Transition matrices of real, periodic, quadratic polynomials

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Background

Given polynomial $f : \mathbb{C} \to \mathbb{C}$, we have *critical set*

$$C_f = \{p \in \mathbb{C} : f'(p) = 0\}$$

and *postcritical set*

$$P_f = \cup_{n>0}\ f^n(C_f)$$

The *Julia set* is the complex points with bounded image under iteration of f.

Hubbard tree is the hull of P_f in the Julia set.

Fact: Hubbard tree maps to itself under f.

Example: Douady Rabbit Polynomial

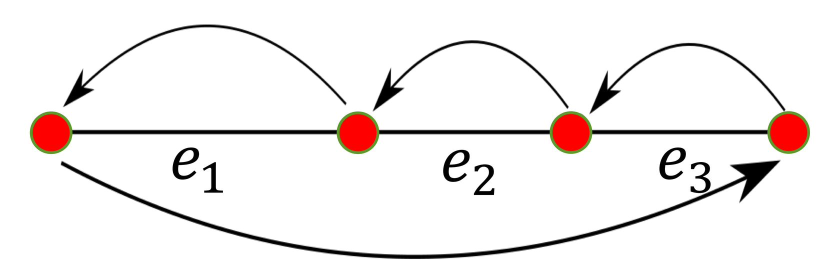
$$f(z) = z^2 - 0.12256 + 0.74486i$$

Objectives

- Obtain a "transition matrix" from a polynomial by looking at the action of f on the edges of the Hubbard tree.
- Examine the transition matrices of polynomials of the form $f(z) = z^2 + c, c \in \mathbb{C}$, where f has a periodic critical point.
- In particular, look at characteristic polyonomials and eigenvalues of the matrices.

Examples and Method

Example 1: $c \approx 1.98542$



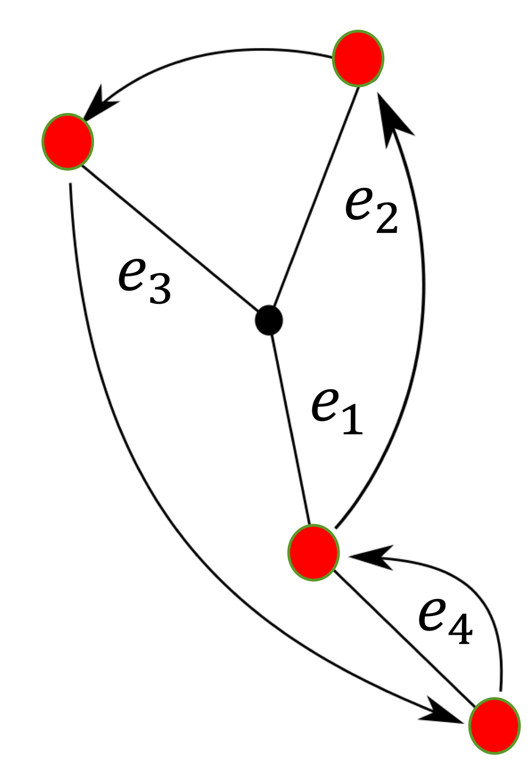
$$\begin{array}{c} \\ \end{array}\begin{matrix} e_1 & e_2 & e_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$$

Characteristic polynomial = $T^3 - T^2 - T - 1$

Largest eigenvalue ≈ 1.8393

Example 1: $c \approx -1.25637 - 0.380321i$



$$\begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{matrix}$$

Characteristic polynomial = $T^4 - 2T - 1$

Largest eigenvalue ≈ 1.3953

- Thurston gives criteria classifying which periodic sequences of post-critical points arise from polynomials.
- We used these criteria to write a program that generates a list containing all possible periodic sequences of a given length and computes the associated transition matrices.

Theorems and Conjectures

Theorems

1. The characteristic polynomial of the transition matrix M for a quadratic polynomial $z^2 + c, c \in \mathbb{R}$, of period $n+1$ is

$$ch_M(T) \equiv T^n + T^{n-1} + \ ... + 1\ (mod\ 2)$$

2. There exists a unique edge of the Hubbard tree that maps over itself.

Corollaries

i) Some edge vector v will give a basis $\{v, Mv, M^2v, \ldots, M^{n-1}v\}$. Equivalently, $ch_M(T)$ is always the minimal polynomial of M.

ii) M is invertible.

iii) Trace of M is 1.

Conjecture

All coefficients of $ch_M(T)$ are ± 1.

Further Questions

- In "Entropy in Dimension One", Thurston classifies all numbers that arise as largest eigenvalues of transition matrices for real polynomials of any degree. Which numbers can arise in the case of real, periodic, quadratic polynomials?
- What can be said about transition matrices when $c \in \mathbb{C} \setminus \mathbb{R}$? What about when 0 is non-periodic?
- Is there a characterization for postcritically finite maps that corresponds to the matrices being reducible, irreducible, or Perron-Frobenius?