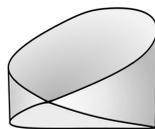


Curves in Non-Orientable Surfaces

Sarah Ruth Nicholls, Wake Forest University; Julia Shneidman, Rutgers University
Mentored by Nancy Scherich, University of Toronto

What are Non-Orientable Surfaces?

The **Möbius band** is a non-orientable surface with only one side and one boundary curve -- a circle.



The **cross-cap** is a reconfiguration of the Möbius band where the boundary circle looks like a true circle.

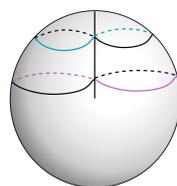


Cross-caps are in 4D, not 3D!

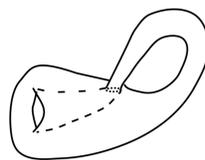


Scan for helpful animation!

Gluing the boundary of a disc to the boundary of a cross-cap yields the **real projective plane**.



Gluing two cross-caps together along their boundary yields the **Klein bottle**.



Classification Theorem

All non-orientable surfaces are formed by gluing some number of cross-caps to a sphere.

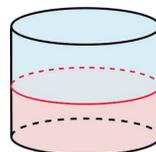


Our Project

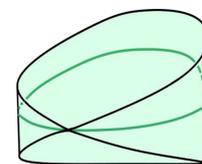
Generalize known results about collections of curves in orientable surfaces to the non-orientable case.

Simple Closed Curves in Surfaces

For this project, curves are homotopy classes of closed loops with no self intersections.



2-sided curves are the core of a cylinder.

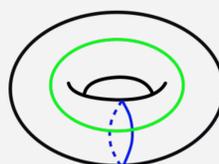


1-sided curves are the core of a Möbius band.

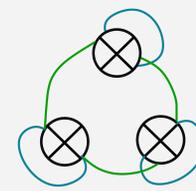
Orientable vs. Non-Orientable

All curves are 2-sided.

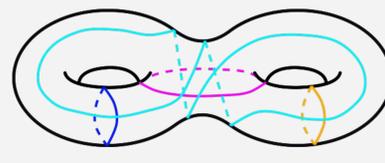
Curves can be either 1-sided or 2-sided.



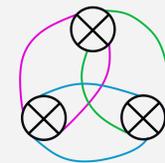
curves in the torus



1-sided curves in N_3



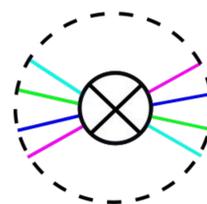
curves in the genus 2 surface



2-sided curves in N_3

Curves Through Cross-Caps

Curves passing through a cross-cap can be disjoint through the entirety of the cross-cap.



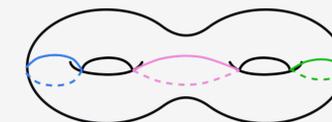
Maximal Collections of Curves

A collection of disjoint curves is **maximal** if there does not exist another curve in the surface disjoint from the collection.

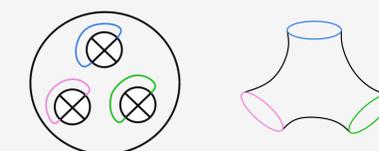
Orientable vs. Non-Orientable

The number curves in a maximal collection depends only on genus.

The number curves in a maximal collection is NOT unique.



Maximal collection of curves on genus 2 surface with 3 curves



Maximal collection in N_3 with 3 curves



Maximal collection in N_3 with 2 curves

Theorem

Orientable Case [Malestein, Rivin, and Theran]

In a genus g surface, the maximum number of curves intersecting at most once is greater than or equal to

$$g^2 + \frac{5}{2}g.$$

Non-Orientable Case

In N_g the maximum number of curves intersecting at most once is greater than or equal to

$$\begin{cases} g^2 + \frac{9}{2}g + 2 & g \text{ is even} \\ g^2 + \frac{5}{2}g + 2 \lfloor \frac{g}{2} \rfloor + 1 & g \text{ is odd.} \end{cases}$$



Acknowledgments

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