LINKING PROOFS OF BROUWER, BORSUK–ULAM, AND POINCARÉ

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1. Introduction

This article is about the Brouwer fixed point theorem, the Borsuk–Ulam theorem, and Poincaré's theorem about vector fields on the sphere. We aim to give intuitive, geometric arguments for all three theorems, at least in dimension two. The main idea we use is to decompose the domain into circles, graph the resulting functions in the solid torus, and then appeal to our intuition about knots and links in 3-space. This approach was conceived by the first author in 1965. While there are dozens of proofs of all three theorems in the literature, we hope that our approach can help shed new light on these theorems and make the proofs accessible to even broader audiences.

The Brouwer fixed point theorem is one of the most famous theorems in mathematics. It has a simple statement and a surprising conclusion. In the statement, D^n is the closed unit ball in $\mathbb{R}^n$.

Brouwer fixed point theorem. *If $f : D^n \to D^n$ is continuous, then f has a fixed point; that is, there is a $p \in D^n$ with $f(p) = p$.*

This theorem is named for L. E. J. Brouwer, who published a proof in 1910. Closely related theorems were proved by Henri Poincaré in 1883 and Piers Bohl in 1904; see the survey by Park [2] for more of the fascinating history of the theorem.

The Brouwer fixed point theorem has profound implications across mathematics and science. One application is the Nash equilibrium

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theorem from the far-flung field of game theory. Nash's theorem says that for any non-cooperative game with a finite number of players and a finite set of actions, there is a choice of action for all the players so that no player can improve their position by changing their action only. A famous special case of this is the prisoner's dilemma.

A down-to-earth demonstration of Brouwer's theorem might read as follows: Take a map of the world on a rectangular piece of paper. Take a second copy of the map, crumple it (without tearing!) and smush the crumpled map onto the first map, making sure that each point of the crumpled map is somewhere above the first map. Then, no matter what you do, there is at least one point in the crumpled map that is directly above the corresponding point in the first(!).

The Borsuk–Ulam theorem, a close relative of the Brouwer fixed point theorem, was proved by Karol Borsuk in 1933. In his paper, Borsuk credits the formulation of the problem to Stanisław Ulam. Again, the theorem has a simple statement and a surprising conclusion. In the statement, S^n is the unit sphere in $\mathbb{R}^{n+1}$.

Borsuk–Ulam theorem. *If $f : S^n \to \mathbb{R}^n$ is continuous, then there is a $p \in S^n$ with $f(p) = f(-p)$.*

Here, the standard down-to-earth demonstration reads: at any given moment, there is a pair of antipodal points on Earth where both the temperature and the atmospheric pressure are equal(!).

The Brouwer fixed point theorem can be deduced from the Borsuk–Ulam theorem; see the article by Francis Su [3]. Of the many other applications, a particularly famous one is the ham sandwich theorem: given a sandwich made of bread, ham, and cheese, we can always find a way to cut the sandwich—with one straight cut of the knife—so that all three ingredients are cut exactly in half. See the book by Matousek for much more about the Borsuk–Ulam theorem [1].

Finally, we come to the aforementioned theorem of Poincaré, who proved the $n = 2$ case in 1885. The general case was proved by Brouwer in 1912. This theorem goes by many names; we have chosen an obscure one. In the statement, a vector field on S^2 is a choice, for each $p \in S^2$, a tangent vector to S^2 at p.

Can't-comb-a-coconut theorem. *Let n be even. Every continuous vector field on S^n has a zero.*

We think of a vector field as a way of combing (and trimming) the hair on a sphere, or, a coconut. The theorem says that there is no way to do this continuously without trimming one of the hairs completely, so that it has zero length. As a down-to-earth consequence, there is always a place on Earth where the wind is not blowing.

It is not too hard to see that Poincaré's theorem fails for $n = 1$: at each point take a unit tangent vector that points in the clockwise direction (for example). To see why it fails for $n = 3$, we encourage the reader to learn about the Hopf fibration of S^3.

We now proceed to prove all three theorems using the strategy outlined at the start. We hope that the proofs appeal to anyone familiar with the kind of magic trick where linked rings impossibly become unlinked.

