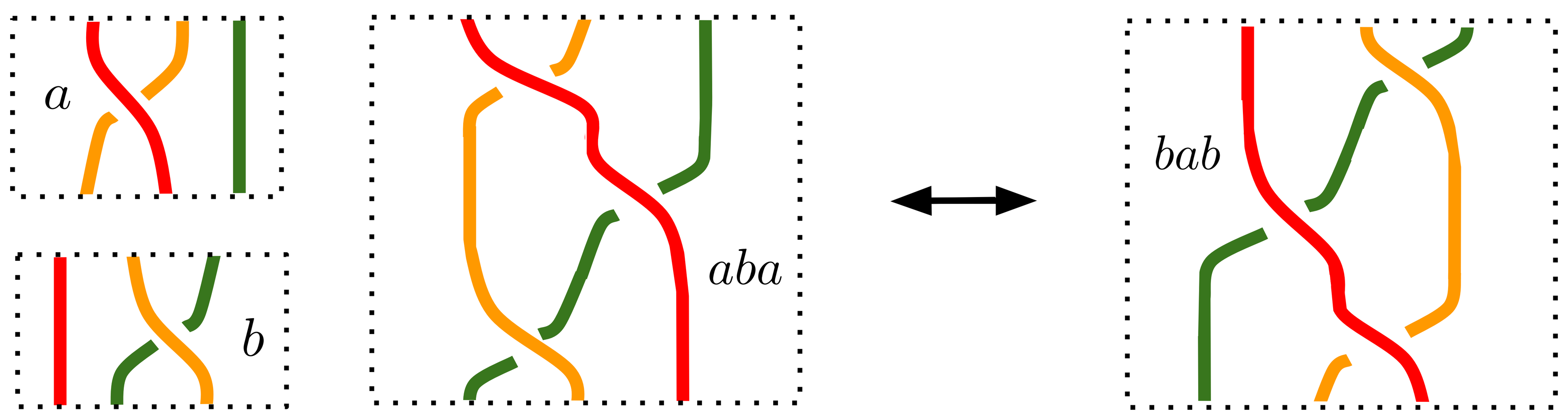
On Congruence Subgroups of the Braid Group

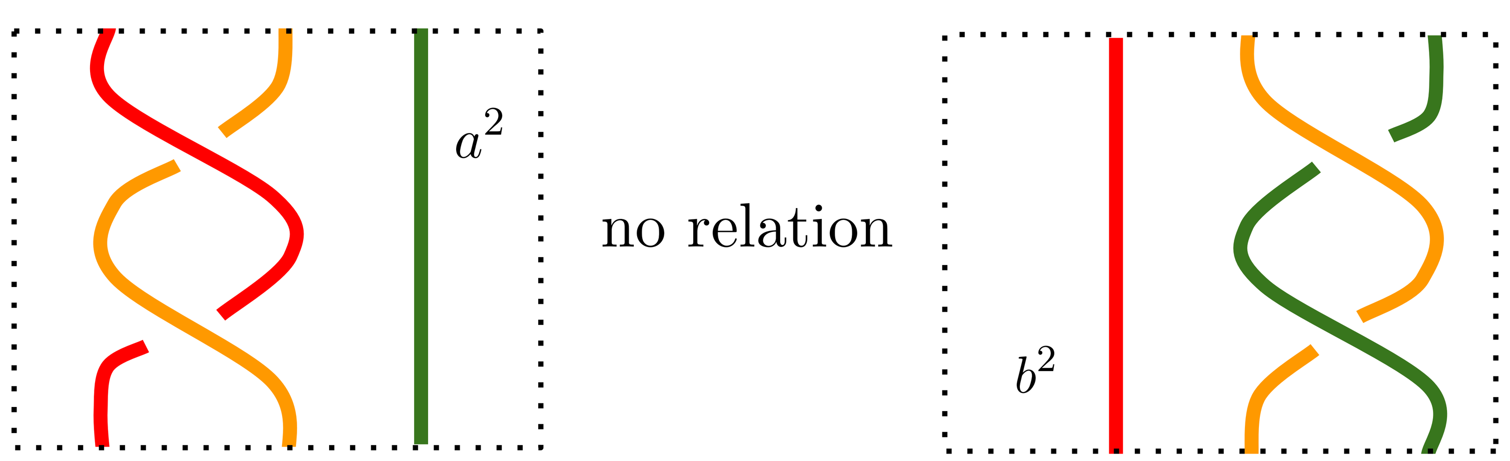
Jessica Appel[1], Katie Gravel[2], Annie Holden[3]

