

DAN MARGALIT: CAREER DEVELOPMENT PLAN
GROUP-THEORETICAL, DYNAMICAL, AND COMBINATORIAL
ASPECTS OF MAPPING CLASS GROUPS

1. BACKGROUND

The *mapping class group* of a surface is the group of homotopy classes of orientation-preserving homeomorphisms of the surface. Among other things, the mapping class group encodes isomorphism types of surface bundles over arbitrary spaces, and is the orbifold fundamental group of moduli space. Dehn and Nielsen first studied this group from the algebraic and geometric points of view in the early part of the twentieth century. The work on mapping class groups proposed here has much of its roots in Thurston's far-reaching work in the 1970's. The techniques spring primarily from hyperbolic geometry, combinatorial topology, and geometric group theory, and the theorems and conjectures have applications and connections to dynamics, algebraic geometry, and 3-manifold theory.

For the closed surface S_g of genus g , the mapping class group $\text{Mod}(S_g)$ acts on $H_1(S_g, \mathbb{Z})$ preserving the algebraic intersection number, which is a symplectic form. The kernel of this action is called the *Torelli group* (after the nineteenth century algebraic geometer), and we denote it by $\mathcal{I}(S_g)$. We have the following short exact sequence:

$$1 \rightarrow \mathcal{I}(S_g) \rightarrow \text{Mod}(S_g) \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1.$$

As the linear symplectic group $\text{Sp}(2g, \mathbb{Z})$ is relatively well-understood, we think of $\mathcal{I}(S_g)$ as encoding the deeper and more mysterious aspects of $\text{Mod}(S_g)$. One theme is that questions about $\text{Mod}(S_g)$ often easily reduce to questions about the Torelli group. Interestingly, questions about the Torelli group often then lead to fundamental questions about the first homology of a surface.

There are many basic unanswered questions about the mapping class group. In this proposal, we put forward a three-pronged attack, researching its group theoretical, the dynamical, and the combinatorial aspects.

2. PRIOR NSF SUPPORT

Over the last six years, the PI has been supported by two NSF grants. We give a summary of the results here, and explain some of the theorems in greater depth in the research component of the proposal below.

“Algebra and topology of the Johnson filtration”
DMS-0926144, \$95,559, Sep 2007–Aug 2010

Homology of Torelli groups. In [BBM], Bestvina, Bux, and the PI showed that the cohomological dimension of $\mathcal{I}(S_g)$ is $3g - 5$, answering a question of Mess [Mes90]. We also showed that $H_{3g-5}(\mathcal{I}(S_g))$ is infinitely generated, answering a special case of Question 1 of the proposal for DMS-0926144, and partly answering questions of Mess [Kir97, Problem 2.9(B)] and of Farb [Far06, Problem 5.14]. We also gave a combinatorial proof of a theorem of Mess, describing $\mathcal{I}(S_2)$ (see Theorem 3.8 below), which was Problem 5 of the proposal for DMS-0926144. Finally, we determined the cohomological dimension of the Johnson kernel, answering a question of Farb [Far06, Problem 5.9]. Our proof involves the introduction of a new kind of combinatorial complex for $\mathcal{I}(S_g)$. A variant of this complex was later studied by Hatcher [Hat].

Dilatations and 3-manifolds. In [FLM], we consider the infinite set of low entropy pseudo-Anosov homeomorphisms. Farb, Leininger, and the PI prove that, up to Dehn surgery, the resulting set of mapping tori is finite. As a consequence, we obtain a relation between dilatation and hyperbolic volume. This answers a question posed by C. McMullen [McM].

Group theory of $\text{Mod}(S)$. Schleimer and the PI [MS09], proved that Dehn twists about nonseparating curves in S_g have nontrivial roots. Our work inspired a paper by McCullough–Rajeevsarathy [MS09], which classifies all possible orders of roots.

Leininger and the PI [LM] showed that if a and b are arbitrary elements of the pure braid group PB_n , then either a and b commute, or they generate a free group of rank 2. This completely answers a question of Paris [Par04].

In [BM08], Brendle and the PI extend the work of [BM04] to all possible cases, simultaneously extending the work of Farb–Ivanov [FI05] to all possible cases (see below).

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DMS-0402601, \$108,000, Sep 2004–Aug 2007

Homology of the Torelli group for $\text{Out}(F_n)$. In [BBM07], Bestvina, Bux, and the PI proved that the cohomological dimension of the Torelli group for $\text{Out}(F_n)$ is $2n - 4$. We also showed that, in this top dimension, the homology is infinitely generated; this settles one case of a question of Vogtmann [Vog02]. Also, this recovers a result of Krstić–McCool that the Torelli group for $\text{Out}(F_3)$ is not finitely presented [KM97].

Pseudo-Anosovs and the Torelli group. Farb, Leininger, and the PI [FLM08] showed that, contrary to the case for $\text{Mod}(S_g)$, there is a universal lower bound to the entropy for a pseudo-Anosov element of $\mathcal{I}(S_g)$. We further gave asymptotic upper and lower bounds for the least entropy of pseudo-Anosov elements of the terms of the Johnson filtration.