2. Brouwer

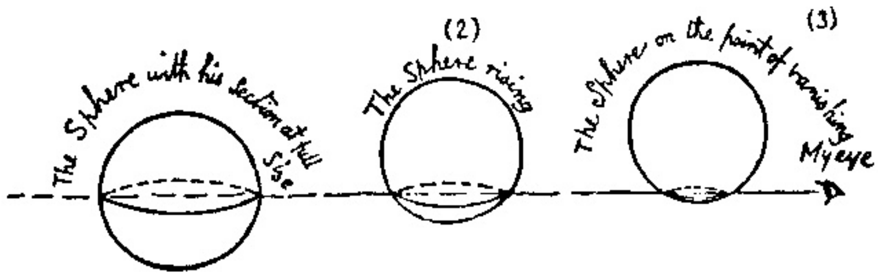
In the case $n = 1$, the Brouwer fixed point theorem says that any continuous $f : [0, 1] \to [0, 1]$ has a fixed point. This case follows from the intermediate value theorem. Indeed, if $i : [0, 1] \to [0, 1]$ is the identity map, and $g(x) = f(x) - i(x)$, then $g(0) \geq 0$ and $g(1) \leq 0$. By the intermediate value theorem, there is a point x where $g(x) = 0$. In other words, $f(x) - i(x) = f(x) - x = 0$, or $f(x) = x$.

We can visualize the argument as follows: since the graph of f must start (at $x = 0$) above the graph of the identity function i and must end (at $x = 1$) above the graph of i, there must be a point where the two graphs cross. The crossing point gives the desired fixed point. We would like to give an argument for the two-dimensional case of the theorem that capitalizes on the same kind of intuition.

So, let D^2 be the closed unit disk in $\mathbb{R}^2$, and let $f : D^2 \to D^2$ be continuous. We would like to show that f has a fixed point. In other words, we want to show that there is a point p in D^2 so that $f(p) = p$.

In the one-dimensional case, our intuition is aided by the fact that we can *see* the graphs of f and i, and we can compare them visually. In higher dimensions, this will not work in the most naïve sense. Indeed, for the two-dimensional case, the graph of f is a two-dimensional subset of the four-dimensional space $D^2 \times D^2$.

In Edwin Abbott Abbott's classic novella *Flatland*, the main character A Square is led to understand the shape of a higher-dimensional visitor (a sphere in 3-space) by viewing one slice at a time.



Taking *Flatland* as inspiration, we can better understand the graph of f by viewing it one slice at a time. Specifically, we will consider the restriction of f to the circles of radius r centered at the origin. Let us denote each such circle by S_r. We thus obtain a one-parameter family of functions:

$$\{f_r : S_r \to D^2 \,|\, 0 < r \leq 1\}.$$

All of these functions together carry the same information as f itself. (We could also include the information of $f(0)$, but this is technically not needed since f is continuous.)

Each S_r can be identified with S^1. Therefore, the graph of each f_r is a subset of $S^1 \times D^2$, the solid torus. As any doughnut or bagel lover knows, we can embed the solid torus in $\mathbb{R}^3$. This allows us to readily visualize the graphs of the various f_r.

What do these graphs in the solid torus look like? To each $\theta \in S^1$, there is an associated disk $\theta \times D^2$ in the solid torus (a bagel chip?). For a given r, the graph of f_r is the set of points $(\theta, f_r(\theta))$. So on each slice $\theta \times D^2$ we see exactly one point of the graph of f_r, namely, $f_r(\theta)$.

As θ changes continuously, the point $f_r(\theta)$ moves continuously. In this way, the graph of f_r traces out a closed curve in the solid torus. Since each slice $\theta \times D^2$ is hit exactly once, this curve goes around the torus exactly once.

Now that we have a way to visualize the graph of f we proceed similarly to the one-dimensional case: we will "draw" the graph of the identity map $i : D^2 \to D^2$, and then argue that the graph of f must cross the graph of i.

As in our discussion of the $n = 1$ case, we first want to have a sense for the graph of the identity map i. We will consider slices i_r, defined analogously to the f_r. For very small r, the graph of i_r is close to $S^1 \times \{i(0)\} = S^1 \times \{0\}$; this is the so-called core curve of the solid torus.

