Geometric Homomorphisms from Surface Groups to Free Groups

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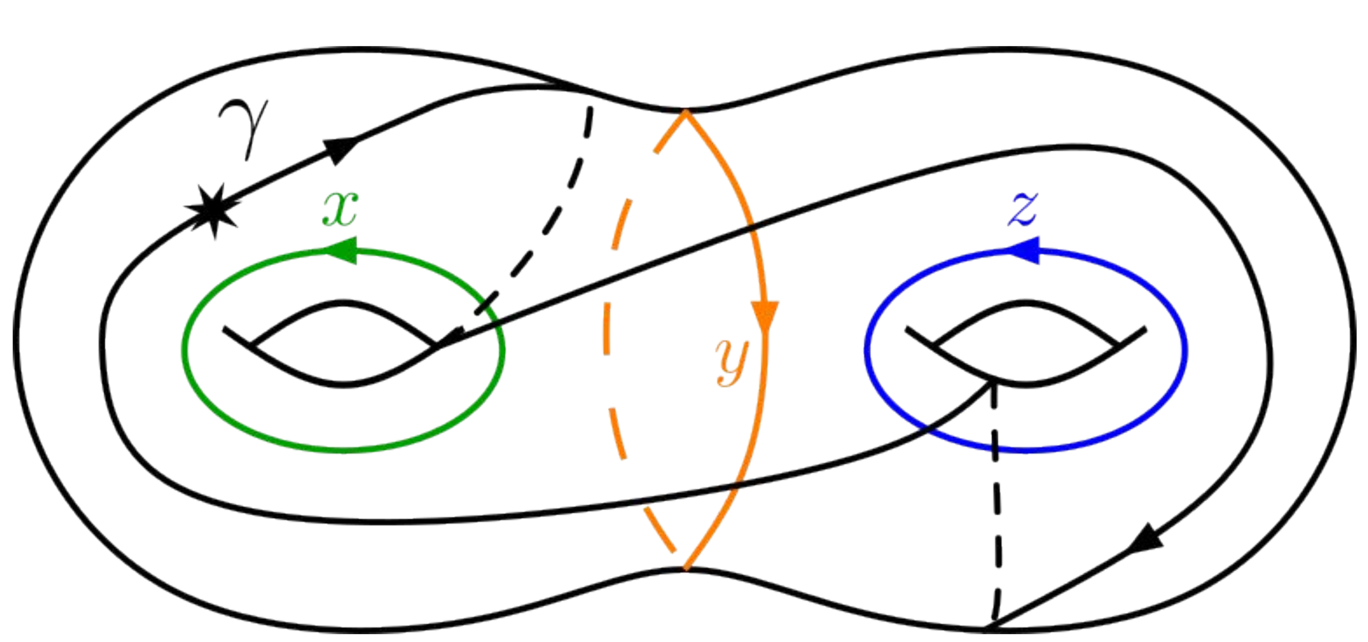
Our Project

This projects aims to determine which homomorphisms $\pi_1(\Sigma_g) \to \mathbb{F}_n$ are *geometric*.

Geometric Homomorphisms

Crisp-Wiest Construction:

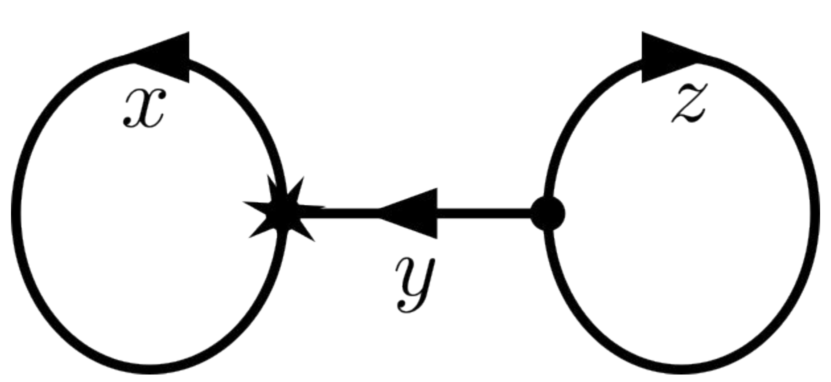
$$\pi_1(\Sigma_2) \to \mathbb{F}_3 = \langle x, y, z \rangle$$



