{cd}(\operatorname{PMod}(\overline{R}_i))$. But by Proposition 4.5, the latter is at most $p_i - 3$.

We now have

$$\begin{aligned}
\operatorname{cd}(\operatorname{Stab}_{\mathcal{SI}(S_g)}(M)) &= \operatorname{cd}(G) \\
&\leq \sum_{i=1}^{P+Z} \operatorname{cd}(G_i) \\
&\leq \sum_{i=1}^{P} (g_i - 1 + p_i) + \sum_{i=P+1}^{P+Z} (p_i - 3) \\
&= \sum_{i=1}^{P} g_i + \sum_{i=1}^{P+Z} p_i - P - 3Z \\
&= (g - P + 1) + 2|\overline{M}| - P - 3Z \\
&= g - 1 - Z + 2\left(|\overline{M}| + 1 - P - Z\right) \\
&= g - 1 - Z.
\end{aligned}$$

The first equality and first inequality follow from the above discussion. The second inequality is the content of the claim. The third equality follows from Lemma 4.8 and the fifth equality from Lemma 4.9. The other two equalities are just algebra. □

4.3. **Finishing the proof of Main Theorem 1.**

Combining Proposition 2.2 and Lemma 4.11 we obtain the following.

Proposition 4.12. *Assume that Main Theorem 1 is true for all genera between 1 and $g-1$ inclusive. For any cell τ in $\mathcal{SB}_x(S_g)$, we have*

$$\operatorname{cd}(\operatorname{Stab}_{\mathcal{SI}(S_g)}(\tau)) + \dim(\tau) \leq g - 1.$$

We can now obtain the following upper bound for $\mathrm{cd}(\mathcal{SI}(S_g))$ by induction on g and applying Propositions 4.3 and 4.12.

Proposition 4.13. *For* $g \geq 1$*, we have* $\mathrm{cd}(\mathcal{SI}(S_g)) \leq g-1$.

Propositions 4.2 and 4.13 immediately imply Main Theorem 1.

4.4. The hyperelliptic Johnson filtration.

We now give a variant of Proposition 4.2 which, together with our Main Theorem 1 and Fact 4.1, gives Theorem 1.1.

Proposition 4.14. *For* $g \geq 1$ *and* $k \geq 1$*, we have* $\mathrm{cd}(\mathcal{SN}_k(S_g)) \geq g-1$.

Proof. Let $c_1, \ldots, c_{g-1}$ denote the separating simple closed curves from the proof of Proposition 4.2. We can find disjoint nonseparating simple closed curves $a_1, \ldots, a_{g-1}$ fixed by s and with the properties that the geometric intersection numbers $i(a_i, c_i)$ are all equal to 2 and $i(a_i, c_j) = 0$ for $i \neq j$. For each i, define $d_i = T_{a_i}(c_i)$, where T_{a_i} is the Dehn twist about a_i. By construction, each d_i is fixed by s. Also, we have $i(c_i, d_i) = 4$ and $i(c_i, d_j) = 0$ when $i \neq j$.

Fix some $k \geq 1$ and some i. As in Farb's proof of the lower bound $\mathrm{cd}(\mathcal{N}_k(S_g)) \geq g-1$ [10, Theorem 5.10], some nontrivial element $\gamma_{i,k}$ of the group $\langle T_{c_i}, T_{d_i} \rangle$ lies in $\mathcal{N}_k(S_g)$. Since each $\gamma_{i,k}$ lies in $\mathrm{SMod}(S_g)$, the group $\langle \gamma_{1,k}, \ldots, \gamma_{g-1,k} \rangle$ lies in $\mathcal{SN}_k(S_g)$. Since the $\gamma_{i,k}$ all have infinite order and are supported on pairwise disjoint subsurfaces of S_g, we in fact see that this group is a free abelian group of rank $g-1$. The proposition now follows from Fact 4.1. □

5. INFINITE GENERATION OF TOP HOMOLOGY

In this section, we prove Main Theorem 2. The basic strategy is to employ the following fact, which is a consequence of the Cartan–Leray spectral sequence [2, Fact 8.2].

Proposition 5.1. *Suppose a group* G *acts without rotations on a contractible cell complex* X*. Suppose that for each cell* τ *of* X *we have*

$$\mathrm{cd}(\mathrm{Stab}_G(\tau)) + \dim(\tau) \leq D.$$

Then for any vertex v *of* X*, the group* $H_D(\mathrm{Stab}_G(v); \mathbb{Z})$ *injects into* $H_D(G; \mathbb{Z})$.

We will apply Proposition 5.1 to the case of the action of $\mathcal{SI}(S_g)$ on $\mathcal{SB}_x(S_g)$. By Proposition 4.12, it suffices to show that the group $H_{g-1}(\mathrm{Stab}_{\mathcal{SI}(S_g)}(v);\mathbb{Z})$ is infinitely generated for some choice of vertex v of $\mathcal{SB}_x(S_g)$.