Now we consider $r = 1$. The graph of i_1 is the set of points

$$\theta \times (1, \theta)$$

where $0 \leq \theta \leq 2\pi$, and the second coordinate is given in polar coordinates on D^2. This is the $(1, 1)$-curve on the boundary of the solid torus, so named because it travels once around each S^1-factor of this torus, thought of as $S^1 \times S^1$.

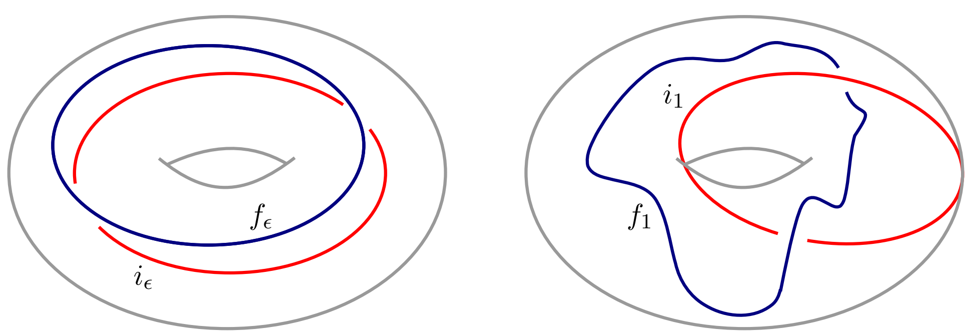


FIGURE 1. The graphs of f and i for small r and for $r = 1$

We now focus on the graph of our function f. As with the identity, the graph of f_r is close to the curve

$$S^1 \times f(0)$$

for very small r. We may as well assume that $f(0)$ is not 0 (for otherwise we would have our fixed point!). Therefore, assuming we made r small enough, the graphs of f_r and i_r are disjoint planar circles lying in parallel planes, as in the left-hand side of Figure 4.

The key point for us is that the graphs of f_r and i_r are *unlinked*. That is, we can pull them away from each other without the curves ever crossing. Now, suppose we let r increase. The graphs of i_r and f_r will change in a continuous way. If we can show that the graphs of i_1 and f_1 are *linked*—as suggested in Figure 4—then it must be the case that there is an r where i_r and f_r intersect.

To show that the graphs of i_1 and f_1 are linked we will deform the graph of f_1—without crossing the graph of i_1—in such a way that the deformed curve is clearly linked with the graph of i_1. More specifically, we will deform the graph of f_1 to the core curve of the solid torus. This core curve is certainly linked with the graph of i_1, the $(1, 1)$-curve on the boundary of the solid torus.

How do we do this deformation? We may assume that the graph of f_1 does not intersect the graph of i_1 (otherwise, we would already see a fixed point!). Still, we don't know too much about f_1. However, we do know that i_1 is a curve on the boundary of the solid torus. Therefore, on each disk $\theta \times D^2$, we can push the graph of f_1 (which is a single point in this disk) in a straight line to the center of the disk. Doing this on each disk all at the same time achieves our goal.

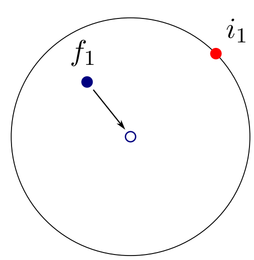


FIGURE 2. Pushing f_1 towards the core curve

The core curve and the $(1, 1)$-curve are linked. Indeed, there is a disk in $\mathbb{R}^3$ with the core curve as its boundary, and the $(1, 1)$-curve pierces this disk in a single point. Counting intersections of curves with disks is in fact one way to formally define linking numbers. But for the purposes of this discussion, we hope that your intuition tells you that these curves are linked. In other words, if they were made of metal, you could not pull them far apart from each other.

If two curves are linked, and we deform one (or both) of them in a continuous way, without introducing any crossings along the way, then the deformed curves are also linked. It follows that the graphs of f_1 and i_1 are linked. Going from our very small r to 1, the graphs of i_r and f_r go from unlinked to linked. Since there is no magic involved, it follows that there must be an r where i_r and f_r intersect. This gives us the desired fixed point, and the theorem is proved.