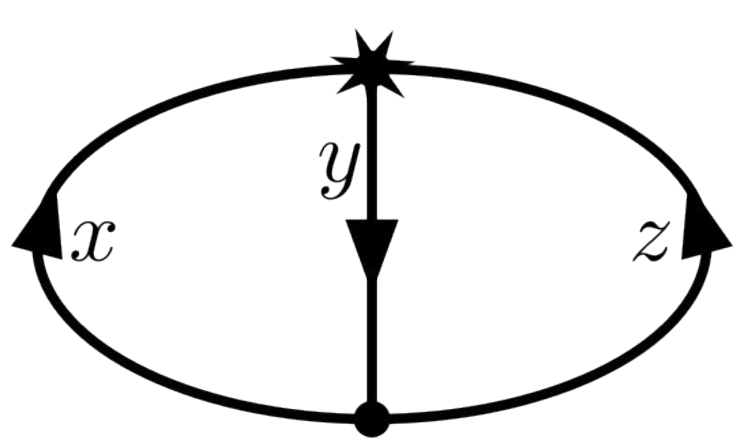
$$\varphi(\gamma) = xy^{-1}zy$$

Subgroups of $\mathbb{F}_n$ from Graphs

The fundamental group of a **labeled trivalent graph** on *n* vertices is a subgroup of $\mathbb{F}_n$.



$\pi_1(\Gamma) = \langle x, y^{-1}zy \rangle$

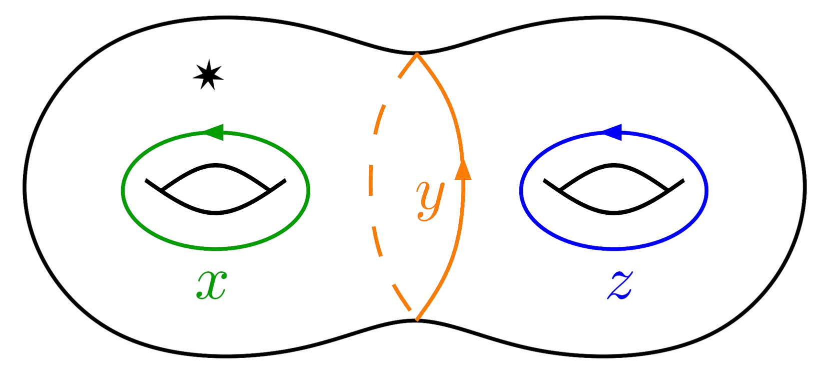


$\pi_1(\Gamma) = \langle yx, yz \rangle$

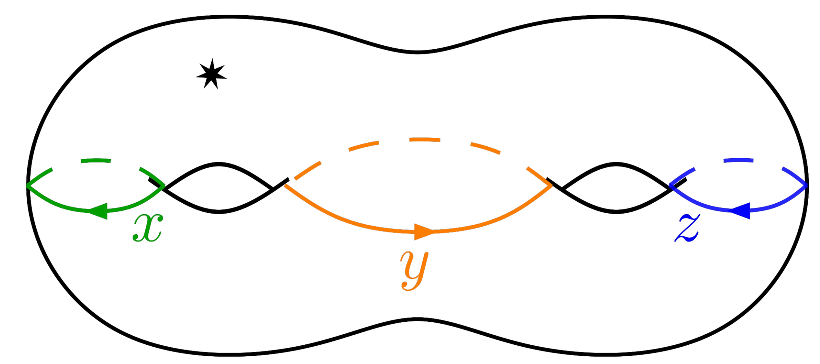
Theorem

A homomorphism, $\pi_1(\Sigma_g) \to \mathbb{F}_{3g-3}$, is *geometric* iff $\mathrm{im}(\varphi)$ corresponds to the fundamental group of a labeled trivalent graph, Γ, on 2*g* - 2 vertices.

Two types of image on a genus 2 surface:



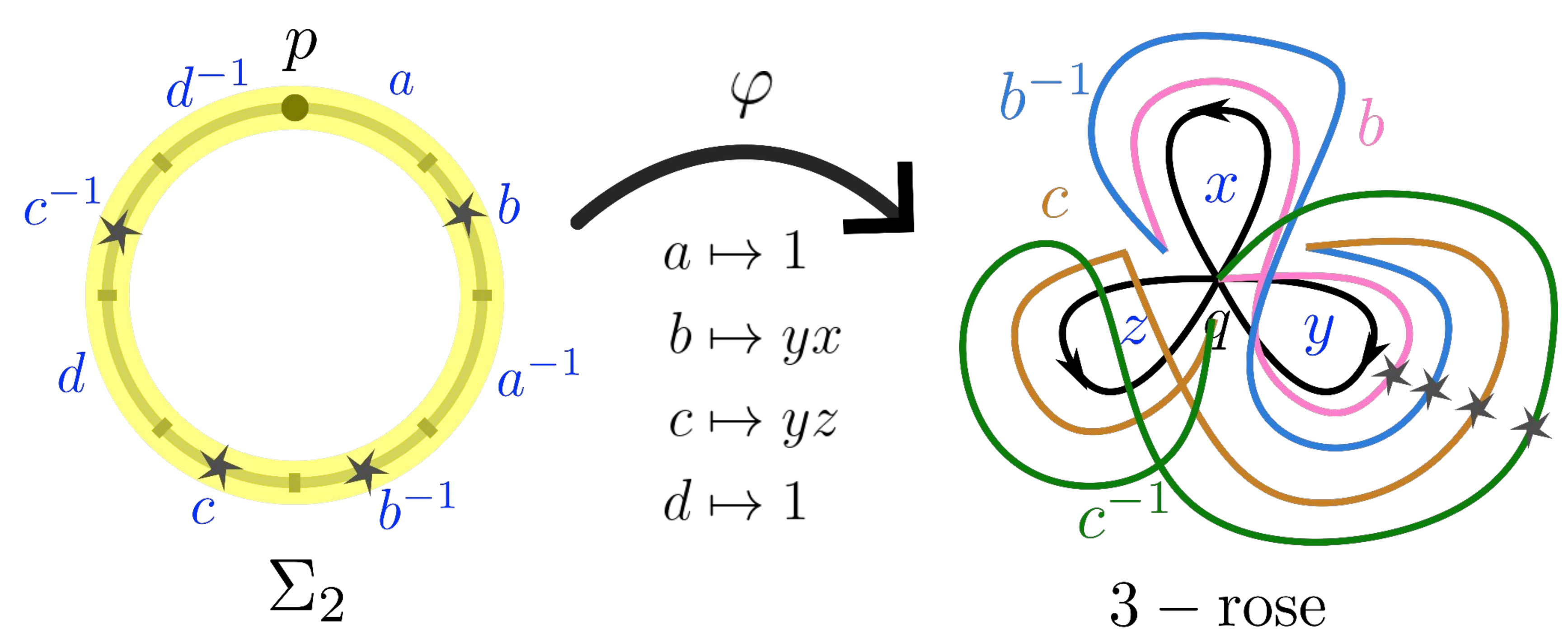
$\mathrm{im}(\varphi) = \langle x, y^{-1}zy \rangle$



$\mathrm{im}(\varphi) = \langle yx, yz \rangle$

Construct Geometric Homomorphisms

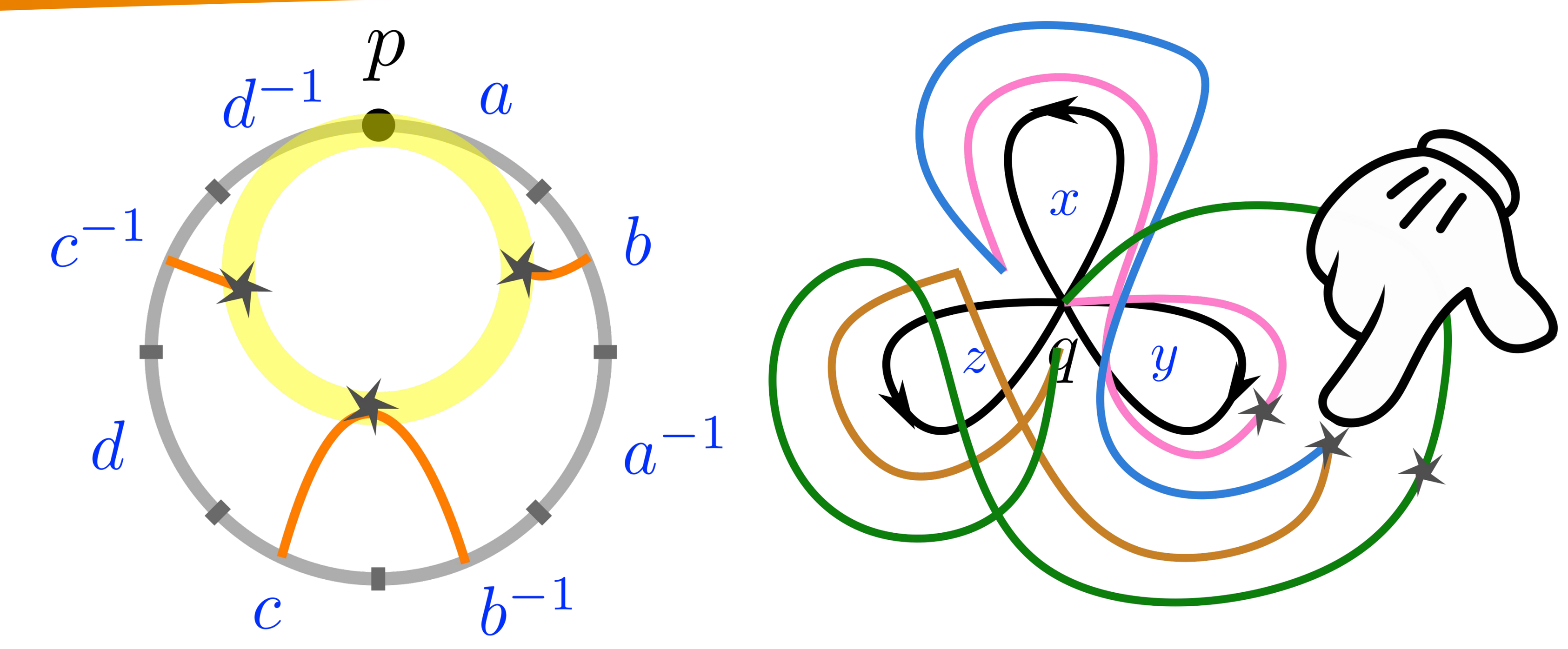
For example, $\varphi : \pi_1(\Sigma_2) \to \mathbb{F}_3$



