Research Prospectus

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My research is on the group theoretical, combinatorial, and dynamical aspects of the mapping class group, which is the group of homeomorphisms of a topological surface, modulo isotopy. We think of the mapping class group as the group of symmetries of a surface. One reason that the mapping class group is important is that it connects in a deep way to many other areas of mathematics, including dynamics, 3- and 4-manifold theory, algebraic geometry, group theory, representation theory, number theory, algebraic topology, and complex analysis, to name a few.

Dehn and Nielsen studied the mapping class group from the algebraic and geometric points of view in the early part of the 20th century. The richness of the subject today is due in large part to the work of William Thurston on the classification of surface homeomorphisms in the 1970s.

My work draws techniques and ideas from—and answers questions raised by—the work of Dehn, Nielsen, Birman, McMullen, Thurston, and others, and makes use of Morse theoretic arguments à la Bestvina–Brady, combinatorial methods of Ivanov, and the tools of geometric group theory, representation theory, 3-manifold theory, and Teichmüller geometry.

My program of research has six prongs. The first concerns the finiteness properties of an important subgroup of the mapping class group called the Torelli group. The second investigates representations of braid groups induced by the mapping class group. The third direction is about the symmetries of the mapping class group itself, that is, the symmetries of the symmetries of a surface. The fourth studies the dynamics of individual elements of the mapping class group. The fifth is on relations in the mapping class group. Finally, the last concerns the theory of 4-manifolds, in particular Lefschetz fibrations and surface bundles over surfaces.

1. Finiteness properties of Torelli groups

The mapping class group $\mathrm{MCG}(S_g)$ of a surface S_g of genus g has a symplectic representation. The kernel $\mathcal{I}(S_g)$ is called the Torelli group:

$$1 \to \mathcal{I}(S_g) \to \mathrm{MCG}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1.$$

We think of the Torelli group $\mathcal{I}(S_g)$ as capturing the non-arithmetic (or, more mysterious) aspects of the mapping class group $\mathrm{MCG}(S_g)$.

A fundamental open problem is to determine if the Torelli group is finitely presentable when $g \geq 3$; in other words, does the Torelli group have a finite algebraic description?

A more general, and much-studied, problem is to determine which of the homology groups $H_k(\mathcal{I}(S_g))$ are finitely generated. Bestvina, Bux, and I determined that the cohomological dimension of $\mathcal{I}(S_g)$ is $3g-5$ (in particular, all homology groups $H_k(\mathcal{I}(S_g))$ are all trivial for $k > 3g-5$), answering a 20 year old question of Mess and Farb.

Theorem 1.1. *For $g \geq 2$, we have* $\mathrm{cd}(\mathcal{I}(S_g)) = 3g - 5$.

We further showed that $H_{3g-5}(\mathcal{I}(S_g))$ is infinitely generated, sharpening a theorem of Akita, answering a question of Mess from Kirby's problem list, and solving a problem of Farb.

Theorem 1.2. *For $g \geq 2$, the group $H_{3g-5}(\mathcal{I}(S_g))$ is infinitely generated.*

We in particular recover, and significantly extend, a celebrated theorem of Mess which says that $\mathcal{I}(S_2)$ is an infinitely generated free group.

We also studied the subgroup $\mathcal{K}(S_g)$ of $\mathcal{I}(S_g)$ that is generated by Dehn twists about separating curves. Johnson showed that $\mathcal{K}(S_g)$ has infinite index in $\mathcal{I}(S_g)$. We proved that the cohomological dimension of $\mathcal{K}(S_g)$ is $2g-3$, also answering a question of Farb.

Theorem 1.3. *For $g \geq 2$, we have* $\mathrm{cd}(\mathcal{K}(S_g)) = 2g - 3$.

