Automorphisms of the pants complex

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1 Introduction

In the theory of mapping class groups, "curve complexes" assume a role similar to the one that buildings play in the theory of linear groups: Ivanov, Korkmaz, and Luo showed that the automorphism group of the curve complex for a surface is generally isomorphic to the extended mapping class group of the surface. In this paper, we show that the same is true for the pants complex.

Throughout, S will be an orientable surface whose Euler characteristic $\chi(S)$ is negative, while $\Sigma_{g,b}$ will denote a surface of genus g with b boundary components. Also, $\mathrm{Mod}(S)$ will mean the *extended mapping class group* of S (the group of homotopy classes of self-homeomorphisms of S).

The pants complex of S, denoted $\mathrm{C_P(S)}$, has vertices representing pants decompositions of S, edges connecting vertices whose pants decompositions differ by an elementary move, and 2-cells representing certain relations between elementary moves (see Section 2). Its 1-skeleton $\mathrm{C^1_P(S)}$ is called the pants graph, and was introduced by Hatcher-Thurston. We give a detailed definition of the pants complex in Section 2.

Brock proved that $\mathrm{C^1_P(S)}$ models the Teichmüller space endowed with the Weil-Petersson metric, $\mathcal{T}_{WP}(S)$, in that the spaces are quasi-isometric [1]. Our results further indicate that $\mathrm{C^1_P(S)}$ is the "right" combinatorial model for $\mathcal{T}_{WP}(S)$, in that $\mathrm{Aut}\,\mathrm{C^1_P(S)}$ (the group of simplicial automorphisms of $\mathrm{C^1_P(S)}$) is shown to be $\mathrm{Mod}(S)$. This is in consonance with the result of Masur-Wolf that the isometry group of $\mathcal{T}_{WP}(S)$ is $\mathrm{Mod}(S)$ [10].

There is a natural action of $\mathrm{Mod}(S)$ on $\mathrm{C^1_P(S)}$; we prove that *all* automorphisms of $\mathrm{C^1_P(S)}$ are induced by $\mathrm{Mod}(S)$. The results of this paper can be summarized as follows:

$$\operatorname{Aut} \mathrm{C_P(S)} \cong \operatorname{Aut} \mathrm{C^1_P(S)} \cong \mathrm{Mod}(S)$$

for most surfaces S.

Theorem 1. *If $S \neq \Sigma_{0,3}$ is an orientable surface with $\chi(S) < 0$, and $\theta : \mathrm{Mod}(S) \longrightarrow \operatorname{Aut} \mathrm{C_P(S)}$ is the natural map, then:*

- *θ is surjective.*
- *$\ker(\theta) \cong \mathbb{Z}_2$ for $S \in \{\Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}\}$, $\ker(\theta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $S = \Sigma_{0,4}$, and $\ker(\theta)$ is trivial otherwise.*

In short, Theorem 1 says that the natural map θ is an isomorphism for most S. The nontrivial kernels in Theorem 1 are generated by hyperelliptic involutions [8]. Note that $\mathrm{C_P}(\Sigma_{0,3})$ is empty.

Theorem 2. *If S is an orientable surface with $\chi(S) < 0$, then:*

$$\operatorname{Aut} \mathrm{C_P(S)} \cong \operatorname{Aut} \mathrm{C^1_P(S)}$$

In terms of simplicial automorphisms, Theorem 2 says that the pants complex carries no more information than its 1-skeleton.

In order to prove Theorem 1, we apply the corresponding theorem for a different simplicial complex, the curve complex:

Theorem 3 (Ivanov, Korkmaz, Luo). *If $S \neq \Sigma_{0,3}$ is an orientable surface with $\chi(S) < 0$, and $\eta : \mathrm{Mod}(S) \longrightarrow \operatorname{Aut} \mathrm{C}(S)$ is the natural map, then:*

- *η is surjective when $S \neq \Sigma_{1,2}$*
- *$\ker(\eta) \cong \mathbb{Z}_2$ for $S \in \{\Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}\}$, $\ker(\eta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $S = \Sigma_{0,4}$, and $\ker(\eta)$ is trivial otherwise.*
- *$\mathrm{Im}(\eta) = \operatorname{Aut}^\star \mathrm{C}(S) \lneqq \operatorname{Aut} \mathrm{C}(S)$ when $S = \Sigma_{1,2}$.*

In the theorem, $\mathrm{C}(S)$ is the curve complex for S (defined in Section 2), and $\operatorname{Aut}^\star \mathrm{C}(S)$ is the subgroup of $\operatorname{Aut} \mathrm{C}(S)$ which preserves the set of vertices of $\mathrm{C}(S)$ representing nonseparating curves. The surjectivity statement implies that $\operatorname{Aut} \mathrm{C}(S)$ is the same as $\operatorname{Aut}^\star \mathrm{C}(S)$ for $S \neq \Sigma_{1,2}$. The reason $\Sigma_{1,2}$ is exceptional is that it has a hyperelliptic involution ρ with the property that the projection $\Sigma_{1,2} \to \Sigma_{1,2}/\langle\rho\rangle \sim \Sigma_{0,5}$ is bijective on curve complexes, but $\mathrm{Mod}(\Sigma_{1,2})/\ker(\eta) \ncong \mathrm{Mod}(\Sigma_{0,5})$.

Theorem 3 for $S \neq \Sigma_{1,2}$ is originally due to Ivanov for high genus [6] and Korkmaz for low genus [7]. Luo gave a new proof for all genera, and also settled the case of $S = \Sigma_{1,2}$ [8].

Theorem 1 is a refinement of Theorem 3 for two reasons. Firstly, $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ is a thin subcomplex of the dual of $\mathrm{C}(S)$, so a priori it has more automorphisms. Also, there are no exceptional cases to the surjectivity statement in Theorem 1.

The key idea for the proof of Theorem 1 is that there is a correlation between marked Farey graphs in $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ and vertices in $\mathrm{C}(S)$. An automorphism of $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ induces a permutation of these Farey graphs, and hence gives rise to an automorphism of $\mathrm{C}(S)$, at which point Theorem 3 applies.

Theorem 2 actually follows from Theorem 1. However, we give an independent, elementary proof in Section 4. We show that the 2-cells of $\mathrm{C}_\mathrm{P}(\mathrm{S})$, which are defined via topological relationships on S, can equivalently be characterized using only the combinatorics of $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$. For example, square 2-cells of $\mathrm{C}_\mathrm{P}(\mathrm{S})$ are originally defined as a commutator of two moves on disjoint subsurfaces on S (see Figure 4). We prove that square 2-cells can equivalently be defined as loops with 4 edges in $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ which have the property that consecutive edges do not lie in a common Farey graph. Note that in the second definition there is no reference to S, only $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$. Therefore, any automorphism of $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ must preserve these square 2-cells of $\mathrm{C}_\mathrm{P}(\mathrm{S})$.

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This work is dedicated in loving memory of my mother, Batya.

2 The complexes

2.1 Curve complex

Curves. A simple closed curve on S (homeomorphic embedding of the circle) is *nontrivial* if it is *essential* (not null homotopic) and *non-peripheral* (not homotopic to a boundary component). Throughout, we will use *curve* to mean *homotopy class of simple closed curves.*

Any mention of *intersection* between two curves α and β will refer to the *geometric intersection number* $\mathrm{i}(\alpha, \beta)$ (the minimum number of intersection points between two representative curves of the respective homotopy classes).

Curve complex. The *curve complex* of S is an abstract simplicial complex denoted $\mathrm{C}(S)$ with vertices corresponding to nontrivial (homotopy classes of simple closed) curves on S.