3. Borsuk–Ulam

Just like the Brouwer fixed point theorem, the $n = 1$ case of the Borsuk–Ulam theorem follows from the intermediate value theorem. Indeed, let $f : S^1 \to \mathbb{R}$ be a continuous map. Using θ-coordinates on S^1, we define $\bar{f} : S^1 \to \mathbb{R}$ by $\bar{f}(\theta) = f(\theta + \pi)$ (if we use $\mathbb{R}^2$-coordinates, then $\bar{f}(p) = f(-p)$). We have

$$f(0) - \bar{f}(0) = -(f(\pi) - \bar{f}(\pi))$$

To simplify, we restrict the domains of both functions to $[0, \pi]$. Since the function $f - \bar{f}$ changes sign between 0 and π, there must be a θ in $[0, \pi]$ where $f(\theta) - \bar{f}(\theta) = 0$, or $f(\theta) = f(\theta + \pi)$, as desired.

Visually, we see that the graphs of $f(\theta)$ and $\bar{f}(\theta)$ swap places between 0 and π. For instance, if the graph of f is higher than the graph of $\bar{f}$ at $\theta = 0$, then the graph of $\bar{f}$ is higher at $\theta = \pi$. Somewhere along the way, they must cross.

We now turn to the case $n = 2$ of the Borsuk–Ulam theorem. Let $f : S^2 \to D^2$ be a continuous function. We again use the auxiliary function

$$\bar{f}(p) = f(-p).$$

To prove the theorem, we must find a point $p \in S^2$ where $f(p) = \bar{f}(p)$. As in the case of the Brouwer fixed point theorem, we will accomplish this by showing that the graphs must intersect.

Again, to show that the graphs of f and $\bar{f}$ cross, we will graph the restrictions of both functions to slices of S^2. Here, the slices will be the latitudes on S^2. In astronomical coordinates, a line of latitude on S^2 is described by an angle, namely the angle ϕ between the z-axis and any line through the origin which intersects the latitude. To specify a point on a given latitude, an extra coordinate θ is needed. This coordinate gives the longitude of a point.

Our slice functions then will be the restrictions to these circles:

$$\{f_\phi : S^1 \to D^2 \,|\, 0 < \phi \leq \pi/2\} \qquad \{\bar{f}_\phi : S^1 \to D^2 \,|\, 0 < \phi \leq \pi/2\}$$

Here S^1 is naturally identified with the latitudes via the coordinate θ. Again, we can graph each of these functions in the solid torus.

For small ϕ, the situation is very similar to that in the proof of the Brouwer fixed point theorem. Indeed, for such ϕ, the graphs of f_ϕ and $\bar{f}_\phi$ resemble the curves

$$S^1 \times f(N) \qquad \text{and} \qquad S^1 \times f(S)$$

where N and S are the north and south poles of S^2.

As before, we can assume these values are distinct, otherwise we would already see the conclusion of the theorem. In this case the graphs of f_ϕ and $\bar{f}_\phi$ are unlinked for the same reason as before. We would like to show that for some ϕ, the graphs of f_ϕ and $\bar{f}_\phi$ are linked.

For $\phi = \pi/2$, the map f_ϕ is the restriction of f to the equator of S^2. Since the equator is fixed under the antipodal map, $p \mapsto -p$, we have that $\bar{f}_\phi$ is the restriction of $\bar{f}$ to the equator. We write f_{eq} and $\bar{f}_{eq}$ for these maps.

As the antipodal map acts on the equator by a rotation of π, we can write:

$$\bar{f}_{eq}(\theta) = f_{eq}(\theta + \pi)$$

where θ is the angular coordinate around the equator. In other words, the functions $\bar{f}_{eq}(\theta)$ and $f_{eq}(\theta)$ are shifted by a rotation of π in the domain.

