Acylindricity in Higher Rank

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(joint work with T.Fernos)

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Goal: Extend the study to products of hyperbolic spaces

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Definition (Acylindrical action)

The action is *acylindrical* if for every $\epsilon > 0$, there exist constants $R = R(\epsilon), N = N(\epsilon)$ such that for any pair points $x, y \in X$ with $d(x, y) \geq R$, we have that

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AU-acylindricty refers to "Ambiguous Uniformity"

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Study homomorphism $\rho : \Gamma \to \mathrm{Aut}(\mathbb{X})$ such that the action is AU-acylindrical

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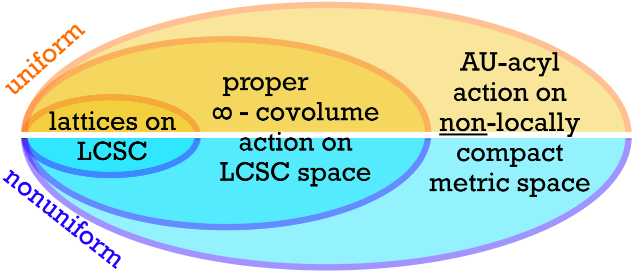
6. Colorable HHGs
7. Burger-Mozes groups, Bestvina-Brady kernels
8. ...



A semi-simple world: from proper finite covolume actions to AU-acylindricity

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Corollary

Let Γ be a group acting acylindrically on a hyperbolic space X. Then each isometry is elliptical or loxodromic.

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Definition (Regular elements)

Let $\rho : \Gamma \to \mathrm{Aut}(\mathbb{X})$ be a homomorphism. An element $g \in \Gamma$ is called *regular* if $\rho(g)$ acts as a loxodromic element in every factor.

Loxodromics to regular elements

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Let Γ act on a space (X, d). Recall that for a loxodromic element γ, its *elementary subgroup* $E_\Gamma(\gamma) := \{g \in \Gamma \mid d_{Hau}(l, gl) < \infty\}$, where l is the quasigeodesic axis of γ.

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Theorem (DGO)

Let Γ be a group acting acylindrically on a hyperbolic space X. Then for any loxodromic element $\gamma \in \Gamma$, $E_\Gamma(\gamma)$ is virtually $\mathbb{Z}$.

Given $\Gamma \to Aut(\mathbb{X})$ and $\gamma \in \Gamma$ a regular element, the associated *elementary subgroup* is

$$E_\Gamma(\gamma) := E_\Gamma(\gamma^-, \gamma^+) := \bigcap_{i=1}^{D} \mathrm{Fix}_\Gamma\{\gamma_i^-, \gamma_i^+\}$$

where $\gamma^-, \gamma^+ \in \partial_{reg}\mathbb{X}$ are respectively the repelling and attracting fixed points for γ.

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Theorem (BF: Stabilizers of regular elements)

If $\Gamma \to \mathrm{Aut}(\mathbb{X})$ is AU-acylindrical and not elliptic, and $\xi^-, \xi^+ \in \partial_{reg}\mathbb{X}$ with distinct factors then $E_\Gamma(\xi^-, \xi^+)$ is virtually $\mathbb{Z}^k$ where $1 \leq k \leq D$.

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In particular, if γ is a regular element, then $E_\Gamma(\gamma)$ is virtually $\mathbb{Z}^k$.

Theorem (BF)

Suppose that $\xi \in \partial_{reg}\mathbb{X}$*, where* $\Gamma \to \mathrm{Aut}(\mathbb{X})$ *is acylindrical. If* $\mathrm{Fix}_{\Gamma}\{\xi\}$ *is infinite then there exists an* $\xi' \in \partial_{reg}\mathbb{X}$ *such that* $\xi \neq \xi'$ *and* $\mathrm{Fix}_{\Gamma}\{\xi\} = \mathrm{Fix}_{\Gamma}\{\xi, \xi'\}$*.*

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Moreover, if $D' = |\{i : \xi_i \neq \xi'_i\}|$ *then* $\mathrm{Fix}_\Gamma\{\xi\}$ *is virtually* $\mathbb{Z}^k$ *for some* $1 \leq k \leq D' \leq D$, *where* D *is the number of factors of* $\mathbb{X}$.