A set of $k+1$ vertices is the 0-skeleton of a k-simplex in $\mathrm{C}(S)$ if there are representative curves from the corresponding curve classes which are simultaneously disjoint. For example, edges correspond to pairs of disjoint curves.

It is a standard fact that if a set of homotopy classes of curves have pairwise intersection number zero then there is a single set of representative curves which are simultaneously disjoint. In other words, every complete graph on $k+1$ vertices in $\mathrm{C}(S)$ is the 1-skeleton of a k-simplex in $\mathrm{C}(S)$. One way to see this is to fix a hyperbolic metric on S and take the representative curves to be the unique geodesics in each homotopy class.

The curve complex was first defined by Harvey [3]. Harer proved that it is homotopy equivalent to a wedge of spheres [2]. Ivanov used the theorem that $\mathrm{Aut}\,\mathrm{C}(S) \cong \mathrm{Mod}(S)$ to give a new proof of Royden's theorem that $\mathrm{Isom}(\mathcal{T}(S)) \cong \mathrm{Mod}(S)$ (where $\mathcal{T}(S)$ is the Teichmüller space of S with the Teichmüller metric) [6]. Masur-Minsky showed that $\mathrm{C}(S)$ is δ-hyperbolic [9].

The curve complex has an altered definition in two cases. For $\Sigma_{0,4}$ and $\Sigma_{1,1}$, since there is no pair of distinct simple closed curves with intersection number zero, two vertices are connected by an edge when the curves they represent have minimal intersection (2 in the case of $\Sigma_{0,4}$, and 1 in the case of $\Sigma_{1,1}$). It turns out that in both cases, the curve complex is a modular configuration, or *Farey graph* (see Figure 1) [11].

2.2 Pants complex

Pants decompositions. A *pants decomposition* of S is a maximal collection of distinct nontrivial simple closed curves on S which have pairwise intersection number zero. In other words, pants decompositions correspond to maximal simplices of the curve complex. A pants decomposition always consists of $3g-3+b$ curves (where $S = \Sigma_{g,b}$). The complement in S of the

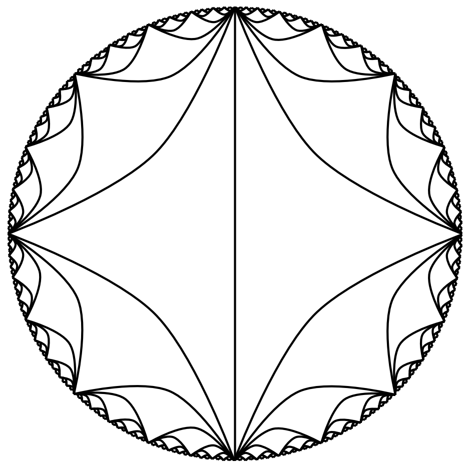

Figure 1: A Farey graph.

curves of a pants decomposition is $2g - 2 + b$ thrice punctured spheres, or *pairs of pants*. A pants decomposition is written as $\{\alpha_1, \ldots, \alpha_n\}$, where the α_i are curves on S.

Elementary moves. Two pants decompositions p and p' of S *differ by an elementary move* if p' can be obtained from p by replacing one curve in p, say α_1, with another curve, say α_1', such that α_1 and α_1' intersect *minimally.* If α_1 lies on a $\Sigma_{0,4}$ in the complement of the other curves in p, then minimally means $\mathrm{i}(\alpha_1, \alpha_1') = 2$; if α_1 lies on a $\Sigma_{1,1}$ in the complement of the rest of p, then we want $\mathrm{i}(\alpha_1, \alpha_1') = 1$. These are the only possibilities, corresponding to whether α_1 is the boundary between two pairs of pants on S or is in a single pair of pants.

An elementary move will be denoted $\{\alpha_1, \ldots, \alpha_n\} \to \{\alpha_1', \alpha_2, \ldots, \alpha_n\}$, or $\alpha_1 \to \alpha_1'$. Note that there are countably many elementary moves $\alpha_1 \to \star$.

Pants graph. The *pants graph* of S, denoted $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$, is the abstract simplicial complex with vertices corresponding to pants decompositions of S, and edges joining vertices whose associated pants decompositions differ by an elementary move.

Note that the pants graphs for $\Sigma_{0,4}$ and $\Sigma_{1,1}$ have the same definitions as (the 1-skeletons of) the curve complexes for these surfaces—all four are

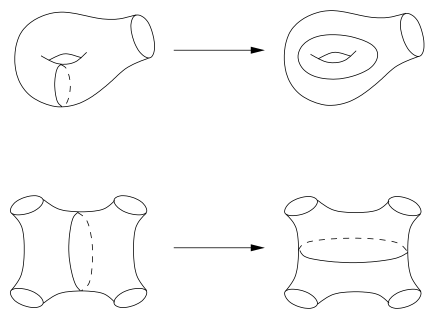

Figure 2: Elementary moves between pants decompositions.

Farey graphs.

Pants complex. The *pants complex* of S, denoted $\mathrm{C_P(S)}$, has $\mathrm{C^1_P(S)}$ as its 1-skeleton, and also has 2-cells representing specific relations between elementary moves which are given by topological data on S, as depicted in Figures 3-6.

The pants complex was first introduced by Hatcher-Thurston as a tool for constructing a finite presentation of $\mathrm{Mod}(S)$ [5]. Hatcher-Lochak-Schneps gave the pants complex its present form, and in particular showed that it is connected and simply connected [4].

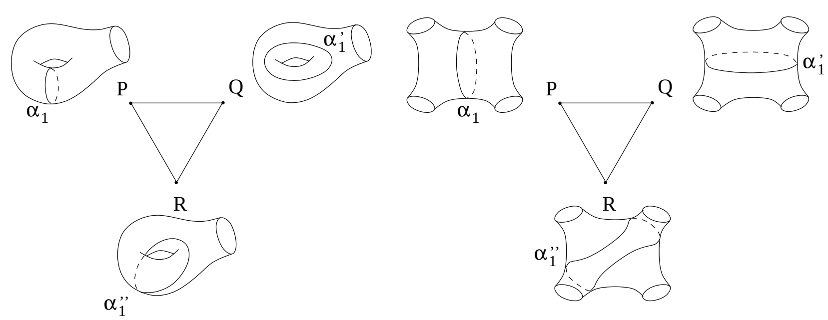

Figure 3: Triangular 2-cells in the pants complex.

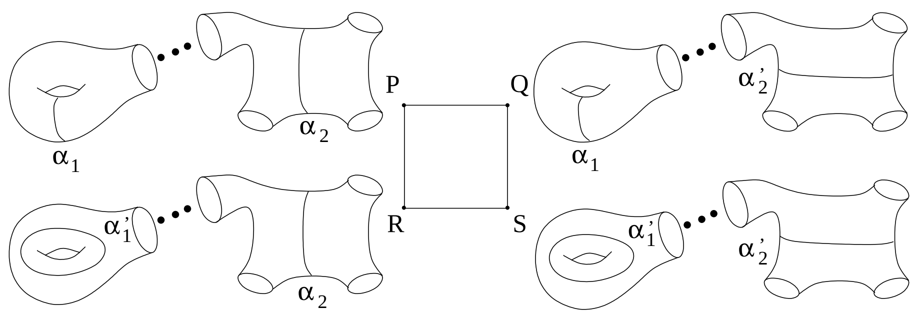

Figure 4: Square 2-cells in the pants complex (moves can be of either type).

3 Proof of Theorem 1

Let $S = \Sigma_{g,b}$ ($\neq \Sigma_{0,3}$) be an orientable surface with $\chi(S) < 0$, and let $n = 3g - 3 + b$ be the number of curves in a pants decomposition of S.