We proceed by induction on g. By Mess's theorem that $\mathcal{I}(S_2)$ is an infinite rank free group [22, Proposition 4], Main Theorem 2 holds for $g = 2$. Now assume that $g \geq 3$.

Let v be a vertex of $\mathcal{SB}_x(S_g)$ corresponding to a skew-symmetric nonseparating curve (or, a skew-symmetric pair where the two curves in the pair are homotopic), and let $\mathrm{Stab}_{\mathcal{SI}(S_g)}(v)$ denote the stabilizer of v in $\mathcal{SI}(S_g)$. There is a splitting

$$\mathrm{Stab}_{\mathcal{SI}(S_g)}(v) \cong \mathcal{SI}(S_{g-1}) \ltimes K,$$

where K is an infinite rank free group [6, Theorem 4.11 plus Lemma 5.8]. What is more, K contains a Dehn twist T_c, where c is a symmetric separating curve in S_g cutting off a handle containing v. It follows from the explicit description of the splitting that T_c is fixed by the action of $\mathcal{SI}(S_{g-1})$.

Now, whenever we have a semidirect product of groups $Q \ltimes K$, where $p = \mathrm{cd}(Q)$ and $q = \mathrm{cd}(K)$ are finite, we have an isomorphism $H_{p+q}(Q \ltimes K) \cong H_p(Q, H_q(K))$. This follows, for instance, from the Hochschild–Serre spectral sequence; see [15] and [2, Fact 8.1]. We thus obtain

$$H_{g-1}(\mathrm{Stab}_{\mathcal{SI}(S_g)}(v);\mathbb{Z}) \cong H_{g-2}(\mathcal{SI}(S_{g-1}); H_1(K;\mathbb{Z})).$$

Johnson defined a homomorphism that maps $\mathcal{K}(S_g)$ to a free abelian group and maps each Dehn twist in $\mathcal{K}(S_g)$ nontrivially [23, Proposition 1.1]. Since $K < \mathcal{SI}(S_g) < \mathcal{K}(S_g)$, it follows that $A = \langle [T_c] \rangle$ is a free submodule of $H_1(K;\mathbb{Z})$.

Since A is torsion free, the universal coefficients theorem gives us

$$H_{g-2}(\mathcal{SI}(S_{g-1}); A) \cong H_{g-2}(\mathcal{SI}(S_{g-1}); \mathbb{Z}) \otimes A.$$

Because A is a trivial $\mathcal{SI}(S_{g-1})$-module, the latter is infinitely generated by induction.

It thus remains to show that $H_{g-2}(\mathcal{SI}(S_{g-1}); A)$ injects into the group $H_{g-2}(\mathcal{SI}(S_{g-1}); H_1(K;\mathbb{Z}))$. The short exact sequence

$$1 \to A \to H_1(K;\mathbb{Z}) \to H_1(K;\mathbb{Z})/A \to 1$$

induces a long exact sequence of homology groups:

$$\begin{aligned} \cdots \to H_{g-1}(\mathcal{SI}(S_{g-1}); H_1(K;\mathbb{Z})/A) &\to H_{g-2}(\mathcal{SI}(S_{g-1}); A) \\ &\to H_{g-2}(\mathcal{SI}(S_{g-1}); H_1(K;\mathbb{Z})) \to \cdots . \end{aligned}$$

By Main Theorem 1, the first term shown is trivial.

Thus, $H_{g-2}(\mathcal{SI}(S_{g-1}); H_1(K;\mathbb{Z})) \cong H_{g-1}(\mathrm{Stab}_{\mathcal{SI}(S_g)}(v);\mathbb{Z})$ is infinitely generated. By Propositions 5.1 and 4.12, our Main Theorem 2 is proven.

Application to the Burau representation. Let v be a vertex of $\mathcal{SB}_x(S_g)$ corresponding to a skew-symmetric nonseparating curve. There are isomorphisms

$$\begin{aligned} \mathrm{Bur}_{2g+1} &\cong \mathcal{SI}(S_g) \times \mathbb{Z} \qquad\qquad \text{and} \\ \mathrm{Bur}_{2g+2} &\cong \mathrm{Stab}_{\mathcal{SI}(S_{g+1})}(v) \times \mathbb{Z} \end{aligned}$$

when $g \geq 2$; see [6, Lemma 5.8] and [20].

The group $\mathrm{Stab}_{\mathcal{SI}(S_{g+1})}(v)$ is isomorphic to $\mathcal{SI}(S_g) \ltimes F_\infty$. Thus, by Main Theorem 1 and Fact 4.4, we have $\mathrm{cd}(\mathrm{Stab}_{\mathcal{SI}(S_{g+1})}(v)) \leq g$. On the other hand, we showed above that $H_g(\mathrm{Stab}_{\mathcal{SI}(S_{g+1})}(v);\mathbb{Z})$ is infinitely generated, so in fact $\mathrm{cd}(\mathrm{Stab}_{\mathcal{SI}(S_{g+1})}(v)) = g$. Theorem 1.3 now follows immediately from the Künneth formula.

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