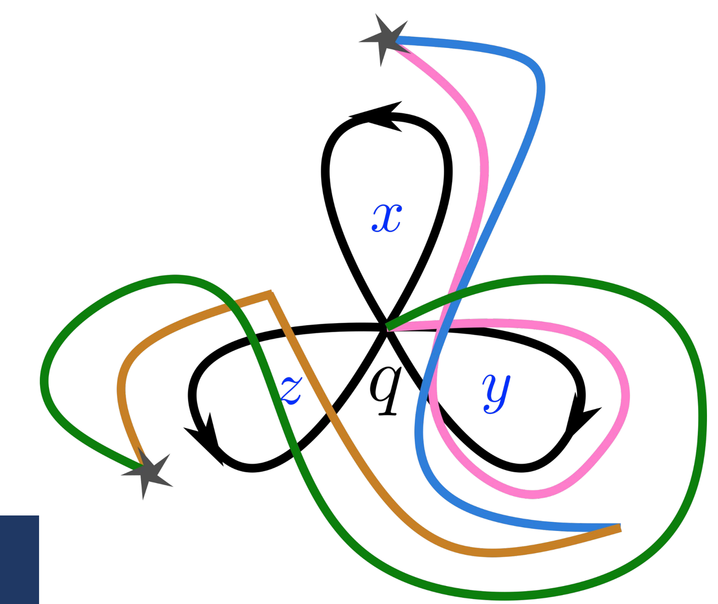
The preimage of *y* includes 4 points on the boundary.

Idea: Use the one-to-one correspondence between

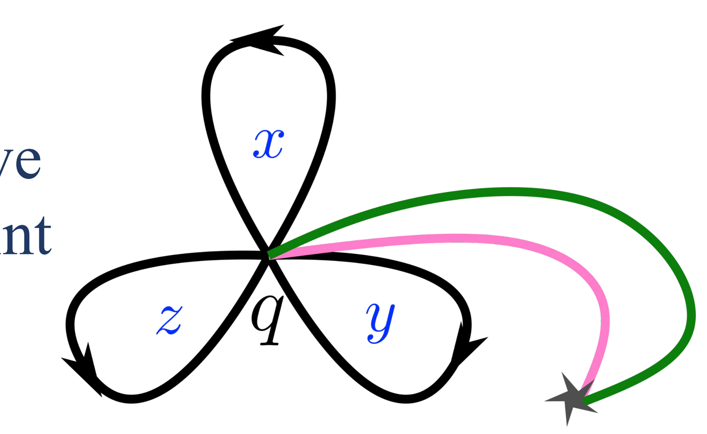
$$\left[\Sigma_2, \bigvee_3 S^1\right] \longleftrightarrow \mathrm{Hom}(\pi_1(\Sigma_2), \mathbb{F}_3)$$