Spallone and the PI [MS07] gave an explicit construction of pseudo-Anosovs, using an improved version of a homological criterion of Casson–Bleiler.

Automorphisms of groups, spaces, and complexes associated to the mapping class group. The Johnson kernel $\mathcal{K}(S_g)$ is the normal subgroup of $\text{Mod}(S_g)$ generated by Dehn twists about separating curves. In [BM04], Brendle and the PI computed its automorphism group and its abstract commensurator group for $g \geq 4$. Let $\text{Mod}^\pm(S_g)$ denote the *extended mapping class group*, the group of homotopy classes of all homeomorphisms of S_g . We showed that $\text{Comm}(\mathcal{K}(S_g)) \cong \text{Aut}(\mathcal{K}(S_g)) \cong \text{Mod}^\pm(S_g)$, answering a question of Farb [Far]. We also showed that $\text{Comm}(\mathcal{I}(S_g)) \cong \text{Aut}(\mathcal{I}(S_g)) \cong \text{Mod}^\pm(S_g)$, simplifying the proof of Farb–Ivanov [FI05] and extending their result to $g \geq 4$.

In [BM07a] and [BM06], the PI worked with Behrstock and Bell to complete the classification of abstract commensurations of mapping class groups, in particular identifying the only nongeometric examples. A key step is to finish the classification of superinjective maps of curve complexes begun by Irmak [Irm04] and Ivanov [Iva97].

In the author’s thesis [Mar04], the PI showed that the automorphism group of the pants complex is isomorphic to $\text{Mod}^\pm(S)$, and under this fellowship, Brock and the PI [BM07b] used this to show that the isometry group of Teichmüller space with the Weil–Petersson metric is isomorphic to $\text{Mod}^\pm(S)$.

Braid groups. With Leininger, the PI determined the abstract commensurator of the braid group explicitly [LM06], and with McCammond, the PI [MM09] discovered a new, intuitive presentation of the pure braid group.

3. RESEARCH PLAN

In separate subsections, we discuss the three aspects of our research proposal: homological properties of Torelli groups, structure of low entropy pseudo-Anosovs, and group-theoretical properties of hyperelliptic Torelli groups.

3.1. Torelli groups and their finiteness properties. The research described in this section concerns the work in the following papers:

[BBM] “The dimension of the Torelli group,” with Mladen Bestvina and Kai-Uwe Bux, to appear in *Journal of the American Mathematical Society*.

[BBM07] “Dimension of the Torelli group for $\text{Out}(F_n)$ ”, with Mladen Bestvina and Kai-Uwe Bux, *Inventiones Mathematicae* 170 (2007), no. 1, 1–32.

The following basic question about $\mathcal{I}(S_g)$ remains open.

Problem 3.1. *Determine which groups $H_k(\mathcal{I}(S_g), \mathbb{Z})$ are finitely generated.*

In particular one would like to know about second homology, for if the second homology of a group is infinitely generated then the group is infinitely presented.

Problem 3.2. *Determine if the groups $\mathcal{I}(S_g)$ are finitely presented for $g \geq 3$.*

The PI plans to continue the long-term project of attacking Problems 3.1 and 3.2 with Mladen Bestvina of the University of Utah and Kai-Uwe Bux of Bielefeld University in Germany. Before outlining our contributions towards these problems, we will recall some earlier work on this subject.

Very little is known about $\mathcal{I}(S_g)$, as compared to, say, the mapping class group itself. In the 1920s, Dehn proved that $\mathcal{I}(S_1)$ is trivial. In 1983, Johnson [Joh83] proved that $\mathcal{I}(S_g)$ is finitely generated for $g \geq 3$, and in 1986, Mess showed that $\mathcal{I}(S_2)$ is an infinitely generated free group (Theorem 3.8 below). In 2001, Akita [Aki01] showed that $H_*(\mathcal{I}(S_g))$ is not finitely generated for $g \geq 7$.

Our first theorem gives the *cohomological dimension* of $\mathcal{I}(S_g)$, that is, the highest dimension in which $\mathcal{I}(S_g)$ has nontrivial cohomology. Mess [Mes90] proved that the cohomological dimension $\text{cd}(\mathcal{I}(S_g))$ is greater than or equal to $3g - 5$. In [BBM], we proved the opposite inequality:

Theorem 3.3 (Bestvina–Bux–Margalit). *For $g \geq 2$, we have*

$$\text{cd}(\mathcal{I}_g) = 3g - 5.$$

The virtual cohomological dimension $\text{vcd}(\text{Mod}(S_g))$ of the full mapping class group was shown by Harer [Har86] to be $4g - 5$, and so there is a gap of g between this and $\text{cd}(\mathcal{I}(S_g))$. Let $\mathcal{I}^k(S_g)$ denote the subgroup of $\text{Mod}(S_g)$ fixing the homology of k particular handles. As an interpolation, we have:

Conjecture 3.4 (Bestvina–Bux–Margalit). *For $g \geq 2$ and $0 \leq k \leq g$, we have $\text{vcd}(\mathcal{I}^k(S_g)) = 4g - 5 - k$.*

We also showed:

Theorem 3.5 (Bestvina–Bux–Margalit). *The group $H_{3g-5}(\mathcal{I}_g, \mathbb{Z})$ is infinitely generated for $g \geq 2$.*

The key idea in our work is the construction of a new contractible complex on which the Torelli group acts. This “complex of minimizing cycles,” denoted $\mathcal{B}(S_g)$, encodes all of the simple representatives of any fixed integral homology class. The main feature is the following.

