Mapping Class Groups of Infinite-Type Surfaces

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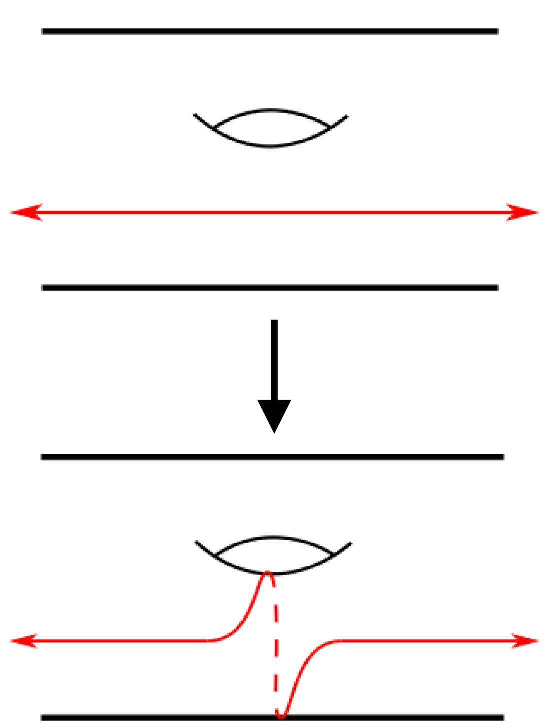
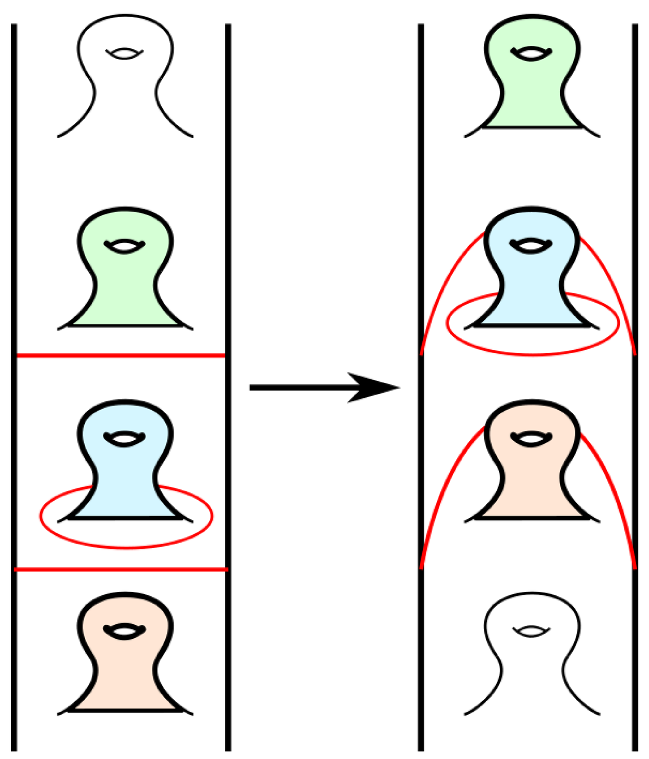
Examples of Mapping Classes

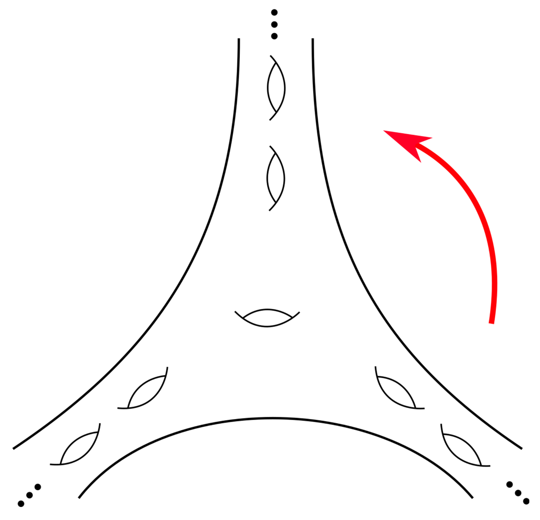
A mapping class is a symmetry of a surface.