The main contribution is a new combinatorial model for $\mathcal{I}(S_g)$ called the complex of minimizing cycles. This idea inspired subsequent papers by Hatcher and Irmer, as well as a paper of Hatcher and myself, where we give a simple proof of the classical theorem of Birman and Powell that the Torelli group is generated by bounding pair maps.

Analogous to $\mathrm{MCG}(S_g)$ is the group $\mathrm{Out}(F_n)$ (the group of outer automorphisms of a free group). We can think of $\mathrm{Out}(F_n)$ as the mapping class group of a graph. There is a corresponding Torelli group $\mathcal{I}(F_n)$, first studied by Nielsen and Magnus. Bestvina, Bux, and I showed the following.

Theorem 1.4. *For $n \geq 3$, we have* $\mathrm{cd}(\mathcal{I}(S_n)) = 2n - 4$.

Theorem 1.5. *For $n \geq 3$, the group $H_{2n-4}(\mathcal{I}(S_n))$ is infinitely generated.*

This answers a question of Bridson and Vogtmann, and extends/sharpens a theorem of Smillie and Vogtmann. In the case $n = 3$, the last result generalizes the famous theorem of Krstić and McCool that $\mathcal{I}(F_3)$ is not finitely presented. We also give a geometric proof of the seminal theorem of Magnus that $\mathcal{I}(F_n)$ is finitely generated.

Despite the similarity in the statements of our theorems about $\mathcal{I}(S_g)$ and $\mathcal{I}(F_n)$, the approaches are decidedly orthogonal. For instance, the complex of minimizing cycles for a graph is a single point. On the other hand, the proof for $\mathcal{I}(F_n)$ makes use of combinatorial Morse theory, applied to Culler–Vogtmann's Outer space; and analogous Morse functions on Teichmüller space do not yield our theorem about $\mathcal{I}(S_g)$.

2. The Burau representation and the Torelli group

The braid groups B_n are isomorphic to the mapping class groups of punctured disks. What is more, there are injective homomorphisms $\mathrm{B}_n \to \mathrm{MCG}(S_g)$. As a direct consequence, we obtain a symplectic representation of $\mathrm{B}_{2g+1} \to \mathrm{Sp}(2g, \mathbb{Z})$. This representation is nothing other than the Burau representation evaluated at $t = -1$. The kernel (modulo its center) is isomorphic to the hyperelliptic Torelli group $\mathcal{SI}(S_g)$ the centralizer in the Torelli group of a hyperelliptic involution.

Hain conjectured that $\mathcal{SI}(S_g)$ is generated by Dehn twists. At first Hain's conjecture seemed overly optimistic: one cannot expect that an infinite index subgroup of $\mathcal{I}(S_g)$ is generated by the generators of $\mathcal{I}(S_g)$ lying in that subgroup. Worse, there are other simple elements of $\mathcal{SI}(S_g)$, and at first it was not clear how to factor them into Dehn twists that lie in $\mathcal{SI}(S_g)$.

I started working on Hain's conjecture in 2005, first breaking the problem into three steps, and then completing the steps between 2005 and 2013 in two papers with Brendle and one paper with Brendle and Putman. In fact, we proved a theorem that is stronger than Hain conjectured.

Theorem 2.1. *For $g \geq 0$, the group $\mathcal{SI}(S_g)$ is generated by Dehn twists about separating curves that are preserved by the hyperelliptic involution and have genus at most two.*

As a consequence of Theorem 2.1, we obtain topological information about an algebro-geometric space, the branch locus of the period mapping from Torelli space to the Siegel upper half-plane. Let $\mathcal{H}_g^c$ denote the space obtained from this branch locus by adjoining curves of compact type.

Theorem 2.2. *For $g \geq 0$, the space $\mathcal{H}_g^c$ is simply connected.*

Brendle, Childers, and I proved two other theorems about $\mathcal{SI}(S_g)$.

Theorem 2.3. *For $g \geq 1$, we have* $\mathrm{cd}(\mathcal{SI}(S_g)) = g - 1$.