Theorem 3.6 (Bestvina–Bux–Margalit). *$\mathcal{B}(S_g)$ is contractible.*

Given Theorem 3.6, Theorems 3.3 and 3.5 are proven using the spectral sequences associated to the action of $\mathcal{I}(S_g)$ on $\mathcal{B}(S_g)$. Our proof suggests an approach to the following.

Conjecture 3.7. *For $g \geq 2$, the groups $H_k(\mathcal{I}(S_g))$ are infinitely generated for $2g - 3 \leq k \leq 3g - 5$.*

In genus 2, the picture is very explicit; see Figure 1. Using this picture, we also give a new proof of the following theorem. Mess’s original proof of the theorem

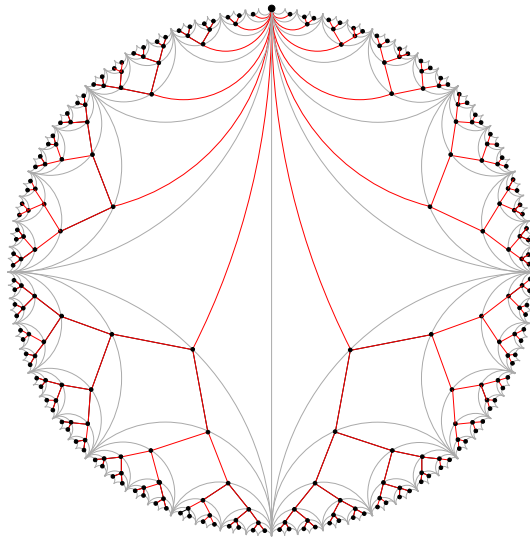


FIGURE 1. The quotient $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ is obtained by gluing countably many of these graphs together, along the distinguished vertex.

is rooted in algebraic geometry (using work of Torelli), and our argument gives a new perspective on his work.

Theorem 3.8 (Mess, new proof Bestvina–Bux–Margalit). *The group $\mathcal{I}(S_2)$ is an infinitely generated free group with one Dehn twist generator for each homology splitting of $H_1(S_2, \mathbb{Z})$.*

Problem 3.9. *Find an explicit generating set for $\mathcal{I}(S_2)$.*

There is a natural normal subgroup of $\mathcal{I}(S_g)$, denoted $\mathcal{K}(S_g)$, which is the group generated by Dehn twists about separating curves. Using the action of $\mathcal{K}(S_g)$ on $\mathcal{B}(S_g)$, we find:

Theorem 3.10 (Bestvina–Bux–Margalit). *For $g \geq 2$, we have*

$$\text{cd}(\mathcal{K}(S_g)) = 2g - 3.$$

The group $\mathcal{K}(S_g)$ is even less well understood than $\mathcal{I}(S_g)$. For instance:

Question 3.11. *For $g \geq 3$, is $\mathcal{K}(S_g)$ finitely generated? Is $H_1(\mathcal{K}(S_g))$ finitely generated? Is $H_{2g-3}(\mathcal{K}(S_g))$ infinitely generated?*

The groups $\mathcal{I}(S_g)$ and $\mathcal{K}(S_g)$ are the first two groups in a filtration of $\mathcal{I}(S_g)$ by normal subgroups, called the *Johnson filtration*, denoted $\{\mathcal{N}^k(S_g)\}$. We would like to understand what happens, at least asymptotically, with respect to the aforementioned questions, as we proceed down the filtration. For instance, work of Farb [Far06], plus Theorem 3.10 above, imply that

$$g - 1 \leq \text{cd}(\mathcal{N}_k(S_g)) \leq 2g - 3$$

for $g \geq 2$ and $k \geq 2$, and we have:

Problem 3.12. *Determine the homological finiteness properties of $\mathcal{N}_k(S_g)$.*

There is a symbiotic relationship between the theory of mapping class groups and the theory of $\text{Out}(F_n)$, the group of outer automorphisms of a free group. We can even think of $\text{Out}(F_n)$ as the mapping class group of a graph; indeed, it is the group of homotopy equivalences of a rank n graph Γ_n , up to homotopy. By considering the action of $\text{Out}(F_n)$ on $H_1(\Gamma_n, \mathbb{Z})$, we obtain an analogous Torelli group $\mathcal{I}(F_n)$:

$$1 \rightarrow \mathcal{I}(F_n) \rightarrow \text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z}) \rightarrow 1.$$

By 1934, Nielsen and Magnus had shown that \mathcal{T}_n is finitely generated [Nie24, Mag34]. Culler–Vogtmann proved in 1986 that $\text{vcd}(\text{Out}(F_n)) = 2n - 3$ (so $\text{cd}(\mathcal{I}(F_n)) \leq 2n - 3$) [CV86]. Smillie–Vogtmann showed, for $n > 2$ even and $2 < n < 100$, that $H_*(\mathcal{I}(F_n), \mathbb{Z})$ is not finitely generated [SV87a, SV87b]. In 1997, Krstić–McCool proved $\mathcal{I}(F_3)$ is not finitely presented [KM97].