We assume that there is no value of θ where $\bar{f}_{eq}(\theta) = f_{eq}(\theta)$, for otherwise we would have nothing to show. Thus, for each θ, we can draw a line segment ℓ_θ in the disk $\{\theta\} \times D^2$ connecting $\bar{f}_{eq}(\theta)$ to $f_{eq}(\theta)$.

As θ varies from 0 to 2π, the ℓ_θ changes in a continuous way, and hence sweeps out a strip in the solid torus. The two edges of this strip are the graphs of f_{eq} and $\bar{f}_{eq}$. To show that the graphs are linked, we will aim to understand how this strip is twisted.

Since $\bar{f}_{eq}$ and f_{eq} differ by the π-shift in the domain, the line segments ℓ_0 and ℓ_π are the same, but with endpoints reversed. Thus, ℓ_θ rotates by an angle of $k\pi$, where k is an odd integer, as θ varies from 0 to π. Again, because of the π-shift, the segment ℓ_θ again rotates by an angle of $k\pi$ (in the same direction) as θ varies from π to 2π.

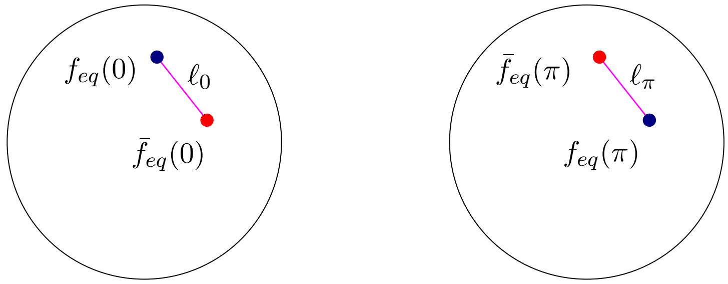


FIGURE 3. f_{eq} and $\bar{f}_{eq}$ at $\theta = 0$ and $\theta = \pi$

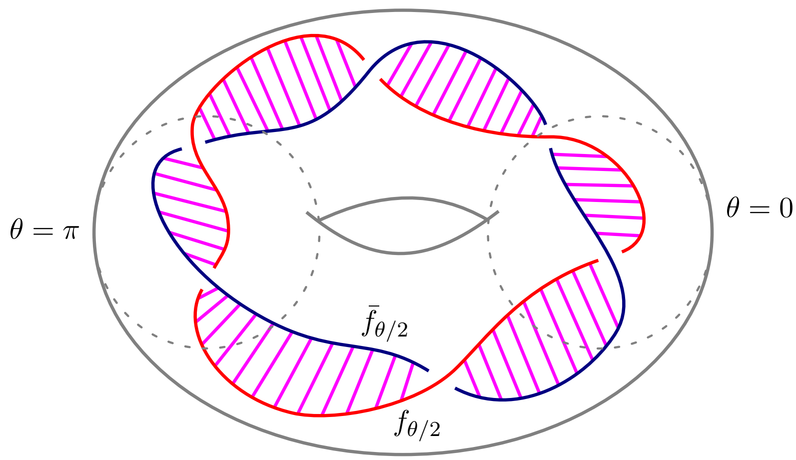


FIGURE 4. The graphs of f_{eq} and $\bar{f}_{eq}$, connected by line segments

The total rotation of ℓ_θ between 0 and 2π is then $2k\pi$, which is nonzero since k is odd. It follows that the graphs of $f_{eq}(\theta)$ and $\bar{f}_{eq}(\theta)$ are linked. (To convince yourself of this, take a strip of paper, give one end some number of full twists, and tape a loop of string to each edge. Then remove the tape, and check that you cannot pull the strings apart.)

Since f_ϕ and $\bar{f}_\phi$ go from linked to unlinked as ϕ varies from a very small positive number to $\pi/2$, there must be a point where they intersect, and this gives the point p where $f(p) = \bar{f}(p)$, or $f(p) = f(-p)$, and the theorem is proved.

In this argument, we have appealed to the reader's intuition that if we take a strip of paper, twist the strip by a nonzero number of full twists, and then glue the ends, then the two curves on the boundary are linked. The authors have not seen a magician perform this version of the linked rings trick, but would encourage such an experiment.