Theorem 2.4. *For $g \geq 2$, the group $H_{g-1}(\mathcal{SI}(S_g))$ is infinitely generated.*

In particular, $\mathcal{SI}(S_3)$ is not finitely presented. It it not known if $\mathcal{SI}(S_g)$ is finitely generated (or has finitely generated first homology) for $g \geq 3$.

There is a mod m version of $\mathcal{SI}(S_g)$: the level m subgroup of the braid group is the kernel of the composition

$$\mathrm{B}_{2g+1} \to \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z}).$$

Arnol'd proved that the level 2 subgroup of the braid group is exactly the pure braid group. Brendle and I prove the following.

Theorem 2.5. *For any n, the level 4 subgroup of B_n is equal to the subgroup generated by squares of Dehn twists, and is also equal to PB_n^2.*

3. Algebraic and geometric models for mapping class groups

Ivanov proved that $\operatorname{Aut}\mathrm{MCG}(S_g) \cong \mathrm{MCG}(S_g)$ when $g \geq 3$. He further proved that the abstract commensurator of $\mathrm{MCG}(S_g)$ (the group of isomorphisms between finite index subgroups of $\mathrm{MCG}(S_g)$) is again $\mathrm{MCG}(S_g)$. His method was to translate the problem into a combinatorial topology statement about the complex of curves, a combinatorial model for $\mathrm{MCG}(S_g)$.

The complex of curves has one vertex for each isotopy class of curves in S_g, and Ivanov's theorem is that its automorphism group is $\mathrm{MCG}(S_g)$. By restricting attention to separating curves, Brendle and I showed the group $\mathcal{K}(S_g)$, the subgroup of $\mathrm{MCG}(S_g)$ generated by Dehn twists about separating curves, also has automorphism group and abstract commensurator group isomorphic to $\mathrm{MCG}(S_g)$, confirming a conjecture of Farb. So the entire algebraic structure of $\mathrm{MCG}(S_g)$ is determined by this small subgroup.

Theorem 3.1. *For $g \geq 3$, we have*

$$\mathrm{Comm}(\mathcal{K}(S_g)) \cong \mathrm{Aut}(\mathcal{K}(S_g)) \cong \mathrm{MCG}(S_g).$$

Among the consequences of this work, we recover the theorem of Farb–Ivanov that the abstract commensurator of $\mathcal{I}(S_g)$ is $\mathrm{MCG}(S_g)$. Our main theorem was generalized to non-closed surfaces by Kida. Our results were extended to the terms of the Johnson filtration—even smaller subgroups—by Bridson, Pettet, and Souto.

In my thesis, I proved another version of Ivanov's theorem about automorphisms of the complex of curves, namely, that the automorphism group of the pants complex $\mathcal{P}(S)$ is again $\mathrm{MCG}(S)$. In the process, I showed that the 2-skeleton of $\mathcal{P}(S)$ is completely encoded in the 1-skeleton.

Theorem 3.2. *If S is any hyperbolic surface, then $\mathrm{Aut}(\mathcal{P}(S)) \cong \mathrm{MCG}(S)$.*

Another result that Ivanov obtained using his theorem about the complex of curves is Royden's theorem that the isometry group of Teichmüller space is isomorphic (again) to $\mathrm{MCG}(S)$. Brock and I used the last theorem to give a new proof of the theorem of Masur and Wolf that the isometry group of Teichmüller space with the Weil–Petersson metric is $\mathrm{MCG}(S)$. Our proof covers all remaining cases left open by Masur and Wolf.

In response to these works and others, Ivanov formulated the following.

Metaconjecture. *Every object naturally associated to a surface S and having a sufficiently rich structure has $\mathrm{MCG}(S)$ as its group of automorphisms.*

Some care is needed: there are examples of curve complexes that admit exchange automorphisms, automorphisms that interchange two vertices and fix the rest. Such automorphisms clearly do not come from $\mathrm{MCG}(S_g)$.