In [BBM07], we use combinatorial Morse theory to prove the following two theorems, which are analogues of Theorems 3.3 and 3.5 above.

Theorem 3.13 (Bestvina–Bux–Margalit). *For $n \geq 2$, we have*

$$\text{cd}(\mathcal{I}(F_n)) = 2n - 4.$$

Theorem 3.14 (Bestvina–Bux–Margalit). *For $n \geq 3$, the group $H_{2n-4}(\mathcal{I}(F_n), \mathbb{Z})$ is infinitely generated.*

In the case $n = 3$, Theorem 3.14 implies the Krstić–McCool result.

To prove Theorem 3.14, we define a complex \mathcal{TM}_n called the “toy model”. This is a subcomplex of \mathcal{Y}_n , the Eilenberg–MacLane space for $\mathcal{I}(F_n)$ given by Culler–Vogtmann [CV86]. The toy model is homotopy equivalent to an $(n - 2)$ -fold product of a “swiss cheese space” (\mathbb{R}^2 minus open disks). We give a classical Morse theoretic argument that $H_k(\mathcal{TM}_n, \mathbb{Z})$ is not finitely generated for $1 < k \leq 2n - 4$, and we then show that $H_{2n-4}(\mathcal{TM}_n, \mathbb{Z})$ injects into $H_{2n-4}(\mathcal{Y}_n, \mathbb{Z})$. We believe that the last statement is true more generally.

Conjecture 3.15. *For $n \geq 3$, $H_k(\mathcal{TM}_n, \mathbb{Z})$ injects into $H_k(\mathcal{Y}_n, \mathbb{Z})$ for all k .*

An affirmative answer for $k = 2$ would imply that $\mathcal{I}(F_n)$ is not finitely presented.

For $n = 3$, we have two proofs of Theorem 3.13 (one in [BBM07] and one unpublished). Even so, we still do not know a generating set for $H_2(\mathcal{T}_3, \mathbb{Z})$.

Problem 3.16. *Find a generating set for $H_2(\mathcal{I}(F_3), \mathbb{Z})$.*

3.2. Pseudo-Anosovs and their dilatations. This section of the research proposal is related to the work in the following two papers.

[FLM] “Small dilatation pseudo-Anosovs and 3-manifolds,” with Benson Farb and Chris Leininger, *submitted*.

[FLM08] “The lower central series and pseudo-Anosov dilatations,”

with Benson Farb and Chris Leininger, *The American Journal of Mathematics* 130(3): 799–827, 2008.

Following Penner [Pen91], we consider, for any $H < \text{Mod}(S)$, the (discrete) set

$$\text{spec}(H) = \{\log(\lambda(f)) : f \in H \text{ is pseudo-Anosov}\} \subset \mathbb{R}$$

where $\lambda(f)$ is the dilatation of f (the asymptotic growth rate of lengths of curves). The number $\log(\lambda(f))$ is equal to the minimal topological entropy of a representative for f . Also, the set $\text{spec}(H)$ is the length spectrum of the cover of moduli space corresponding to H .

The set $\text{spec}(H)$ has a least element, which we call $L(H)$. Penner [Pen91] proved that $L(\text{Mod}(S_g)) \asymp 1/g$. In particular, if one considers the mapping class groups of surfaces of arbitrarily large genus, there are pseudo-Anosov elements with dilatation arbitrarily close to 1. Penner’s examples of such elements are products $\phi\rho$, where ϕ is supported on a genus 2 subsurface of S_g , and ρ is a rotation of order g ; see Figure 2.

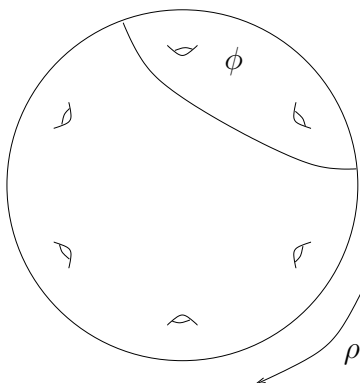


FIGURE 2. A construction of pseudo-Anosovs due to Penner; a model for the Symmetry Conjecture.

Problem 3.17. *Understand the structure of pseudo-Anosov mapping classes, in particular those of small/least dilatation.*

The PI proposes to continue the pursuit of this problem with Benson Farb (University of Chicago) and Chris Leininger (University of Illinois at Urbana–Champaign). The following conjecture, which is a central goal of the project, asserts that all small dilatation pseudo-Anosovs arise from Penner’s construction.

Conjecture 3.18 (Symmetry conjecture, Farb–Leininger–Margalit). *A pseudo-Anosov of least (or, “small”) dilatation is of the form $\phi\rho$, where ρ is periodic and ϕ is reducible and is supported on a subsurface of uniformly small complexity.*

There has been much work devoted to understanding the structure of small dilatation pseudo-Anosovs. However, still today, the least dilatations are unknown for $g \geq 2$.