4. Poincaré

As a final mathematical flourish, we demonstrate how the technique of slicing and graphing can be used to prove the can't-comb-a-coconut theorem. The basic idea of our proof is the same as the previous two: we restrict the domain to circles and graph (a version of) the restrictions.

Let v be a continuous vector field on S^2, that is, a continuous function

$$v : S^2 \to \mathbb{R}^3$$

with $v(p) \perp p$ for all p. We want to find a p in S^2 with $v(p) = 0$. To simplify our pictures, we will assume that all vectors $v(p)$ have length at most 1. You can imagine giving your coconut a haircut if you like.

Now, let $\gamma : S^1 \to S^2$ be a curve where $\gamma'(\theta)$ is well defined, never zero, and changes continuously with θ. If you walk along any circle on the sphere without taking a rest, you are describing such a curve.

For any curve γ on S^2 as above, we define a function

$$f_\gamma(\theta) : S^1 \to D^2$$

as follows: given θ we take the tangent plane to S^2 at $\gamma(\theta)$ and rigidly place it on the xy-plane in $\mathbb{R}^3$ so that the outward normal vector to S^2 at $f_\gamma(\theta)$ ends up pointing along the positive z-axis and so that $\gamma'(t)$ is placed onto the positive x-axis. Then the vector $v(\gamma(\theta))$ gives us a point in D^2 (remember the haircut!). It is important here that all vectors are based at the origin.

The function f_γ is continuous because γ' and v are continuous. Its graph is thus a curve in the solid torus $S^1 \times D^2$. We are in familiar territory! As before, the graph of each f_γ is a curve that intersects each slice $\theta \times D^2$ in one point.

The core curve of the solid torus corresponds to zero vectors. If our v wants to not have zeros, the graphs of all the f_γ must avoid the core.

Let γ be a small circular loop in S^2. For concreteness, we take γ to be a loop around a line of latitude very close to the north pole, in the easterly direction. Will describe the graph of f_γ in detail.

We may as well assume v is nonzero at the north pole (otherwise we would be done!). Since v is continuous, we can zoom into the north pole until the vector field v looks as close to a constant vector field as we want. Therefore, if we choose γ to be small enough, then the

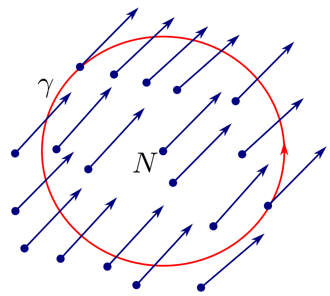


FIGURE 5. A continuous vector field near the north pole N, and a small loop γ

picture will look similar to the one in Figure 5 (in the picture, we have shifted each $v(p)$ so its base is at p).

For this curve γ, we can describe the graph of f_γ very precisely. We can see from the picture that as we walk around γ, the vector field does one full turn in the clockwise direction, and the lengths of the vectors stay approximately constant; say the lengths are approximately R. From this we conclude that the graph of f_γ is very close to the $(1,1)$-curve on the torus obtained by taking the circles of radius R on each disk.

And now (to mix magical metaphors) we pull an ace out of our sleeve:

the curve γ can be continuously deformed to its reverse

(and moreover we can do this in such a way that each intermediate curve has constant speed). To do this we stretch and pull γ so that it is deformed to the Tropic of Cancer, the equator, the Tropic of Capricorn and then to a small loop around the south pole. Then, keeping the loop small, we move it up along a longitude and over the north pole, back to its original position. At the end, we obtain $\bar{\gamma}$, the reverse of γ, given by the formula $\bar{\gamma}(\theta) = \gamma(-\theta)$.

Now for the finishing touch. The graph of $f_{\bar{\gamma}}$ is the $(1,-1)$-curve on the same torus of radius R; this is like the $(1,1)$-curve, but as we travel around the first factor of the torus $S^1 \times S^1$, we travel around the second factor in the opposite direction from before. But we deformed continuously from γ to $\bar{\gamma}$, which means the graphs deform continuously. For the same reason that it is impossible to take two linked metal rings and to flip one over without moving the other, we know there must be a moment during the deformation where the graph crosses the core curve. As above, this means that v has a zero right there. Ta da!

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