Outline. The idea for the proof of Theorem 1 is to construct an isomorphism ϕ so that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Mod}(S) & = & \mathrm{Mod}(S) & = & \mathrm{Mod}(S) \\ \theta\downarrow & & \downarrow & & \downarrow\eta \\ \mathrm{Aut}\,\mathrm{C_P(S)} & \xrightarrow{\iota} & \mathrm{Aut}\,\mathrm{C^1_P(S)} & \xrightarrow{\phi} & \mathrm{Aut}\,\mathrm{C}(S) \end{array}$$

The surjectivity of θ (Theorem 1) then follows from the surjectivity of η (Theorem 3) and the injectivity of the natural map ι. Note that ι must also be surjective, so ι is an isomorphism (Theorem 2). The description of $\ker(\theta)$ (Theorem 1) also follows from Theorem 3.

For the case $S = \Sigma_{1,2}$, a separate argument will be needed to show that $\mathrm{image}(\phi) \subset \mathrm{image}(\eta) = \mathrm{Aut}^*\,\mathrm{C}(S)$ (Section 5).

In order to construct ϕ, we will develop the following natural surjective map:

$$\{\text{abstract marked Farey graphs in } \mathrm{C^1_P(S)}\} \twoheadrightarrow \{\text{vertices of } \mathrm{C}(S)\}$$

Here, an *abstract Farey graph* is any subgraph of $\mathrm{C^1_P(S)}$ abstractly isomorphic to a Farey graph; and a *marked* graph (F, X) is a graph F with a distinguished vertex X.

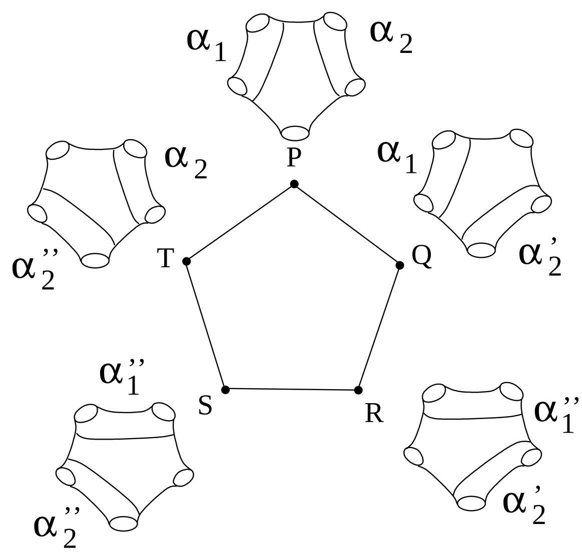

Figure 5: Pentagonal 2-cells in the pants complex.

3.1 Definition of ϕ

To begin, we completely characterize triangles in $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$, since they are the building blocks of Farey graphs. By *triangle*, we mean a subgraph of $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ which is a complete graph on 3 vertices. The following lemma implies that the three pants decompositions associated to the vertices of any triangle are of the form $\{\star, \alpha_2, \ldots, \alpha_n\}$.

Lemma 1. *Every triangle in* $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ *is the boundary of a triangular 2-cell of* $\mathrm{C}_\mathrm{P}(\mathrm{S})$.

Proof. Suppose P, Q, and R are the vertices of a triangle in $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$. Since the pants decompositions associated to P and Q differ by an elementary move, they must differ by exactly one curve. Say P and Q are associated to $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\alpha_1', \alpha_2, \ldots, \alpha_n\}$. A pants decomposition associated to R must have exactly $n-1$ curves in common with each of these, so it must in fact contain $\alpha_2, \ldots, \alpha_n$ (otherwise, it would have to contain α_1 and α_1', which can't happen since $\mathrm{i}(\alpha_1, \alpha_1') > 0$. Hence R is associated to $\{\alpha_1'', \alpha_2, \ldots, \alpha_n\}$ for some α_1''.

The curves α_1, α_1', and α_1'' lie on a common subsurface S' (either a $\Sigma_{1,1}$ or a $\Sigma_{0,4}$) in the complement of $\{\alpha_2, \ldots, \alpha_n\}$. Thus the triangle PQR can

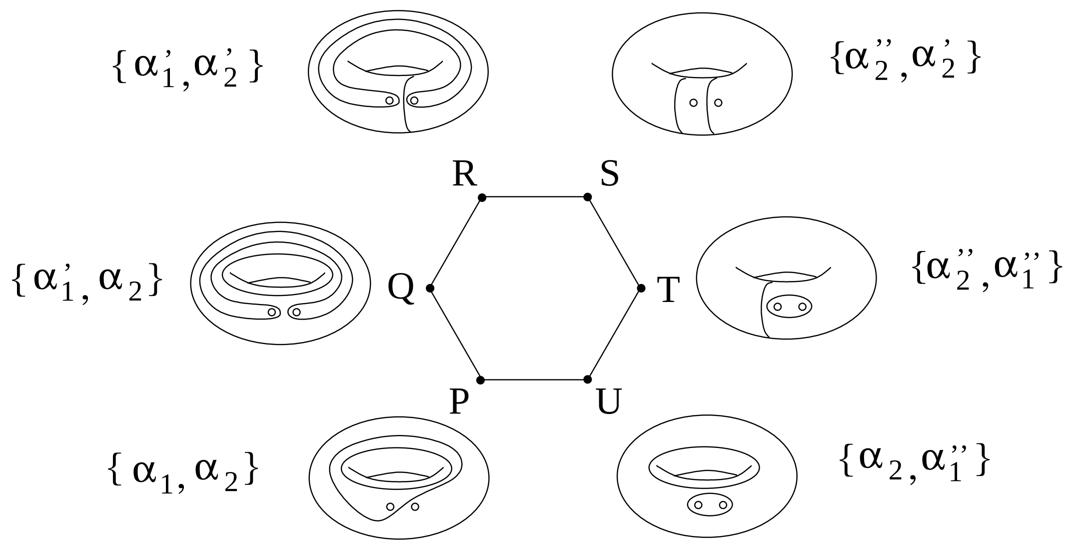

Figure 6: Hexagonal 2-cells in the pants complex.

be thought of as one of the triangles in $\mathrm{C}^1_{\mathrm{P}}(S')$, which correspond exactly to triangular 2-cells.

□

By piecing triangles together, we can characterize Farey graphs in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$:

Lemma 2. *There is a natural surjective map from the set of marked abstract Farey graphs in* $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ *to the set of vertices of* $\mathrm{C}(S)$.

Proof. Let (F, X) be a marked abstract Farey graph in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$. Since F is chain-connected (any two triangles can be connected by a sequence of triangles so that consecutive triangles share an edge), and since the pants decompositions associated to any triangle are of the form $\{\alpha_1^1, \alpha_2, \dots, \alpha_n\}$, $\{\alpha_1^2, \alpha_2, \dots, \alpha_n\}$, $\{\alpha_1^3, \alpha_2, \dots, \alpha_n\}$, it follows that there are $n-1$ *fixed curves* $(\alpha_2, \dots, \alpha_n)$ and one *moving curve* (the α_1^i's) in the pants decompositions associated to the vertices of F. The vertex X distinguishes one of the α_1^i. Hence, there is a unique vertex $v_{(F,X)}$ of $\mathrm{C}(S)$ corresponding to (F, X).

To show that the map defined above is surjective, we will now find a marked Farey graph corresponding to a given vertex v of $\mathrm{C}(S)$. If v is associated to the curve α_1 on S, then choose a vertex X of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ associated to some pants decomposition $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ containing α_1. Since the complement of $\alpha_2, \dots, \alpha_n$ in S is a number of pants and either a $\Sigma_{0,4}$ or $\Sigma_{1,1}$, the set of pants decompositions of the form $\{\star, \alpha_2, \ldots, \alpha_n\}$ corresponds to a Farey graph $F_v \cong \mathrm{C}^1_{\mathrm{P}}(\Sigma_{0,4}) \cong \mathrm{C}^1_{\mathrm{P}}(\Sigma_{1,1})$ in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$, (F_v, X) corresponds to v. Note that $v_{(F_v,X)} = v$.