[1]University of Kentucky [2]Massachusetts Institute of Technology [3]Colby College

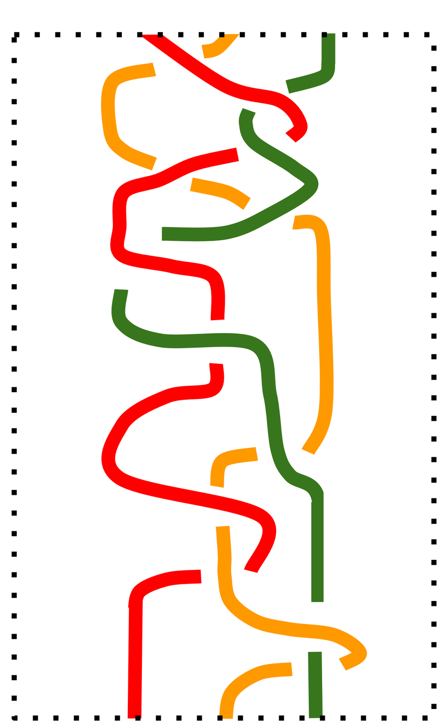
Braids

Goal: Understand the structure of congruence subgroups of the braid group.





no relation



a level four braid

Integral Burau Representation

$$\rho_{-1} : B_n \to GL(n, \mathbb{Z})$$

$$\sigma_i \mapsto I_{i-1} \oplus \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1}$$

r_N is the usual mod N reduction map

$$B_n[N] = ker(r_N \circ \rho_{-1})$$

Problem I: Generating Sets

Question: What is a natural generating set for $B_n[4]$? How big is it?

Margalit and Kordek: Size lower bounded by

$$\binom{n}{2} + 3\binom{n}{3} + 3\binom{n}{4} \sim \mathrm{O}(n^4)$$

Schreier's method $\rightsquigarrow$ exponential generating set

Use recurrence relation to reduce generating set

Theorem.

$$\#\text{ generators of } B_n[4] \sim \mathrm{O}(n^5)$$

Acknowledgements

We would like to thank Dr. Wade Bloomquist, Dr. Dan Margalit, Georgia Tech, and the National Science Foundation.

Problem II: PB_n^ℓ and $B_n[2\ell]$

Question: What is the relationship between PB_n^ℓ and $B_n[2\ell]$ for varying ℓ?

Brendle and Margalit: $PB_n^2 = B_n[4]$

Theorem.

For $\ell = 2^k$, $PB_n^\ell \subset B_n[2\ell]$

For $\ell = 6, 10, 12$ or ℓ odd, $PB_n^\ell \not\subset B_n[2\ell]$

Conjecture.

$\ell = 2^k \iff PB_n^\ell \subset B_n[2\ell]$

Problem III: Quotients

Question: What can we say about quotients of Burau levels?

Artin: $B_n/PB_n \cong S_n$

Stylianakis: $B_n[p]/B_n[2p] \cong S_n$ for p prime

Theorem.

$B_n[\ell]/B_n[2\ell] \cong S_n$ for odd ℓ

$B_n[\ell]/B_n[2\ell] \cong (\mathbb{Z}/2\mathbb{Z})^{\binom{n}{2}}$ for even ℓ

Georgia Tech School of Mathematics College of Sciences

NSF