A forthcoming paper with Brendle resolves Ivanov's metaconjecture for a wide class of simplicial complexes. I will concentrate here on a special case that gives the main idea. Let A be a finite list of compact, connected surfaces. Let $\mathcal{D}_A(S_g)$ be the graph whose vertices are isotopy classes of nonseparating subsurfaces of S_g homeomorphic to some element of A, and whose edges connect vertices with disjoint representatives. Define $\kappa(A)$ to be the smallest integer so that each element of A can be realized as a subsurface of $S^1_{\kappa(A)}$.

Theorem 3.3. *For A a set of compact, connected surfaces, the natural map*

$$\mathrm{MCG}(S_g) \to \mathrm{Aut}(\mathcal{D}_A(S_g))$$

is an isomorphism if $g \geq 3\kappa(A) + 1$.

Our theorem is the first of its kind to treat more than one complex with a single argument. In the spirit of Ivanov's metaconjecture, our proof passes through Ivanov's original theorem (in fact, the proof passes through a sequence of complexes interpolating between $\mathcal{D}_A(S_g)$ and $\mathcal{C}(S_g)$).

The general version of our theorem allows for connected subsurfaces of S_g that are separating in S_g. Also, it gives a necessary *and* sufficient condition for the resulting complex to have automorphism group isomorphic to $\mathrm{MCG}(S_g)$, namely, that the complex admits no exchange automorphisms.

Using our theorem we make progress towards the algebraic version of the metaconjecture: any sufficiently rich subgroup of $\mathrm{MCG}(S_g)$ has abstract commensurator isomorphic to $\mathrm{MCG}(S_g)$. A special case is the aforementioned theorem announced by Bridson–Pettet–Souto.

Using Ivanov's approach via curve complexes, Leininger and I computed the abstract commensurator of the braid group, Behrstock and I computed the abstract commensurators of mapping class groups of genus 1 surfaces (this is the only place where there are non-geometric commensurators), and Bell and I classified injective maps between various Artin groups of finite type, generalizing work of Charney–Crisp.

In a somewhat different direction, Birman, Menasco, and I studied the geometry of the complex of curves. By the seminal work of Masur and Minsky, the complex of curves is a hyperbolic space of infinite diameter. Leasure, Shackleton, and Webb gave algorithms for computing the distance between two vertices, but the running times were too long to be practical. We gave a new simple approach to computing the distance. Our main contribution is a new class of geodesics connecting all pairs of vertices, called efficient geodesics, which are analogous to the tight geodesics of Masur and Minsky. As one application of our methods, Menasco led a group of REU students to show that the minimum intersection number for two vertices of distance four in genus two is twelve.

4. Dynamical aspects of pseudo-Anosov mapping classes

The Nielsen–Thurston classification theorem says that every homeomorphism of a surface is homotopic to one in a certain canonical form: there is a (possibly empty) invariant 1-submanifold, and on each complementary component the homeomorphism either acts as a periodic map or a pseudo-Anosov map, that is, a homeomorphism that is locally modeled (away from a finite set) on the action of a hyperbolic element of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$.

A basic measure of complexity of a pseudo-Anosov map is its topological entropy. This number measures the amount of mixing being effected. A central open question is: which real numbers are entropies of pseudo-Anosov maps of surfaces?

For a fixed surface, the set of entropies of pseudo-Anosov maps is closed and discrete; in particular, there is a smallest one. For $H \leqslant \mathrm{MCG}(S_g)$, write $L(H)$ for the smallest entropy of a pseudo-Anosov element of H.

Penner showed that if we allow the genus of our surface to go to infinity we can find pseudo-Anosov maps with smaller and smaller entropies. More precisely, he gave the asymptotics

$$L(\mathrm{MCG}(S_g)) \asymp \frac{1}{g}.$$

Farb, Leininger, and I set out to show that $L(\mathcal{I}(S_g))$ goes to zero at a slower rate than $L(\mathrm{MCG}(S_g))$. What we found is much stronger: the set of entropies in the collection of all pseudo-Anosov elements of $\mathcal{I}(S_g)$ is bounded away from zero, independently of g.