In [FLM] and [FLM08], we discover two new universality phenomena for pseudo-Anosov dilatations, both of which lend credence to the conjecture. And, in [FLM], our results give an algorithm for computing the least dilatation in each genus. More precisely, we fix some P and consider

$$\Psi_P = \{\phi : S \rightarrow S : \chi(S) < 0, \phi \text{ pseudo-Anosov, and } \lambda(\phi) \leq P^{\frac{1}{|\chi(S)|}}\}.$$

It is important in the definition of Ψ_P that S is allowed to vary. By Penner's result, Ψ_P is an infinite set for large enough P .

Given a pseudo-Anosov homeomorphism $\phi : S \rightarrow S$, let $S_\phi^\circ = S^\circ \subset S$ be the surface obtained by removing the singularities of the stable and unstable foliations for ϕ and let $\phi|_{S^\circ} : S^\circ \rightarrow S^\circ$ denote the restriction. The set of pseudo-Anosov homeomorphisms

$$\Psi_P^\circ = \{\phi|_{S^\circ} : S^\circ \rightarrow S^\circ \mid (\phi : S \rightarrow S) \in \Psi_P\}$$

is therefore also infinite.

The main discovery in [FLM] is a universal finiteness phenomenon for all small dilatation pseudo-Anosovs: they are all “generated” by a finite number of examples. For the first statement, let $\mathcal{T}(\Psi_P^\circ)$ denote the homeomorphism classes of mapping tori of elements of Ψ_P° .

Theorem 3.19. *The set $\mathcal{T}(\Psi_P^\circ)$ is finite.*

McMullen [McM96] has given a construction of low dilatation pseudo-Anosovs which complements that of Penner. The idea is to start with a given pseudo-Anosov ϕ and to construct its mapping torus M_ϕ . By work of Thurston [Thu86], the mapping torus fibers in infinitely many ways, that is, we can write it as a mapping torus $\phi_k : S_k \rightarrow S_k$ for infinitely many genera k . McMullen showed that, by choosing the ϕ_k appropriately, we can obtain

$$\lim_{k \rightarrow \infty} \lambda(\phi_k) = 1.$$

We can say that the single mapping torus M_ϕ *generates* the low dilatation pseudo-Anosovs ϕ_k .

Theorem 3.19 is a converse to McMullen's construction—starting with all low-dilatation pseudo-Anosovs, we show there are finitely many mapping tori which generate them. We have the analogy:

$$\begin{array}{l} \text{Symmetry Conjecture : Penner's construction ::} \\ \text{Theorem 3.19 : McMullen's construction} \end{array}$$

Problem 3.20. *Find a set of mapping tori that generates Ψ_P° (“up to Dehn filling”). Find one mapping torus that generates least dilatation pseudo-Anosovs for infinitely many genera (such a mapping torus exists by Theorem 3.19).*

In [FLM08] we give a completely different approach towards understanding small dilatation pseudo-Anosovs. The universality phenomenon discovered in [FLM08] can be described as: algebraic complexity implies dynamical complexity

for pseudo-Anosov homeomorphisms. Here, algebraic complexity is determined by the action on $\pi_1(S)$, and dynamical complexity is specified by the dilatation. Specifically, we find the following contrast to Penner’s result.

Theorem 3.21 (Farb–Leininger–Margalit). *Suppose $g \geq 2$.*

$$.197 < L(\mathcal{I}(S_g)) < 4.127$$

The point is that the bounds are independent of g , i.e., $L(\mathcal{I}(S_g)) \asymp 1$.

The *Johnson filtration* is the sequence of groups

$$\mathcal{N}_k(S) = \text{kernel}(\text{Mod}(S) \rightarrow \text{Out}(\Gamma/\Gamma_k))$$

where Γ_k is the k^{th} term of the lower central series for $\Gamma = \pi_1(S)$. The following theorem exemplifies the theme of [FLM08].

Theorem 3.22 (Farb–Leininger–Margalit). *Given $k \geq 1$, there exist $M(k)$ and $m(k)$, where $m(k) \rightarrow \infty$ as $k \rightarrow \infty$, so that*

$$m(k) < L(\mathcal{N}_k(S_g)) < M(k)$$

for every $g \geq 2$.

If we fix the surface S , then a compactness argument shows that there exist M and m with the above properties. Again, the entire point is that the bounds are independent of the surface S . The idea is that if f is a pseudo-Anosov in a deep term of the Johnson filtration, then $i(c, f(c))$ is large for every curve c (independently of genus); for instance $f \in \mathcal{K}(S)$ implies $i(c, f(c)) \geq 4$.

As a consequence of our work, we can currently prove the weakest form of the Symmetry Conjecture: a pseudo-Anosov of least dilatation is the product of a periodic element and an element which fixes a single curve. It is not hard to see that “most” mapping classes cannot be written as such a product, so this result certainly represents progress.

