DEHN TWISTS HAVE ROOTS

DAN MARGALIT AND SAUL SCHLEIMER

Let S_g denote a closed, connected, orientable surface of genus g, and let $\mathrm{Mod}(S_g)$ denote its mapping class group, that is, the group of homotopy classes of orientation preserving homeomorphisms of S_g.

Fact. If $g \geq 2$, then every Dehn twist in $\mathrm{Mod}(S_g)$ has a nontrivial root.

It follows from the classification of elements in $\mathrm{Mod}(S_1) \cong \mathrm{SL}(2, \mathbb{Z})$ that Dehn twists are primitive in the mapping class group of the torus.

For Dehn twists about separating curves, the fact is well-known: if c is a separating curve then a square root of the Dehn twist T_c is obtained by twisting one side of c through an angle of π. In the case of nonseparating curves, the issue is more subtle. We give two equivalent constructions of roots below.

Geometric construction. Fix $g \geq 2$. Let P be a regular $(4g-2)$-gon. Glue opposite sides to obtain a surface $T \cong S_{g-1}$. The rotation of P about its center through angle $2\pi g/(2g-1)$ induces a periodic map f of T. Notice that f fixes the points $x, y \in T$ that are the images of the vertices of P. Let T' be the surface obtained from T by removing small open disks centered at x and y. Define $f' = f|T'$.

Let A and B be annular neighborhoods of the boundary components of T'. Modify f' by an isotopy supported in $A \cup B$ so that

- $f'|\partial T'$ is the identity,
- $f'|A$ is a $g/(2g-1)$–left Dehn twist, and
- $f'|B$ is a $(g-1)/(2g-1)$–right Dehn twist.

Identify the two components of $\partial T'$ to obtain a surface $S \cong S_g$ and let $h : S \to S$ be the induced map. Then h^{2g-1} is a left Dehn twist along the gluing curve, which is nonseparating.

Algebraic construction. Let $c_1, \ldots, c_k$ be curves in S_g where c_i intersects c_{i+1} once for each i, and all other pairs of curves are disjoint. If k is odd, then a regular neighborhood of $\cup c_i$ has two boundary components, say, d_1 and d_2, and we have a relation in $\mathrm{Mod}(S_g)$ as follows:

$$(T_{c_1}^2 T_{c_2} \cdots T_{c_k})^k = T_{d_1} T_{d_2}.$$

This relation comes from the Artin group of type B_n, in particular, the factorization of the central element in terms of standard generators; it also follows from the D_{2p} case of [2, Proposition 2.12(i)]. In the case $k = 2g-1$, the curves d_1 and d_2 are isotopic nonseparating curves; call this isotopy class d. Using the fact that T_d commutes with each T_{c_i}, we see that

$$[(T_{c_1}^2 T_{c_2} \cdots T_{c_{2g-1}})^{1-g} T_d]^{2g-1} = T_d.$$

Date: October 30, 2008.

Roots of half-twists. Let $S_{0,2g+2}$ be the sphere with $2g+2$ punctures (or cone points) and let d be a curve in $S_{0,2g+2}$ with 2 punctures on one side and $2g$ on the other. On the side of d with 2 punctures, we perform a left half-twist, and on the other side we perform a $(g-1)/(2g-1)$–right Dehn twist by arranging the punctures so that one puncture is in the middle, and the other punctures rotate around this central puncture. The $(2g-1)^{\text{st}}$ power of the composition is a left half-twist about d. Thus, we have roots of half-twists in $\mathrm{Mod}(S_{0,2g+2})$ for $g \geq 2$. There is a 2-fold orbifold covering $S_g \to S_{0,2g+2}$ where the relation from our algebraic construction above descends to this relation in $\mathrm{Mod}(S_{0,2g+2})$ [1, Theorem 1 plus Corollary 7.1]. A slight generalization of this construction gives roots of half-twists in any $\mathrm{Mod}(S_{0,n})$ with $n \geq 5$.

Roots of elementary matrices. If we consider the map $\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ given by the action of $\mathrm{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z})$, we also see that elementary matrices in $\mathrm{Sp}(2g, \mathbb{Z})$ have roots; for instance, we have

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By stabilizing, we obtain cube roots of elementary matrices in $\mathrm{Sp}(2g, \mathbb{Z})$ for $g \geq 2$.

Roots of Nielsen transformations. Let F_n denote the free group generated by $x_1, \ldots, x_n$, let $\mathrm{Aut}(F_n)$ denote the group of automorphisms of F_n, and assume $n \geq 2$. A Nielsen transformation is an element of $\mathrm{Aut}(F_n)$ conjugate to the one given by $x_1 \mapsto x_1 x_2$ and $x_k \mapsto x_k$ for $2 \leq k \leq n$. The following automorphism is the square root of a Nielsen transformation in $\mathrm{Aut}(F_n)$ for $n \geq 3$.

$$\begin{array}{rcl} x_1 & \mapsto & x_1 x_3 \\ x_2 & \mapsto & x_3^{-1} x_2 x_3 \\ x_3 & \mapsto & x_3^{-1} x_2 \end{array}$$

Passing to quotients, this gives a square root of a Nielsen transformation in $\mathrm{Out}(F_n)$ and, multiplying by $-\mathrm{Id}$, a square root of an elementary matrix in $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$. Also, our roots of Dehn twists in $\mathrm{Mod}(S)$ can be modified to work for punctured surfaces, thus giving "geometric" roots of Nielsen transformations in $\mathrm{Out}(F_n)$.

Other roots. If $f \in \mathrm{Mod}(S_g)$ is a root of a Dehn twist T_d, then f commutes with T_d. Since $fT_cf^{-1} = T_{f(c)}$ for any curve c, we see that f fixes d. In the complement of d, the class f must be periodic. This line of reasoning translates to $\mathrm{GL}(n, \mathbb{Z})$ and $\mathrm{Aut}(F_n)$: roots correspond to torsion elements in $\mathrm{GL}(n-1, \mathbb{Z})$ and $\mathrm{Aut}(F_{n-1})$, respectively. In all cases, one can show that the degree of the root is equal to the order of the torsion element.

Acknowledgements. We thank Serge Cantat for a useful conversation.

References

[1] Joan S. Birman and Hugh M. Hilden. On the mapping class groups of closed surfaces as covering spaces. In *Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 81–115. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.

[2] Catherine Labruère and Luis Paris. Presentations for the punctured mapping class groups in terms of Artin groups. *Algebr. Geom. Topol.*, 1:73–114 (electronic), 2001.

Dan Margalit, Department of Mathematics, 503 Boston Ave, Tufts University, Medford, MA 02155

E-mail address: dan.margalit@tufts.edu

Saul Schleimer, Department of Mathematics, University of Warwick, Coventry, CV4 7AL, UK

E-mail address: s.schleimer@warwick.ac.uk