□

By a slight abuse of notation, we say that $v_{(F,X)}$ *corresponds to* (F, X), and vice versa (even though the map is not bijective).

Now that we have the correspondence of Lemma 2, we are ready to define the map ϕ.

Definition of ϕ. Let $A \in \mathrm{Aut}\,\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$. We define $\phi(A) : \mathrm{C}^{(0)}(S) \to \mathrm{C}^{(0)}(S)$ (and hence ϕ) by way of saying what $\phi(A)$ does to each vertex of $\mathrm{C}(S)$:

If v is a vertex of $\mathrm{C}(S)$ and (F_v, X) is some marked Farey graph in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ corresponding to v (recall that there is a choice here), then $\phi(A)(v)$ is defined to be $v_{(A(F_v),A(X))}$, the unique vertex of $\mathrm{C}(S)$ corresponding to the marked Farey graph $(A(F_v), A(X))$.

3.2 ϕ is well-defined

In order to show that ϕ is well-defined, we will require two new concepts: alternating sequences and small circuits.

Circuits. A *circuit* is a subgraph of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ homeomorphic to a circle. We define *triangles*, *squares*, *pentagons*, and *hexagons* to be circuits with the appropriate number of vertices.

For the definition of alternating below, note that an edge of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ lies in a unique Farey graph in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$. This fact follows from the proof of Lemma 2.

Alternating sequences. A sequence of consecutive vertices $P_1 P_2 \ldots P_m$ in a circuit is called *alternating* if the unique Farey graph containing the edge $P_{i-1}P_i$ is not the same as the unique Farey graph containing $P_i P_{i+1}$ for $1 < i < m$. By Lemma 2, an equivalent characterization of alternating is that the pants decompositions associated to P_{i-1}, P_1, and P_{i+1} have no set of $n-1$ curves in common.

A useful working definition of an alternating sequence of vertices PQR is that if the elementary move corresponding to PQ is $\star \to \alpha$, then the elementary move corresponding to QR is not of the form $\alpha \to \star$. A circuit in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ with the property that any three consecutive vertices make up an alternating sequence is called an *alternating circuit*.

Since alternating sequences are defined in terms of the combinatorics of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$, we have:

Lemma 3. *Automorphisms of* $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ *preserve alternating sequences.*

Small circuits. A *small circuit* in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ is a circuit with no more than six edges. We give a partial characterization which will be used to show that the map ϕ is well-defined, and to prove the results in Section 4.

A *2-curve small circuit* is a circuit with the property that the pants decompositions associated to its vertices all contain the same set of $n-2$ curves; i.e. they are of the form $\{\star, \star, \alpha_3, \ldots, \alpha_n\}$. For convenience, small circuits which are subgraphs of Farey graphs are also called 2-curve small circuits (by Lemma 2 they are really "1-curve small circuits").

Lemma 4. *Any small circuit which is not a 2-curve small circuit is an alternating hexagon.*

Proof. Let $\mathcal{L}$ be the small circuit, and say one of its vertices is associated to the pants decomposition $p = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Since $\mathcal{L}$ is not a 2-curve small circuit, then (after picking a direction around $\mathcal{L}$) there must be three edges of $\mathcal{L}$ corresponding to moves of the form $\alpha_i \to \star$, $\alpha_j \to \star$, and $\alpha_k \to \star$ with $1 \leq i, j, k \leq n$ distinct. Without loss of generality, we have:

$$\alpha_1 \overset{m_1}{\to} \alpha_1' \quad \alpha_2 \overset{m_2}{\to} \alpha_2' \quad \alpha_3 \overset{m_3}{\to} \alpha_3'$$

In order to make $\mathcal{L}$ a closed loop, there must also be three edges of the form:

$$\star \overset{m_1'}{\to} \alpha_1 \quad \star \overset{m_2'}{\to} \alpha_2 \quad \star \overset{m_3'}{\to} \alpha_3$$

Note that these 6 moves are distinct. In other words, α_i' is not α_j for any j. This is true because $\mathrm{i}(\alpha_i', \alpha_i) > 0$ (they differ by an elementary move), while $\mathrm{i}(\alpha_i, \alpha_j) = 0$ (they both appear in the pants decomposition p). Since $\mathcal{L}$ is a small circuit, there are no further edges.

Further, we claim that each m_i' is given by $\alpha_i' \to \alpha_i$. If, on the contrary, we have for example that m_1' is $\alpha_2' \to \alpha_1$, then $\mathrm{i}(\alpha_2', \alpha_1) > 0$ and $\mathrm{i}(\alpha_2', \alpha_2) > 0$ (it differs from both by an elementary move). Thus, among the set of curves

$\{\alpha_1, \alpha_1', \alpha_2, \alpha_3, \alpha_3'\}$, α_2' can only appear in a pants decomposition with α_1', α_3, α_3'. So the only possibilities for m_1' are:

$$\{\alpha_2', \alpha_1', \alpha_3\} \xrightarrow{m_1'} \{\alpha_1, \alpha_1', \alpha_3\}$$
$$\{\alpha_2', \alpha_1', \alpha_3'\} \xrightarrow{m_1'} \{\alpha_1, \alpha_1', \alpha_3'\}$$
$$\{\alpha_2', \alpha_3', \alpha_3\} \xrightarrow{m_1'} \{\alpha_1, \alpha_3', \alpha_3\}$$

which are all impossibilities since they each contain a pair α_i and α_i', but $\mathrm{i}(\alpha_i, \alpha_i') > 0$. (Note that we ignore the curves $\alpha_4, \ldots, \alpha_n$, as they must appear in each pants decomposition.)

Now $\mathcal{L}$ must be alternating, because otherwise it has a pair of consecutive edges corresponding to m_i and m_i'.

□

An immediate consequence of Lemmas 3 and 4 is the following:

Lemma 5. *If $A \in \mathrm{Aut}\,\mathrm{C}^1_\mathrm{P}(\mathrm{S})$, and $\mathcal{L}$ is a small circuit which is not an alternating hexagon, then $A(\mathcal{L})$ is a 2-curve small circuit.*

Lemma 6. *The map $\phi : \mathrm{Aut}\,\mathrm{C}^1_\mathrm{P}(\mathrm{S}) \longrightarrow \mathrm{Aut}\,\mathrm{C}(S)$ is well-defined.*

Proof. Let v be a vertex in $\mathrm{C}(S)$ associated to the curve α_1 on S. We need to show that if p and p' are two pants decompositions which give rise to two marked Farey graphs (F_v, X) and (F_v', X') corresponding to v, then the two vertices of $\mathrm{C}(S)$ corresponding to $(A(F_v), A(X))$ and $(A(F_v'), A(X'))$ are the same.

Actually, by the connectedness of $\mathrm{C}^1_\mathrm{P}(S - \alpha_1)$, we only need to treat the case when p and p' differ by an elementary move, say $\alpha_2 \to \alpha_2'$.

The idea is as follows: we will find a 2-curve small circuit $\mathcal{L}$ (not an alternating hexagon) such that four of its vertices make up an alternating sequence $WXX'Y$, with $W, X \in (F_v, X)$ and $X', Y \in (F_v', X')$ (see Figure 7).