Problem 3.23. *Show that $L(\mathcal{N}_{k+1}(S_g)) > L(\mathcal{N}_k(S_g))$. In particular, show that $L(\mathcal{K}(S)) > L(\mathcal{I}(S))$. What are the asymptotics of $L(\mathcal{N}_k(S_g))$?*

3.3. Symmetric mapping class groups and braid groups. The following two papers are in preparation.

[BMa] “Generating the hyperelliptic Torelli group,” with Tara Brendle.

[BMb] “The dimension of the hyperelliptic Torelli group,” with Tara Brendle.

A *hyperelliptic involution* of S_g is an order two homeomorphism with $2g + 2$ fixed points. For a subgroup G of $\text{Mod}(S)$, we can define the “hyperelliptic version” of G as the centralizer in G of some fixed hyperelliptic involution ι (the hyperelliptic involutions are all conjugate). For instance, we can consider the full

hyperelliptic mapping class group $\text{SMod}(S_g)$, or the hyperelliptic Torelli group $\mathcal{SI}(S_g)$.

The PI proposes to continue work with Tara Brendle of the University of Glasgow on the following problem.

Problem 3.24. *Determine the group theoretical properties of $\mathcal{SI}(S_g)$, for instance, finite generation, finite presentability, finite generation of homology groups, explicit generating sets, and abelianization.*

The quotient of S_g by the involution ι is a sphere $S_{0,2g+2}$ with $2g + 2$ cone points of order 2. A theorem of Birman–Hilden gives that

$$\text{SMod}(S_g)/\langle \iota \rangle \cong \text{Mod}(S_{0,2g+2}).$$

The groups $\text{Mod}(S_{0,2g+2})$ are closely related to the braid groups. Thus, we can think of $\mathcal{SI}(S_g)$ as a kind of Torelli group for the braid group.

It follows immediately from the work of Johnson that the symmetric Torelli group $\mathcal{SI}(S)$ is a subgroup of the symmetric Johnson kernel $\mathcal{SK}(S_g)$. That is, each element of $\mathcal{SI}(S_g)$ is a product of Dehn twists about separating curves. The following conjecture of R. Hain [Hai06] is much stronger. For the statement, say that a curve in S_g is *symmetric* if it is fixed by ι .

Conjecture 3.25 (Hain). *The symmetric Torelli group $\mathcal{SI}(S_g)$ is generated by the (infinite) set of Dehn twists about symmetric separating curves.*

In genus 2 the conjecture follows from Theorem 3.8, since

$$\mathcal{I}(S_2) = \mathcal{SI}(S_2) = \mathcal{SK}(S_2).$$

Hain has pointed out that, by an algebro-geometric count, namely, by counting even theta nulls in genus 3, Conjecture 3.25 implies a result of Johnson [Joh83] that $\mathcal{I}(S_3)$ is generated by 35 bounding pair maps. Also, the conjecture has the following equivalent, algebro-geometric formulation.

Conjecture 3.26 (Hain). *The branch locus of the period mapping, from Torelli space to the space of principally polarized abelian varieties, has simply connected components.*

We are attacking Hain’s conjecture by considering the action of $\mathcal{SI}(S_g)$ on the complex of curves for $S_{0,2g+2}$. This complex has one vertex for each isotopy class of essential simple closed curve in $S_{0,2g+2}$, and simplices for collections of disjoint curves. Our approach to Conjecture 3.25 consists of the following three steps:

Step 1. Give an algebraic characterization of $\mathcal{SC}(S_g)/\mathcal{SI}(S_g)$.

Step 2. Show that $\mathcal{SC}(S_g)/\mathcal{SI}(S_g)$ is simply connected.

Step 3. Induction: Show that the stabilizer in $\mathcal{SI}(S_g)$ of a vertex of $\mathcal{SC}(S_g)$ is generated by Dehn twists about symmetric separating curves.

So far, we have completed proofs of Steps 1 and 3. In terms of verifying Hain’s conjecture, Step 3 held the most mystery, and in proving it, we have discovered some new, deep relations between Dehn twists in $\mathcal{I}(S_g)$.

Conjecture 3.27 (Brendle–Margalit). *The quotient $\mathcal{SC}(S_g)/\mathcal{SI}(S_g)$ is simply connected.*

It follows from work of Armstrong [Arm65] that if a group acts (without rotations) on a simply connected complex, then the quotient is simply connected if and only if the group is generated by vertex stabilizers. Thus:

Conjecture 3.27 \implies Conjecture 3.25

As part of our work, we give an elementary proof of a result of A’Campo [A’C79] that, using deep work of J. Tits, gives a completely algebraic characterization of the image of $\text{SMod}(S_g)$ in $\text{Sp}(2g, \mathbb{Z})$.

Theorem 3.28 (Brendle–Margalit). *For $g \geq 2$, we have:*

$$\text{cd}(\mathcal{SI}(S_g)) = g - 1$$

Putting Theorem 3.28 together with the above theorems and conjectures, we have the following picture:

$$\begin{aligned} \text{vcd}(\text{Mod}(S_g)) &= 4g - 5 \\ \text{cd}(\mathcal{I}^k(S_g)) &= 4g - 5 - k \quad (\text{conjecture}) \\ \text{cd}(\mathcal{I}(S_g)) &= 3g - 5 \\ \text{cd}(\mathcal{K}(S_g)) &= 2g - 3 \\ \text{cd}(\mathcal{N}_k(S_g)) &= ??? \quad (\text{Problem 3.12}) \\ \text{cd}(\mathcal{SI}(S_g)) &= g - 1 \end{aligned}$$

Theorem 3.29 (Brendle–Margalit). *For $g \geq 2$, the group $H_{g-1}(\mathcal{SI}(S_g))$ is infinitely generated.*

In particular, Theorem 3.29 shows that $\mathcal{SI}(S_3)$ is not finitely presented.