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Theorem (BF: Tits Alternative in higher rank)

Let $\Gamma \to \mathrm{Aut}(\mathbb{X})$ *be acylindrical and not elliptic. Either* Γ *contains a nonabelian free group or* Γ *is virtually* $\mathbb{Z}^k$*, where* $1 \leq k \leq D$*, where* D *is the number of factors of* $\mathbb{X}$.

The semi-simple theory

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Theorem (BF: Canonical product decomposition)

Let Γ is be a finitely generated group admitting an AU-acylindrical action by isometries with general type factors $\Gamma \to \mathrm{Aut}(\mathbb{X})$. Then, up to virtual isomorphism, Γ admits a strongly canonical product decomposition. i.e. there is an internal product decomposition $\Gamma = \Gamma_1 \cdots \Gamma_F$ such that:

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- $\Gamma_1, \ldots, \Gamma_F$ *are strongly irreducible, that is, they do not decompose into a direct product of infinite groups, even up to virtual isomorphism;*

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- *The following are finite index inclusions: $\prod_{i=1}^{F} \mathrm{Aut}(\Gamma_i) \trianglelefteq \mathrm{Aut}(\Gamma)$ and $\prod_{i=1}^{F} Out(\Gamma_i) \trianglelefteq Out(\Gamma)$.*

Lattice embedding	AU-acylindrical action (and associated subdirect products)
Cocompact lattice	Acylindrical action
Nonuniform lattice	Nonuniform acylindrical action
Semi-simple lattice	AU-acylindrical with general type factors
Parabolic subgroup	$\bigcap_{i \in I} \mathrm{Fix}\{\xi_i\}$, for some $I \subset \{1, \ldots, D\}$ and $\xi_i \in \partial X_i$ for $i \in I$
Radical	Amenable radical
Cartan subgroup	The elementary subgroup $E_\Gamma(\gamma)$, $\gamma \in \Gamma$ is regular
Center	Trembling radical (need not be abelian)

Table: Summarizing the "translations" of existing terms in lattice theory to our framework.

A partial resolution to Sela's conjecture

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Theorem (Sela, Levitt)

Let Γ be a one-ended hyperbolic group. Then $Out(\Gamma)$ contains a finite index subgroup $Out'(\Gamma)$ that fits into the following central short exact sequence, with $|F| < \infty$:

$$1 \to \mathbb{Z}^n \to Out'(\Gamma) \to \prod_{s \in F} \mathrm{PMCG}(\Sigma_s) \to 1.$$

Conjecture (Sela)

Let Γ be a colorable HHG and let $\mathbb{X} = \prod_{i=1}^{D} X_i$ be the finite product of δ-hyperbolic spaces guaranteed by Hagen and Petyt. Assume the factor actions of Γ on X_i is *weakly acylindrical*, for $i = 1, \ldots, D$.

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If the associated higher rank Makanin-Razborov diagram is single ended (i.e., contains no splittings along finite groups along it), then there is a finitely generated virtually abelian groups A, A', finitely many 2-orbifolds $O_1, \ldots, O_F$, a finite index subgroup $\mathrm{Out}'(\Gamma) \leq Out(\Gamma)$ and an exact sequence:

$$1 \to A \to \mathrm{Out}'(\Gamma) \to \mathrm{Out}(A') \times \prod_{j=1}^{F} MCG(O_j)$$

Theorem (BF)

Let $\Gamma \to \mathrm{Aut}(\mathbb{X})$ *be acylindrical and cobounded with general type factors such that the projection of* $\Gamma_0 \to \mathrm{Isom}(X_i)$ *is weakly acylindrical for all* $i = 1, \ldots, D$*.*

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Corollary (BF)

Let Γ be an HHG with no eyrie that is a quasi-line and whose eyrie stabilizers act weakly acylindrically. Then Γ is virtually isomorphic to a subdirect product of D-many acylindrically hyperbolic groups, where D is the number of eyries.

Corollary (BF: Partial Resolution of Sela's Conjecture)

A colorable HHG, none of whose eyries are quasi-lines, satisfies the conclusion of Sela's Conjecture if and only if the irreducible factors in its canonical product decomposition satisfy Sela's Conjecture.

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Theorem (BF)

Let $\Gamma \to \mathrm{Aut}(\mathbb{X})$ *be strongly acylindrical with general type factors. Then up to virtual isomorphism,* $\Gamma = \prod_{i=1}^{D} \Gamma_i$ *is the direct product of hyperbolic groups, where* D *is the number of factors of* $\mathbb{X}$.

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Thank you!