Suppose $(A(F_v), A(X))$ corresponds to a vertex of $\mathrm{C}(S)$ representing to the curve β_1 and that $A(X)$ is associated to the pants decomposition $\{\beta_1, \ldots, \beta_n\}$. We show that $(A(F_v'), A(X'))$ also corresponds to the vertex representing to β_1:

Since the edge $A(W)A(X)$ is in $(A(F_v), A(X))$, it corresponds to a move of the form $\star \to \beta_1$. As $A(W)A(X)A(X')$ is alternating (Lemma 3), $A(X)A(X')$

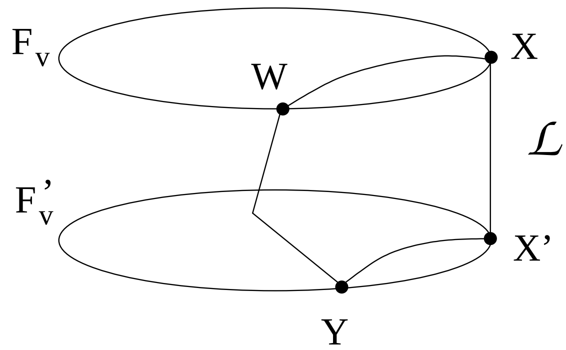

Figure 7: The 2-curve small circuit $\mathcal{L}$.

corresponds to a move $\beta_2 \to \star$. Now, combining the facts that $A(X)A(X')A(Y)$ is alternating (Lemma 3) and that $A(\mathcal{L})$ is a 2-curve small circuit (Lemma 5), it follows that the move corresponding to $A(X')A(Y)$ is of the form $\beta_1 \to \star$, and so the vertex of $\mathrm{C}(S)$ corresponding to $(A(F_v'), A(X'))$ represents β_1.

Finding the 2-curve small circuit. To show that ϕ is well-defined, the only thing left is to show that there always exists a 2-curve small circuit $\mathcal{L}$ as above. There are four cases to consider:

1. α_1, α_2 lie on disjoint subsurfaces
2. α_1, α_2 lie on a $\Sigma_{0,5}$
3. α_1, α_2 lie on a $\Sigma_{1,2}$, and one of α_1, α_2, or α_2' is separating
4. α_1, α_2 lie on a $\Sigma_{1,2}$, and α_1, α_2, and α_2' are nonseparating

Note that a curve is separating on $\Sigma_{1,2} \subset S$ if an only if it is separating on S.

Case 1: Let $\mathcal{L}$ be the boundary of a square 2-cell containing X, X'.

Case 2: Let $\mathcal{L}$ be the boundary of a pentagonal 2-cell containing X, X'.

Case 3: There are only three possibilities for the curves α_1, α_2, and α_2', since a pants decomposition of $\Sigma_{1,2}$ can't have two separating curves, and two separating curves on $\Sigma_{1,2}$ can't differ by an elementary move:

· α_1 is separating, α_2 and α_2' are nonseparating

· α_1 and α_2 are nonseparating, and α_2' is separating

· α_1 is nonseparating, α_2 is separating, and α_2' is nonseparating

Note that the second and third possibilities are equivalent by symmetry.

In any case, choose $\mathcal{L}$ to be the boundary of a hexagonal 2-cell. For the first possibility choose $\mathcal{L}$ so that X and X' correspond to the vertices T and U in Figure 6. For the second possibility, X and X' should correspond to S and T.

Case 4: Since this situation does not occur in any of the circuits bounding 2-cells of $\mathrm{C_P(S)}$, we reduce to Case 3 by showing that any elementary move on a $\Sigma_{1,2}$ of the form $\{\alpha_1, \alpha_2\} \to \{\alpha_1, \alpha_2'\}$ with α_1, α_2, and α_2' all nonseparating can be realized by a pair of elementary moves $\{\alpha_1, \alpha_2\} \to \{\alpha_1, \alpha_2''\} \to \{\alpha_1, \alpha_2'\}$ which fall under Case 3.

Topologically, α_1 and α_2 are as in Figure 8 (the complement of a pair of nonseparating curves on $\Sigma_{1,2}$ is two copies of $\Sigma_{0,3}$, each with one boundary component of the $\Sigma_{1,2}$) . Then, there is one topological possibility for α_2', as α_2' differs from α_2 by an elementary move on $\Sigma_{0,4} = \Sigma_{1,2} - \alpha_1$, and the two boundary components of $\Sigma_{1,2}$ lie on different sides of α_2' (recall α_2' in nonseparating). Thus α_2' is also as in Figure 8. Therefore, we may choose α_2'' as in the same figure.

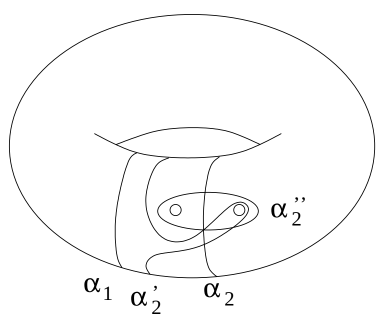

Figure 8: Reduction of Case 4 to Case 3.

□

3.3 ϕ maps into $\mathrm{Aut}\,\mathrm{C}(S)$

Since $\mathrm{C}(S)$ has the property that every set of $k+1$ mutually connected vertices is the 1-skeleton of a k-simplex in $\mathrm{C}(S)$, it follows that $\mathrm{Aut}\,\mathrm{C}(S) \cong$

$\mathrm{Aut}\,\mathrm{C}^1(S)$. Therefore, we only need to check that $\phi(A)$ extends to an automorphism of the 1-skeleton of $\mathrm{C}(S)$, i.e. that $\phi(A)$ takes vertices connected by edges to vertices connected by edges.

Suppose that v and w are vertices of $\mathrm{C}(S)$ associated to curves α and β on S, and let X (a vertex of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$) correspond to some pants decomposition $\{\alpha, \beta, \gamma_3, \dots, \gamma_n\}$. Then let F_v and F_w be the Farey graphs corresponding to the pants decompositions $\{\star, \beta, \gamma_3, \dots, \gamma_n\}$ and $\{\alpha, \star, \gamma_3, \dots, \gamma_n\}$. In this case, (F_v, X) and (F_w, X) are marked Farey graphs corresponding to v and w, and which intersect at one vertex (X). Note that this construction is possible if and only if α and β appear in a common pants decomposition, which is equivalent to the existence of an edge between v and w. Since intersections between Farey graphs are strictly preserved under A, and since $\phi(A)$ is independent of choice of marked Farey graph, it follows that edges of $\mathrm{C}(S)$ are preserved under $\phi(A)$.

3.4 ϕ is an isomorphism

Multiplicativity. Let $A, B \in \mathrm{Aut}\,\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$. We will show that $\phi(AB)v = \phi(A)\phi(B)v$ for any vertex v in $\mathrm{C}(S)$. By definition, $\phi(AB)v$ is the vertex in $\mathrm{C}(S)$ corresponding to $(AB(F_v), AB(X))$, where (F_v, X) is a marked Farey graph in $\mathrm{C}_{\mathrm{P}}(\mathrm{S})$ corresponding to v. On the other hand, $\phi(B)v$ is the vertex w of $\mathrm{C}(S)$ corresponding to $(B(F_v), B(X))$, and $\phi(A)\phi(B)v$ is the vertex of $\mathrm{C}(S)$ corresponding to $(A(F_w), A(Y))$, where (F_w, Y) is some Farey graph corresponding to w. We can choose (F_w, Y) to be $(B(F_v), B(X))$, and so $\phi(A)\phi(B)v$ is the vertex corresponding to $(AB(F_v), AB(X))$, which is the same as $\phi(AB)v$.