A Dehn twist



A handle shift



Order three element

The Mapping Class Group

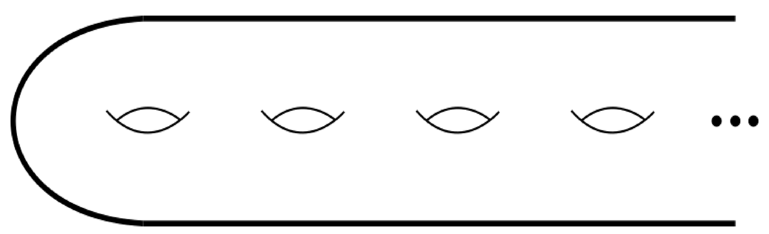
$$\mathrm{Mod}(S) := \frac{\mathrm{Homeo}^{+}(S,\partial S)}{\text{homotopy}}$$

Generating Mapping Class Groups

Theorem (Patel-Vlamis): If S has infinite genus, Dehn twists and handle shifts form a dense generating set of $\mathrm{PMod}(S)$

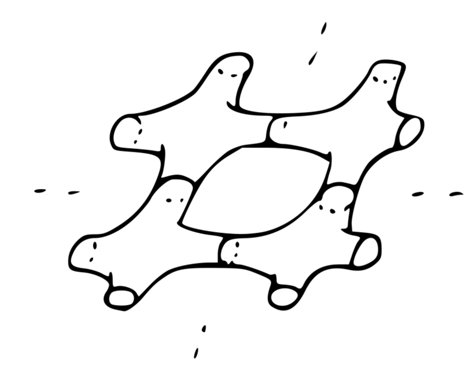
Theorem (A-F-Y): If S has infinite genus, handle shifts are a dense generating set of $\mathrm{PMod}(S)$

Examples of Infinite-Type Surfaces

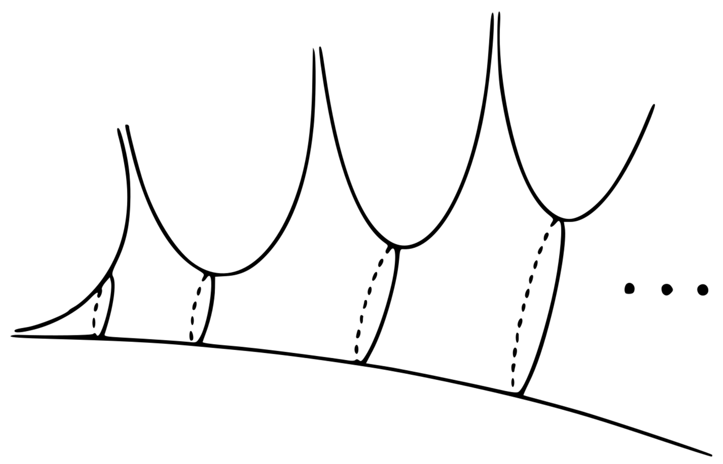


The Loch Ness monster surface

$E(S) = \{*\}$ (One-point discrete space)