Theorem 4.1. *For $g \geq 2$, we have $L(\mathcal{I}(S_g)) \asymp 1$.*

We can generalize this result in two directions. First, we can consider $L(\mathrm{MCG}(S_g), k)$ the smallest entropy of a pseudo-Anosov element of $\mathrm{MCG}(S_g)$ fixing a subspace of $H_1(S_g)$ of dimension at least k. With Agol and Leininger, I recently proved the following, answering a question of Ellenberg.

Theorem 4.2. *For $g \geq 2$ and $0 \leq k \leq 2g$, we have*

$$L(\mathrm{MCG}(S_g), k) \asymp (k+1)/g.$$

This interpolates between Penner's result ($k = 0$) and our result on the Torelli group ($k = 2g$).

In another direction, there is a filtration of $\mathrm{MCG}(S_g)$ called the Johnson filtration. The kth term $\mathcal{N}_k(S_g)$ is the kernel of the action on $\pi_1(S_g)$ modulo the kth term of its lower central series. The first three terms are $\mathrm{MCG}(S_g)$, $\mathcal{I}(S_g)$, and $\mathcal{K}(S_g)$. We have the following theorem with Farb and Leininger.

Theorem 4.3. *Given $k \geq 1$, there exist constants m_k and M_k, with $m_k \to \infty$ as $k \to \infty$, so that*

$$m_k \leq L(\mathcal{N}_k(S_g)) \leq M_k$$

for all $g \geq 2$.

The point here is that m_k and M_k are independent of g, in contrast to Penner's theorem above. We can paraphrase the theme as: algebraic complexity implies dynamical complexity.

Farb, Leininger, and I also proved a theorem that relates small entropies to 3-manifolds. Fix some $L > 0$ and consider the set

$$\Psi(L) = \{\phi : S \to S \text{ pseudo-Anosov} \mid \text{entropy}(\phi) < L/|\chi(S)|\};$$

in this definition, S ranges over all surfaces. Penner's result implies that, for L large enough, this set of *small-entropy pseudo-Anosov maps* is infinite. Any $\phi \in \Psi(L)$ gives rise to a 3-manifold M_ϕ, its mapping torus. Denote by M_ϕ° the 3-manifold obtained from M_ϕ by deleting the orbit of each singular point of ϕ under the suspension flow.

Theorem 4.4. *Fix $L > 0$. The set $\{M_\phi^\circ \mid \phi \in \Psi(L)\}$ is finite.*

In other words, the infinite set of small-entropy pseudo-Anosov maps is "generated" by (or, flow-equivalent to) a finite set of examples (after deleting the singular sets). This answers a question posed by McMullen. Our work inspired a paper by Agol, who (among other things) gave a new proof of our theorem and a paper by Algom-Kfir and Rafi, who gave a version of our theorem for $\mathrm{Out}(F_n)$.

Another point of view is that pseudo-Anosov maps of a surface correspond to closed geodesics in the corresponding moduli space. The length of this geodesic with respect to the Teichmüller metric is the entropy of the pseudo-Anosov map. Building on Theorem 4.4, Leininger and I obtained a description of the location of these geodesics in moduli space.

Let $\mathcal{M}_{g,[\epsilon,R]}$ denote the subset of moduli space $\mathcal{M}_g$ consisting of all surfaces with injectivity radius lying in $[\epsilon, R]$. Let $\mathcal{G}_g(L)$ denote the set of geodesics in $\mathcal{M}_g$ corresponding to elements of $\Psi(L)$; these are the L-short geodesics in moduli space.