Surjectivity. It suffices to show that the diagram at the beginning of this section is commutative. Let $f \in \mathrm{Mod}(S)$, and let v be the vertex of $\mathrm{C}(S)$ associated to a curve α_1 on S. Then $\phi \circ \iota \circ \theta(f)(v) = \phi \circ \theta'(f)(v)$ is the vertex of $\mathrm{C}(S)$ corresponding to $(f(F_v), f(X))$, where (F_v, X) is a marked Farey graph corresponding to v. But if F_v and X correspond to pants decompositions $\{\star, \alpha_2, \dots, \alpha_n\}$ and $\{\alpha_1, \dots, \alpha_n\}$, then $f(F_v)$ and $f(X)$ correspond to $\{\star, f(\alpha_2), \dots, f(\alpha_n)\}$ and $\{f(\alpha_1), \dots, f(\alpha_n)\}$. Thus $(\phi \circ \iota \circ \theta)(f)(v) = \eta(f)(v)$, the vertex of $\mathrm{C}(S)$ representing $f(\alpha_1)$.

Injectivity. Suppose $\phi(A)$ is the identity in $\mathrm{Aut}\,\mathrm{C}(S)$, and let X be the vertex of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ associated to the pants decomposition $\{\alpha_1, \ldots, \alpha_n\}$, where $v_1, \ldots, v_n$ are the vertices of $\mathrm{C}(S)$ associated to the α_i. Denote by F_{v_i} the Farey graph corresponding to the pants decompositions:

$$\{\alpha_1, \ldots, \alpha_{i-1}, \star, \alpha_{i+1}, \ldots, \alpha_n\}$$

The (F_{v_i}, X) correspond to the v_i, and the F_{v_i} all intersect at the vertex X in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$.

Since $A(F_{v_1}), \ldots, A(F_{v_n})$ must be marked Farey graphs corresponding to $\{A(v_i)\} = \{v_i\}$ for $1 \leq i \leq n$ and intersecting at one vertex, it follows that their common intersection is X. Thus $A(X) = X$, and so A is the identity in $\mathrm{Aut}\,\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$.

4 Proof of Theorem 2

Our goal is now to show that it is possible to recognize the 2-cells of $\mathrm{C}_{\mathrm{P}}(\mathrm{S})$ simply by considering the combinatorics of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$, and without reference to the surface S. This will give a complete proof of Theorem 2, and will help prove Theorem 1 for the case $S = \Sigma_{1,2}$.

Again, $S = \Sigma_{g,b}$ is an orientable surface with $\chi(S) < 0$, and $n = 3g-3+b$ is the number of curves in a pants decomposition for S. Recall that a circuit is a subgraph of $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ homeomorphic to a circle, and that triangles, squares, pentagons, and hexagons are circuits with the usual number of vertices.

Triangles. Lemma 1 says that every triangle in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ is the boundary of a triangular 2-cell in $\mathrm{C}_{\mathrm{P}}(\mathrm{S})$.

Squares. We will show that square 2-cells in $\mathrm{C}_{\mathrm{P}}(\mathrm{S})$ can equivalently be characterized as alternating squares in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$.

Lemma 7. *Every alternating square in* $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ *is the boundary of a square 2-cell in* $\mathrm{C}_{\mathrm{P}}(\mathrm{S})$.

Proof. Let P, Q, R and S be the (ordered) vertices of an alternating square in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$. By Lemma 4, $PQRS$ is a 2-curve small circuit, so the associated pants decompositions all contain a common set of $n-2$ curves, say $\alpha_3, \ldots, \alpha_n$ (which we take to be implicit below).

Using the fact that $PQRS$ is alternating, we have that the pattern of curves is as follows:

$$\overset{P}{\{\alpha_1, \alpha_2\}} \rightarrow \overset{Q}{\{\alpha_1, \alpha_2'\}} \rightarrow \overset{R}{\{\alpha_1', \alpha_2'\}} \rightarrow \overset{S}{\{\alpha_1', \alpha_2\}}$$

Note that α_2 must be in the pants decomposition for S since SPQ is alternating.

It remains to show that α_1 and α_2 lie on different subsurfaces in the complement of $\alpha_3, \ldots, \alpha_n$, as per the definition of square 2-cells. Assume that α_1 and α_2 lie on a connected subsurface $S' \subset S - \{\alpha_3, \ldots, \alpha_n\}$. Since S' has a pants decomposition of two curves ($\{\alpha_1, \alpha_2\}$), S' is either $\Sigma_{0,5}$ or $\Sigma_{1,2}$.

There are four topological possibilities for α_1, α_2, and α_2'—on the $\Sigma_{0,5}$ there is only one possibility, and on the $\Sigma_{1,2}$ there are two cases (Cases 3 and 4 of Lemma 6). It is clear that in each of the cases, there is no curve α_1' which intersects α_1 minimally and is disjoint from α_2 and α_2'. This is a contradiction, so α_1 and α_2 lie on different subsurfaces.

□

Pentagons. We now prove that pentagonal 2-cells in $\mathrm{C_P(S)}$ can be characterized as alternating pentagons in $\mathrm{C^1_P(S)}$.

Lemma 8. *Every alternating pentagon in* $\mathrm{C^1_P(S)}$ *is the boundary of a pentagonal 2-cell in* $\mathrm{C_P(S)}$.

Proof. Let P, Q, R, S, and T be the (ordered) vertices of an alternating pentagon in $\mathrm{C^1_P(S)}$. By Lemma 4, the pants decompositions associated to these vertices all have $n-2$ curves in common, say $\alpha_3, \ldots, \alpha_n$ (these curves are implicit in the pants decompositions below).

Because $PQRST$ is an alternating sequence, the pattern of curves in the pants decompositions for those vertices is:

$$\overset{P}{\{\alpha_1, \alpha_2\}} \rightarrow \overset{Q}{\{\alpha_1, \alpha_2'\}} \rightarrow \overset{R}{\{\alpha_1', \alpha_2'\}} \rightarrow \overset{S}{\{\alpha_1', \alpha_2''\}} \rightarrow \overset{T}{\{\alpha_2, \alpha_2''\}}$$

Note that α_2 must be in the pants decomposition for T, since QPT is an alternating sequence.

Since curves in a pants decomposition are disjoint, we have that for the sequence $\alpha_1\ \alpha_1'\ \alpha_2\ \alpha_2'\ \alpha_2''\ \alpha_1$, curves differ by an elementary move if they are adjacent in the sequence, and are disjoint otherwise. Our goal now is to show that these curves must be the ones in Figure 5.

Firstly, α_1 and α_2 do not lie on disjoint subsurfaces (α_1' has nontrivial intersection with both of them). Therefore, α_1 and α_2 must lie on a $\Sigma_{0,5}$ or $\Sigma_{1,2}$ in $S - \{\alpha_3, \ldots, \alpha_n\}$.

In the first case, the curves α_2, α_2', α_2'', α_1, and α_1' must be topologically as in the definition of pentagonal 2-cells in the pants complex (Figure 5). This is because $\alpha_1, \alpha_1'', \alpha_2, \alpha_2'$ is a *regular chain* (curves intersect twice if consecutive, are disjoint otherwise), and regular chains are topologically unique. Then α_2'' is the unique curve intersecting α_1 and α_2' each twice in the complement of the other curves.

We now show that the second case cannot happen, i.e. that there is no such sequence curves on $\Sigma_{1,2}$.

Assume that on $\Sigma_{1,2}$ there is a sequence $\alpha\ \beta\ \gamma\ \delta\ \epsilon\ \alpha$ with the property that consecutive curves intersect minimally and other pairs are disjoint. In such a sequence, there can be at most one curve which is separating on $\Sigma_{1,2}$, since two separating curves on $\Sigma_{1,2}$ intersect at least four times. We will consider two cases:

1. the sequence has a separating curve

2. the sequence has no separating curve

Below, we call a nonseparating curve on $\Sigma_{1,2}$ of *(p, q)-type* if it is a (p, q) curve on the torus obtained by forgetting the two punctures.