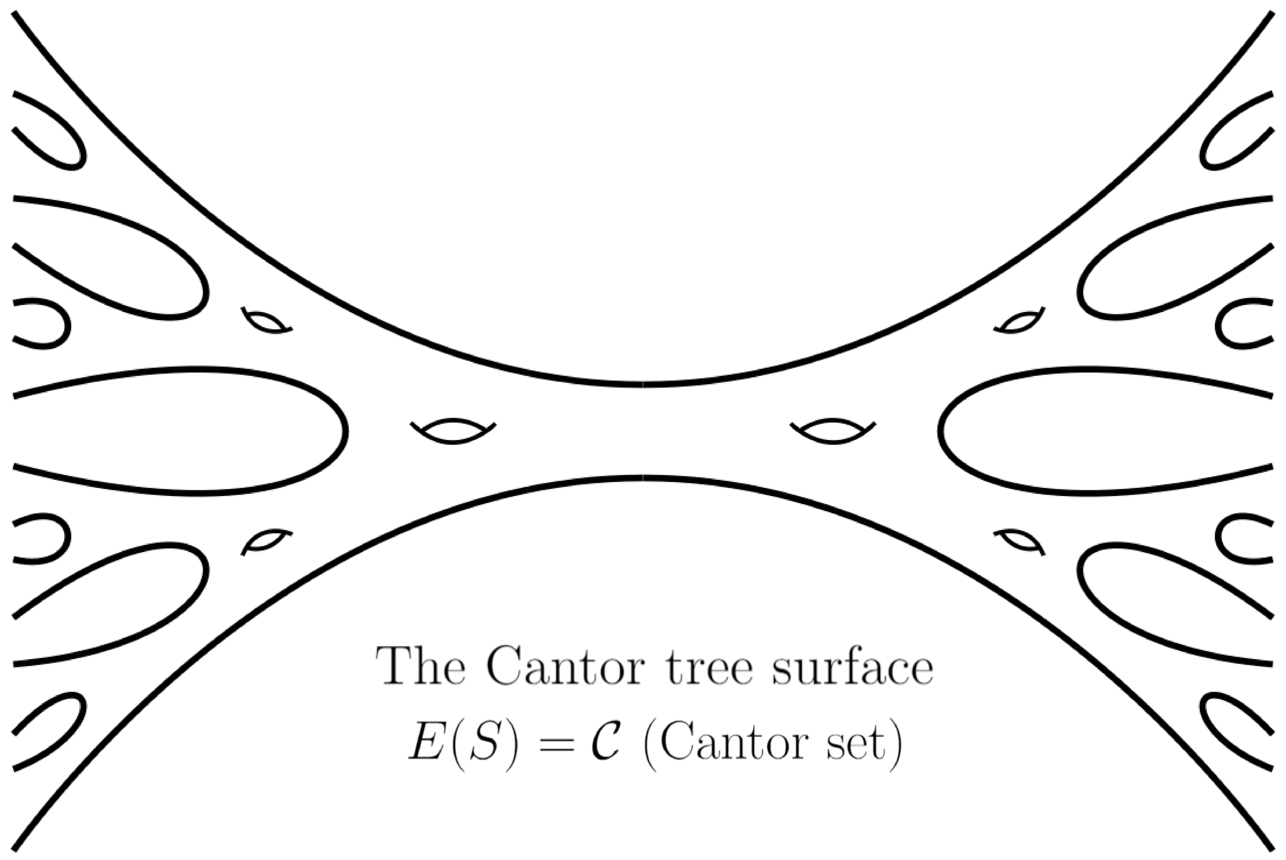


The infinite jail cell surface



The Devil's flute surface

$E(S) = \{1/n : n \in \mathbb{N}\} \cup \{0\}$



The Cantor tree surface

$E(S) = \mathcal{C}$ (Cantor set)