Theorem 4.5. *Let $L > 0$. There exists $R > \epsilon > 0$ so that, for each $g \geq 1$, each element of $\mathcal{G}_g(L)$ lies in $\mathcal{M}_{g,[\epsilon,R]}$.*

We are currently working on the following Symmetry Conjecture, which would give yet another point of view on the small-entropy pseudo-Anosov maps: any pseudo-Anosov map realizing the smallest entropy for a given surface can be decomposed as a homeomorphism supported on a subsurface of uniformly small complexity multiplied by a finite order homeomorphism. Such a theorem would show that Penner's examples are universal.

5. Relations in the mapping class group

I have written a number of other papers on the group structure of the mapping class group and the braid group. For instance:

Theorem 5.1. *Let $a, b \in \mathrm{PB}_n$. Then $\langle a, b\rangle$ is either abelian or free.*

This answers a question of Paris, and the proof uses the theories of hyperbolic manifolds and group actions on trees. In the theory of mapping class groups, one only expects such a theorem "up to powers," that is, one expects that two elements have high powers that either commute or generate a free group. In this case the power required is 1.

The last theorem shows that the pure braid group satisfies a property that is common to all right-angled Artin groups, so one is led to try to understand the extent to which braid groups and mapping class groups are similar to right-angled Artin groups. Matt Clay, Chris Leininger, and I showed that (aside from the obvious exceptions) braid groups and mapping class groups are never quasi-isometric to right-angled Artin groups.

Schleimer and I showed that Dehn twists about nonseparating curves in S_g are not primitive. In particular, they have nontrivial $(2g-1)$st roots. This simple fact came as a surprise to all of the experts we polled. Our work inspired papers by McCullough–Rajeevsarathy, who showed that our examples of roots realize the maximal possible degree, and by Hirose, who studied the analogous problem in the handlebody group.

In my thesis, I showed that if a relation between Dehn twists has the algebraic form of a lantern relation, then the curves in the relation must be configured as in the usual lantern relation.

Theorem 5.2. *If the product of two non-commuting Dehn twists T_x and T_y is equal to a multitwist, then $i(x, y) = 2$ and $\hat{i}(x, y) = 0$.*

This theorem has had several applications, for instance in the study by Aramayona–Souto on homomorphisms between mapping class groups and in the study of planar open books by Kaloti and by Van Horn-Morris and Plamenevskaya.

Finally, in a seminar at Columbia in the early 1980s, Dennis Johnson asked whether the group J generated by simply intersecting pair maps in S_2 is equal to the whole Torelli group $\mathcal{I}(S_2)$. Brendle and I gave a suite of new relations in the Torelli group and answered Johnson's question in the negative.

Theorem 5.3. *$\mathcal{I}(S_2)/J \cong \mathbb{Z}$.*

6. Lefschetz fibrations and surface bundles

One widely studied class of 4-manifolds are the Lefschetz fibrations. These are 4-manifolds equipped with a smooth map onto a surface which is a submersion except at a finite number of critical points (of a certain specific form). Donaldson showed that, after blow-ups, every compact symplectic 4-manifold admits the structure of a Lefschetz fibration. In the case where the critical locus is empty, a Lefschetz fibration is a surface bundle over a surface.

There are two natural ways to build new surface bundles from old ones, namely the fiber sum operation and the section sum operation. The surface bundles that are indecomposable in either sense can be viewed as the building blocks of surface bundles.

Theorem 6.1. *For every $h \geq 2$ and $g \geq 2$ there exist infinitely many 4-manifolds that have the structure of a genus g surface bundle over a genus h surface which is both fiber sum and section sum indecomposable.*

Baykur and I also constructed explicit examples of Lefschetz fibrations and surface bundles over surfaces with monodromy in the Torelli group (Smith proved that there are no such fibration when the base genus is zero). As a byproduct, we give the first higher-genus examples of Lefschetz fibrations that are not fiber sums of holomorphic fibrations (Smith and Stipsicz constructed examples where the base genus is zero).