Case 1. Suppose there is a separating curve in the sequence, say α. It follows that the other curves in the sequence are nonseparating and that α separates $\Sigma_{1,2}$ into a $\Sigma_{0,3}$ and a $\Sigma_{1,1}$. Since γ and δ both have trivial intersection with α and have minimal intersection with each other in the complement of α, they must lie on the $\Sigma_{1,1}$ and intersect once; say γ and δ are of $(1, 0)$ and $(0, 1)$-type, respectively. As β and ϵ are both nonseparating curves on $\Sigma_{1,2}$, and $\mathrm{i}(\beta, \delta) = 0$, β must be of $(0, 1)$-type; likewise ϵ must be of $(1, 0)$-type. This implies that $\mathrm{i}(\beta, \epsilon) > 0$ (curves of different type intersect), so we have a contradiction.

Case 2. Now suppose that all the curves in the sequence are nonseparating. Since all elementary moves involving three nonseparating curves are topologically equivalent, we can assume without loss of generality that α, γ, and δ are the curves in Figure 10.

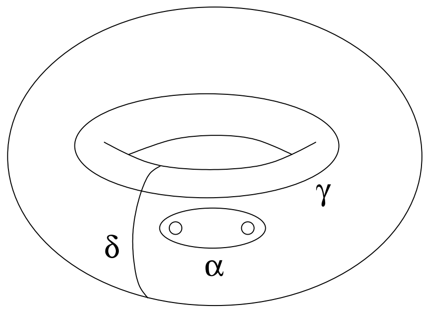

Figure 9: Case 1. The configuration for α separating.

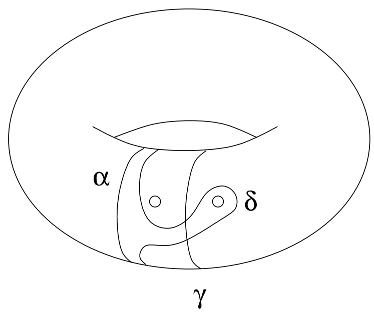

Figure 10: Case 2. The configuration for α nonseparating.

In order to have $\mathrm{i}(\beta, \alpha) = 2$ and $\mathrm{i}(\beta, \delta) = 0$, we must have that when S is cut along α and δ, the two components of β are essential arcs on the two $\Sigma_{0,3}$ components of $S - (\alpha \cup \delta)$. However, it is easy to see that on each of these components, any essential arc with endpoints on α must intersect (a piece of) γ at least twice (in an essential way). Thus $\mathrm{i}(\beta, \gamma) \geq 4$, a contradiction. $\square$

Hexagons. An *almost-alternating hexagon* in $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ is a hexagon with an alternating sequence of 6 vertices, and a sequence of 3 vertices which lie on a square in some Farey graph (and do not lie in a common triangle). Note that the boundary of a hexagonal 2-cell is an almost-alternating hexagon.

Lemma 9. *Every almost-alternating hexagon in* $\mathrm{C}^1_{\mathrm{P}}(\mathrm{S})$ *is the boundary of a hexagonal 2-cell in* $\mathrm{C}_{\mathrm{P}}(\mathrm{S})$.

Proof. Let P, Q, R, S, T, and U be (ordered) vertices of an almost-alternating hexagon, where UPQ lie in a common Farey graph. Then the alternating sequence must be $PQRSTU$.

Since an almost-alternating hexagon is not an alternating hexagon, then by Lemma 4, the pants decompositions associated to the vertices all have a set of $n-2$ curves in common, say $\alpha_3, \ldots, \alpha_n$ (again, we ignore these curves below).

As $PQRSTU$ is alternating, we get the following pattern of curves for the associated pants decompositions:

$$\overset{P}{\{\alpha_1, \alpha_2\}} \to \overset{Q}{\{\alpha_1', \alpha_2\}} \to \overset{R}{\{\alpha_1', \alpha_2'\}} \to \overset{S}{\{\alpha_2'', \alpha_2'\}} \to \overset{T}{\{\alpha_2'', \alpha_1''\}} \to \overset{U}{\{\alpha_2, \alpha_1''\}}$$

Note that the pants decomposition for U must have the curve α_2 since UPQ lies in a Farey graph.

The goal now is to show that the curves must be as the curves corresponding to a hexagonal 2-cell (Figure 6). We take the following steps:

1. α_1, α_2 do not lie on disjoint subsurfaces
2. α_1, α_2 do not lie on a $\Sigma_{0,5}$ (and hence they lie on a $\Sigma_{1,2}$)
3. α_2 is nonseparating on the $\Sigma_{1,2}$
4. α_1 is nonseparating on the $\Sigma_{1,2}$
5. α_1' (and hence α_1'') is separating on the $\Sigma_{1,2}$
6. The choices of $\alpha_1, \alpha_1', \alpha_1'', \alpha_2, \alpha_2', \alpha_2''$ on $\Sigma_{1,2}$ are topologically unique

Step 1. The curves α_1 and α_2 cannot lie on disjoint subsurfaces, since there is a chain of curves connecting them which are disjoint from $\alpha_3, \ldots, \alpha_n$:

$$\alpha_1 \quad \to \quad \alpha_1'' \quad \to \quad \alpha_2' \quad \to \quad \alpha_2$$

Step 2. Assume α_1 and α_2 lie on a $\Sigma_{0,5}$ in the complement of $\alpha_3, \ldots, \alpha_n$. Since P, Q, and U are the vertices of a square in a Farey graph, and α_2 appears in all three associated pants decompositions, the aforementioned Farey graph is $\mathrm{C}_{\mathrm{P}}^1(\Sigma_{0,4})$ where $\Sigma_{0,4} = \Sigma_{0,5} - \alpha_2$. Then since any two squares in $\mathrm{C}_{\mathrm{P}}^1(\Sigma_{0,4})$ are topologically equivalent, it follows that the pants decompositions associated to P, Q, and U are as in Figure 11. (in the figure, a boundary component is represented by a puncture, and a curve is represented by an

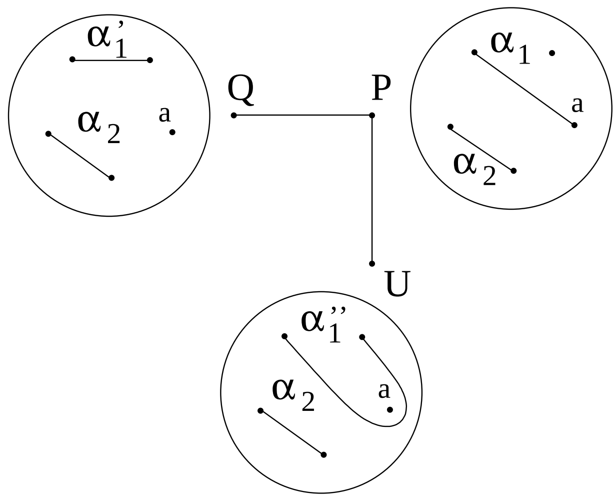

Figure 11: Step 2. The configuration for α_1 and α_2 on a $\Sigma_{0,5}$.

arc; to recover the curve, take a boundary of a small neighborhood of the arc).

The edges QR and UT (note directions) correspond to the elementary moves $\alpha_2 \to \alpha_2'$ and $\alpha_2 \to \alpha_2''$. Since all elementary moves on $\Sigma_{0,5}$ are topologically equivalent, both α_2' and α_2'' must be represented by arcs which have an endpoint at the puncture a (see Figure 11). This implies that $\mathrm{i}(\alpha_2', \alpha_2'') > 0$. This is a contradiction, since α_2 and α_2'' appear together in the pants decomposition associated to S.