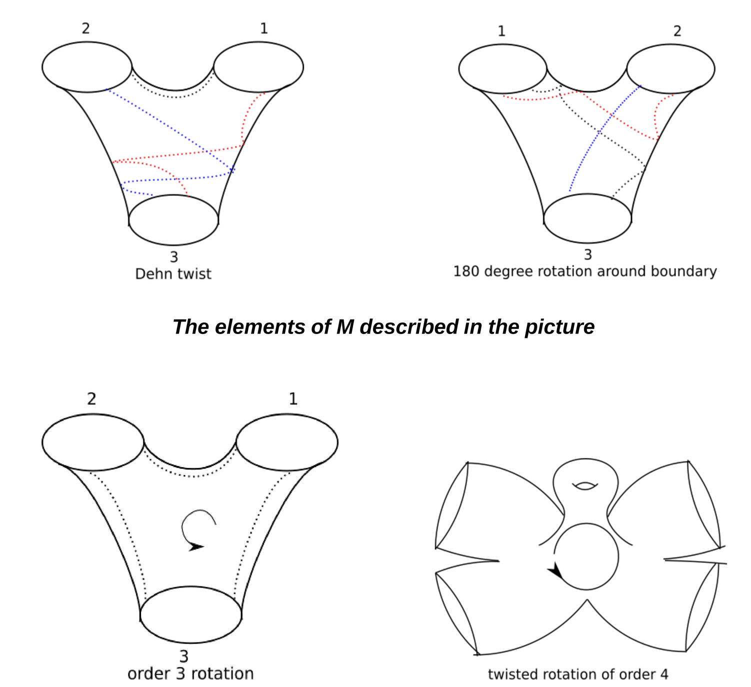
When are two surfaces the same?

Theorem. *(Richards) An orientable, infinite-type surface is uniquely determined by:*

- *The genus* $g \in \mathbb{N} \cup \{0, \infty\}$
- *The number of boundary components* $b \in \mathbb{N} \cup \{0\}$
- *The space of ends* $E(S)$*, where* $E(S)$ *is a closed subset of the Cantor set.*

Homology and Symplectic Structure

Question: Does every automorphism of the homology group of an infinite-type surface come from a mapping class?



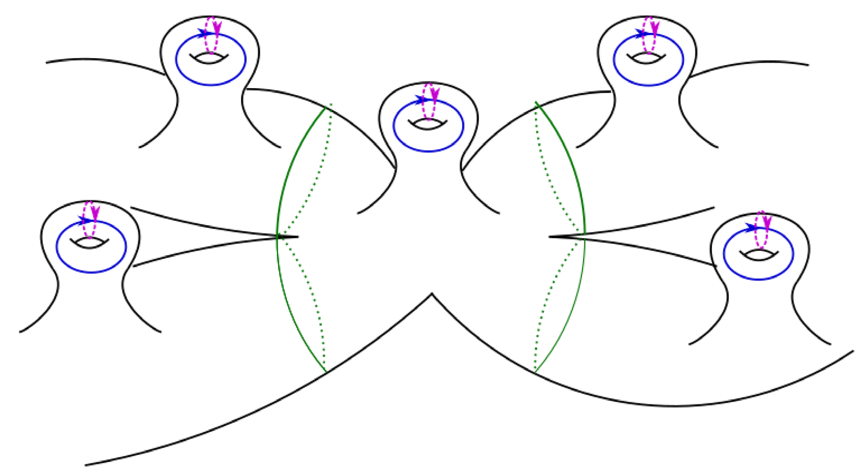
The elements of M described in the picture

A finitely group M whose integral homology is isomorphic to the integral stable homology of the big mapping class group.

Theorem The integral homology M is isomorphic to the stable integral homology of the mapping class group.

The sequence in homology:

$$1 \to PM \to M \to V \to 1$$



The class of blue and pink curves above form a simplectic basis for $H_1(S,R)_s$ *,green form a basis for* $H_1(S,R)_{ns}$

H_r *:Completion of* $H_1(S,R)_{ns}$ *as a Hilbert space*
Inner product $\omega(v,w) = (v, Jw)$ *where* $J^2 = -1$

$$H_1(S,R) = H_1(S,R)_s \oplus H_1(S,R)_{ns}$$

Theorem The symplectic representation of the mapping class group PM extends to a representation $\rho:\ M \to Sp_{res}(H_r)$ into the restricted symplectic group.

Dehn twist and handle shifts generate big mapping class group.
The blue and pink curves are all generators of big mapping class group.

Questions

- Is the group generated by all handle shifts an "on the nose" generating set for $\mathrm{Mod}(S)$?
- Do all automorphisms of the first homology class of the mapping class group arise from Dehn twists and handle shifts?

Acknowledgements

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