The following problem might shed some light on the algebraic structure of $\mathcal{SI}(S_g)$, and also on the second Johnson homomorphism.

Problem 3.30. *Determine the image of $\mathcal{SI}(S_g)$ under the second Johnson homomorphism.*

The following question suggests a far-reaching generalization of Hain’s conjecture.

Question 3.31 (Brendle–Margalit). *Let G be a finite subgroup of $\text{Mod}(S_g)$. Is it true that the centralizer in $\mathcal{I}(S_g)$ of G is generated by Dehn twists about separating curves that are fixed by each element of G ?*

The next question suggests a generalization of Mess’s Theorem 3.8.

Question 3.32 (Brendle–Margalit). *Is it true that $\mathcal{SI}(S_g)$ is generated by a set of Dehn twists, with one Dehn twist for each symmetric symplectic splitting of $H_1(S_g, \mathbb{Z})$? Are there any relations besides the obvious commutation relations?*

The work in [BM04] suggests the following:

Question 3.33. *Is it true that the automorphism group of $\mathcal{SI}(S_g)$ (more generally, the abstract commensurator group) is isomorphic to $\text{SMod}(S_g)$?*

Question 3.34. *Are there any nontrivial elements in the intersection of the group $\text{SMod}(S_g)$ and a deeper term of the Johnson filtration?*

4. EDUCATIONAL PLAN

The PI has been deeply involved in education, as evidenced, for example, by teaching prizes won at both Chicago and Utah, by hands-on work with the Chicago public schools, and by the PI's book projects. In the educational part of this proposal, we discuss several proposed, completed, and ongoing projects that further demonstrate the PI's commitment to pedagogical issues and outreach activities. The centerpiece is a summer workshop designed to give budding topologists a toolkit for navigating a career in mathematics.

4.1. Professional development for beginning topologists. We propose to develop a one week workshop at Tufts for 15 beginning graduate students in the areas of Geometric Group Theory and Topology. The workshop, which would be held during the first and fourth years of the duration of the award, could be called "Things I Wish I Knew When I Was a Graduate Student." It is typical in graduate school that students do not get sufficient training on the nuts and bolts of succeeding in academics. This shortfall causes many otherwise talented students—in particular, those of underrepresented groups—to drop out or become disaffected. Since underrepresented groups are the most in need (by definition), the aim would be to have many of the participants be from underrepresented groups.

Through panel discussions, workshops, and lectures, we propose to cover the following topics:

- (1) Choosing an advisor, and a research topic
- (2) Giving effective research presentations
- (3) Teaching Calculus
- (4) Study habits, reading papers, arXiv
- (5) Work–life balance, raising a family
- (6) Writing papers, writing abstracts, writing grant proposals, LaTeX
- (7) Career options
- (8) Attending conferences, networking, collaborating
- (9) Geometric Group Theory and Topology

For example, the topic of career options would best be addressed via panel discussions. Possible panelists would be from academia, finance, private high schools, the consulting industry, the NSF, and the NSA. For the topic of giving research talks, the workshop format would be most appropriate. The attendees of the

conference would be expected to give a 20 minute talk, which would be critiqued by their peers, a panel of experts (both communications experts as well as mathematicians), and themselves (with the aid of a video recorder). As a benefit, the other young topologists at the conference would learn about the research areas of their future colleagues and collaborators. For the lectures, we would bring in top researchers in the area to explain their cutting edge research. This would serve to be both informational and motivational. Special care would be taken to choose speakers whose research and speaking styles are to be imitated. We also propose to have one or two social events, so the attendees would have a chance to mingle with the outside speakers and each other.

The most important differences between the proposed workshop and similar programs, such as EDGE, IMMERSE, and Cornell's Summer Math Institute are:

- (1) Timing: The aforementioned programs all take place between college and graduate school, and we are unaware of any such programs specifically designed for the middle of graduate school. However, this is the most trying time in graduate school, where many students decide to quit, or leave with a masters degree. Also, most of the topics in the current proposal only become relevant at this point of graduate school.
- (2) Audience: The proposed workshop is aimed specifically at beginning topologists. This gives the discussions the dual purpose of conveying mathematics, as well as encouraging proper technique.
- (3) Focus: In the proposed program, the non-mathematical topics are given equal importance to the mathematical topics. We hope to help build the students' sense of confidence in these areas.

In order to evaluate the program, surveys will be given to the participants at the end of the workshop. Also, participants will be asked to provide "progress reports" in each of the following 3-4 years, to make it possible to see the long-term effects of the training.