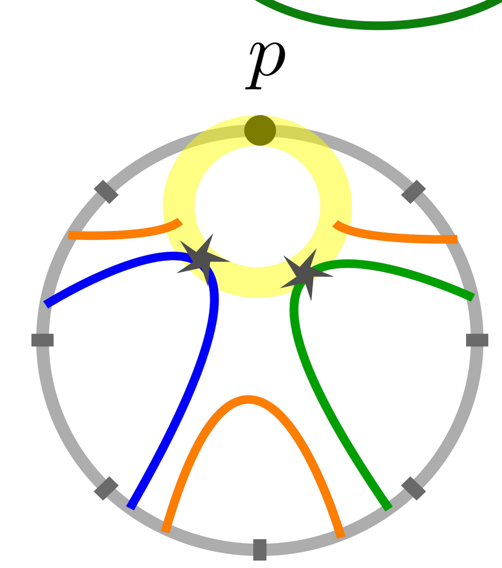


Find the preimage of intersection to retrace the curve.

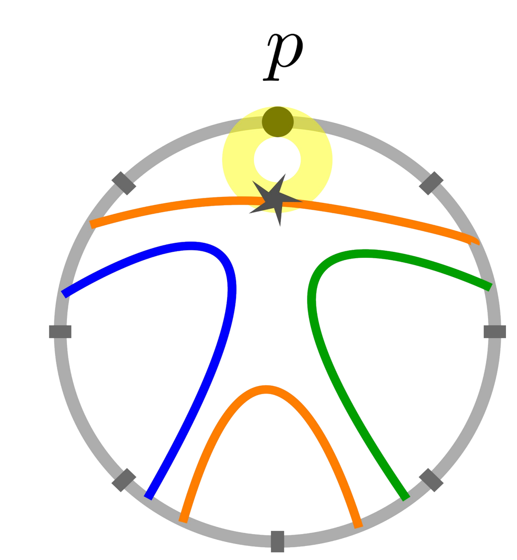


1. Unwrap the curve back to the basepoint

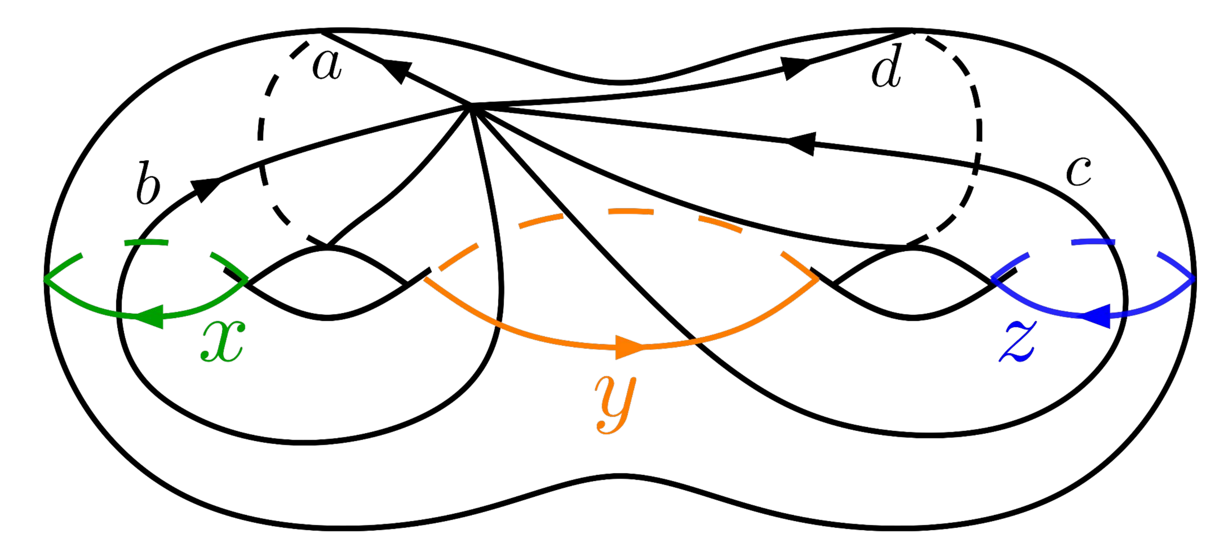




2. This corresponds to shrinking the boundary down to the basepoint



3. Realize curves on the surface



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1: Harvey Mudd College 2: Grinnell College 3: Vanderbilt University 4: Georgia Institute of Technology