Step 3. If we assume α_2 is separating on $\Sigma_{1,2}$, then it separates $\Sigma_{1,2}$ into a $\Sigma_{0,3}$ and a $\Sigma_{1,1}$. Then $\{\alpha_1\}$, $\{\alpha_1'\}$, and $\{\alpha_1''\}$ are pants decompositions of the $\Sigma_{1,1}$, whose associated vertices in $\mathrm{C}^1_{\mathrm{P}}(\Sigma_{1,1})$ lie on a square (but not a triangle). Topologically, then, α_1'', α_1, and α_1' are the $(1,0)$, $(2,1)$, and $(1,1)$ curves on the $\Sigma_{1,1}$, so they are of the same three types on the $\Sigma_{1,2}$ (see Lemma 8). Since α_2 is separating, α_2' and α_2'' must both be nonseparating (they must have intersection number with α_2 no more than 2). Also, because $\mathrm{i}(\alpha_2', \alpha_1') = 0$ (the two form a pants decomposition), it follows that α_2' must be of type $(1,1)$. Likewise, α_2'' must be of type $(1,0)$. However, since α_2' and α_2'' must make up a pants decomposition of $\Sigma_{1,2}$, they must have trivial intersection; but curves of different type always have nontrivial intersection, a contradiction.

Step 4. Assume that α_1 is separating on the $\Sigma_{1,2}$. Since all pants decompositions containing a separating curve are topologically equivalent, we assume that α_1 and α_2 are as in Figure 12.

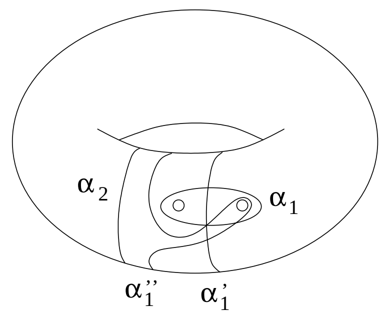

Figure 12: Step 4. The configuration for α_1 separating.

Now, there is a unique choice for α_1', as $\mathrm{i}(\alpha_1', \alpha_1) = 2$ and $\mathrm{i}(\alpha_1', \alpha_2) = 0$. Since α_1'' must be part of a square (but not a triangle) with the vertices of $\mathrm{C}^1_{\mathrm{P}}(\Sigma_{0,4})$ associated to α_1 and α_1', the choice of α_1'' is topologically unique.

The curve α_2' must have trivial intersection with α_1', and must differ from both α_1'' and α_2 by elementary moves. By the same argument as in Case 2 of Lemma 8, there is no such α_2', so we have a contradiction.

Step 5. Without loss of generality, α_1 and α_2 are the curves in Figure 13. If we assume that α_1' is nonseparating on the $\Sigma_{1,2}$, then since $\{\alpha_1, \alpha_2\} \to \{\alpha_1', \alpha_2\}$ is an elementary move, the choice for α_1' is topologically unique.

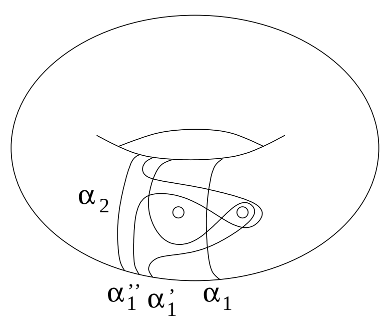

Figure 13: Step 5. The configuration for for α_1' nonseparating.

In order for α_1', α_1, and α_1'' to lie on a square (but not a triangle) in the Farey graph $\mathrm{C}^1_{\mathrm{P}}(\Sigma_{0,4}) = \mathrm{C}^1_{\mathrm{P}}(S - \alpha_2)$, and for α_1 to differ from α_1' and α_1'' by elementary moves, we must have $\mathrm{i}(\alpha_1, \alpha_1') = \mathrm{i}(\alpha_1, \alpha_1'') = 2$, and $\mathrm{i}(\alpha_1', \alpha_1'') = 4$. The only such configuration is shown in Figure 13.

Again, we must have $\mathrm{i}(\alpha_2', \alpha_1') = 0$, and it α_2' must differ from α_1'' and α_2 by elementary moves. By the same argument as in Case 2 of Lemma 8, there is no such α_2', and we have a contradiction.

Step 6. Starting with α_1 and α_2 (both nonseparating), we can assume that they are as in Figure 14. As above, $\mathrm{i}(\alpha_1, \alpha_1') = \mathrm{i}(\alpha_1, \alpha_1'') = 2$, $\mathrm{i}(\alpha_1', \alpha_1'') = 4$, and $\mathrm{i}(\alpha_1', \alpha_2) = \mathrm{i}(\alpha_1'', \alpha_2) = 0$. The only such topological configuration is shown in Figure 14.

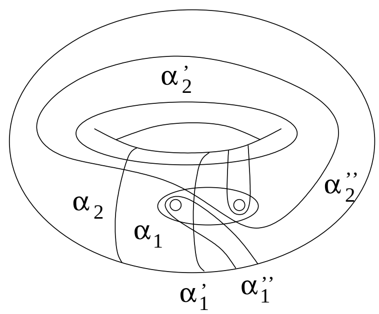

Figure 14: Step 6. The unique configuration for almost-alternating hexagons.

Finally, there are unique choices for α_2' and α_2'', as α_2' must have trivial intersection with α_1' and must have minimal intersection with α_2 and α_1'', while α_2'' must have trivial intersection with α_1'' and must have minimal intersection with α_2 and α_1'.

□

Proof of Theorem 2. Lemmas 1, 7, 8, and 9 say that each kind of 2-cell in $\mathrm{C}_\mathrm{P}^1(\mathrm{S})$ can be recognized completely in terms of the combinatorics of $\mathrm{C}_\mathrm{P}^1(\mathrm{S})$. Therefore, $\mathrm{Aut}\,\mathrm{C}_\mathrm{P}^1(\mathrm{S})$ and $\mathrm{Aut}\,\mathrm{C}_\mathrm{P}(\mathrm{S})$ are canonically isomorphic.

5 Theorem 1 for $S = \Sigma_{1,2}$

As stated in Theorem 3, the exceptional feature of $\Sigma_{1,2}$ is that the natural map $\eta : \mathrm{Mod}(\Sigma_{1,2}) \longrightarrow \mathrm{Aut}\,\mathrm{C}(\Sigma_{1,2})$ is not a surjection. More precisely, the image of η is $\mathrm{Aut}^\star\,\mathrm{C}(\Sigma_{1,2})$, the subgroup of $\mathrm{Aut}\,\mathrm{C}(\Sigma_{1,2})$ consisting of elements which preserve the set of vertices associated to nonseparating curves on $\Sigma_{1,2}$.

Therefore, the only added complication is to show that the image of ϕ (as defined in Section 3.1) lies in $\mathrm{Aut}^\star \mathrm{C}(\Sigma_{1,2})$.

Let v be a vertex of $\mathrm{C}(\Sigma_{1,2})$ representing a nonseparating curve α, and let X in $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ represent $\{\alpha, \beta\}$, a pants decomposition with β nonseparating. This gives rise to a marked Farey graph (F_v, X) corresponding to v. Note that there is a hexagonal 2-cell containing X with the property that X corresponds to the vertex P in Figure 6. The vertex P is distinguished as the middle vertex of the non-alternating sequence in an almost-alternating hexagon. This construction is only possible for α nonseparating. Since almost-alternating hexagons and non-alternating sequences are preserved by automorphisms of $\mathrm{C}^1_\mathrm{P}(\mathrm{S})$ (Lemmas 3 and 9), and since ϕ is independent of choice of marked Farey graphs, we are done.

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