In terms of dissemination, we plan to ask each of the presenters, panelists, and workshop leaders to write essays on the above topics. We plan to disseminate the collection of essays on the internet. Currently, there are very few widely available professional development materials for mathematicians.

4.2. Outreach activities. At Chicago, the PI was involved in the CNA Math-achievement program with Prof. Robert Fefferman. For this program, the PI was paired with several of the most in-need public schools in Chicago's impoverished South Side. The PI came to school one day a week and worked with various mathematics classrooms. The PI also worked with Prof. Paul Sally on his SESAME program, whose goal is to train the elementary school teachers of Chicago's public school system.

At Tufts, the PI is already building a strong relationship with the Center for STEM Diversity. This institution is devoted to promoting the diversity of

students in the fields of Science, Technology, Engineering, and Mathematics. The PI has recently participated in a recruitment program for undergraduates from underrepresented backgrounds. This fall, as the coordinator of our first semester Calculus course, the PI will be working with the Center for STEM Diversity on developing a study program for struggling students.

4.3. Undergraduate education. This summer, the PI is taking the initiative to develop an online component for the first semester Calculus class at Tufts. The PI has had experience in this area, developing online courseware with Professor Thomas Banchoff at Brown University as an undergraduate, having taught with this software at the University of Utah, and having been an early adopter of WebWork at Utah. The PI is currently working with Elena Murray, an undergraduate at Tufts on building a WebWork site for Tufts, which is a way to assign online homework to our students.

This fall, the PI will be teaching a course on Knot Theory, which will be a variant of an REU-style course he gave at Utah. The course will make integral use of computers, in particular for online discussion and problem solving, and for software such as SnapPea and Knot Plot. So far, one-third of the students enrolled are women.

At Utah, the PI led a team of three that developed the “Calculus Carnival.” This event was designed to bring Calculus students at all levels into a forum where math was engaging and fun. Over 180 students participated, including about 65 of the PI’s own students, and it was obvious how much fun the students had. The PI wrote an article, “A Carnival of Calculus,” which appeared in the MAA Focus magazine, and which has inspired educators the contact the PI about running similar events, for instance at Cherry Creek in Colorado and at Gordon College.

The PI has given a number of successful undergraduate colloquia on the mathematics of juggling, most recently at the University of Utah, Brown University, and at a reception for newly admitting students and parents at Tufts. In these talks, the concepts of the Squeeze Theorem, Möbius inversion, and braid groups are applied in the context of juggling.

This summer the PI is working with Diane Herrmann of the University of Chicago, and David Schmitz of North Central College on reworking the MAA guidelines for undergraduate programs. A key component is the recommendations for broadening the diversity.

4.4. Graduate education. In the PI’s first year at Tufts, he ran a topics course for graduate students called “Dynamics of surface of homeomorphisms.” The text was the translation of “Travaux de Thurston sur les surfaces”, written by DJun Kim and the PI. A key part of the course was a collaboration with the Dynamics course taught by Boris Hasselblatt. The semester ended with a “mini-conference”

with the students from both courses. This was so successful, that plans have been made to continue this idea next year at Tufts.

Last summer, the PI participated in the conference “Examples of Groups,” held at Ohio State and run by Indira Chatterji as part of her NSF CAREER award. This was a program for women in mathematics. This summer, the PI is running a research seminar on a paper of Bestvina–Handel, and two women, Anna Vijayan and Yulan Qing, are participating.

4.5. Book projects. The PI is currently in the final stages of two book projects. The first, “A Primer on Mapping Class Groups,” with the PI’s advisor Benson Farb, is an introductory text on mapping class group. The book, which is submitted but not yet published, has already been used by individuals around the world, and for courses at many schools, including UC Davis, Indiana, Utah, Rice, Brandeis, and Maryland, to name a few. The PI has used the book in independent studies with graduate students at Utah and Tufts.

The other book project is a translation into English of the text “Travaux de Thurston sur les surfaces,” with Djun Kim. This book has been accepted for publication by Princeton University Press. The PI has received hundreds of requests for the translation over the years, from many countries.

4.6. Broader impacts. There are several ways in which the proposed activities will impact the broader community. The proposed summer workshop will give second and third year topology students from underrepresented groups important tools and confidence needed for success in mathematics. Women and underrepresented groups are making up more and more of the graduate school applicant pool (certainly at Tufts) and the nation’s work force, and it is vital that we give all students equal opportunity.

Also, the collection of essays, written by the workshop leaders and panelists, will be a help to all young mathematicians, as currently there are few resources on professional development for mathematicians at this particular stage in their careers. As a long term goal, the PI hopes to work with graduate programs to invest more effort in covering the topics of the workshop as a matter of course.

The PI is also working with female mathematicians at all levels. One third of the students in his Knot Theory course in Fall 2009 are women. Two female graduate students are currently working with the PI on a summer research project. Also, part of the proposed research is with a female collaborator. Also, the proposed research has a strong international component as well, as two of his collaborators have positions in Scotland and Germany.

Both book projects have had a substantial impact already, as many students and reading groups have been using the books to learn about the subjects of mapping class groups and surface dynamics. As a consequence, the PI receives frequent emails from beginning students, many international, with questions on these subjects.

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