Median Structures, local cubulations, and foliations

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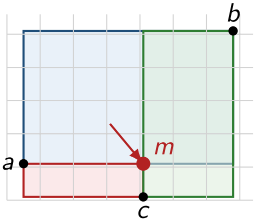
Median spaces – the basic example: $\mathbb{R}^n$

For $a, b, c \in \mathbb{R}^2$, the median is the unique point in the intersection of the three coordinate rectangles.

Equivalently,

$$m(a,b,c) = \big(\mathrm{mid}(a_1, b_1, c_1),\ \mathrm{mid}(a_2, b_2, c_2)\big).$$

The same coordinatewise definition works in $\mathbb{R}^n$.



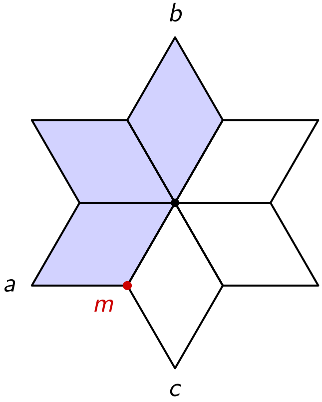
CAT(0) cube complexes are median spaces

Let X be a $\mathrm{CAT}(0)$ cube complex. For $a, b \in X$, let

$$[a, b] = \text{the union of all } \ell^1\text{-geodesics from } a \text{ to } b.$$

Then

$$\{m(a, b, c)\} = [a, b] \cap [a, c] \cap [b, c].$$



$X^{(0)}$ is closed under the median operation.

Definition: median space

Let X be a topological space. A median on X is a continuous map

$$m : X^3 \longrightarrow X, \qquad (a, b, c) \longmapsto abc,$$

satisfying the median identities:

Axioms

$aab = a$	majority rules
$abc = bac = acb$	symmetry
$(axb)xc = ax(bxc)$	associativity

A median space is a pair (X, m).

Examples: $\mathrm{CAT}(0)$ cube complexes and their vertex sets.

Chepoi: A finite median space is the vertex set of a $\mathrm{CAT}(0)$ cube complex.

Intervals and gate maps

For a median space X, define the interval between $a, b \in X$ by

$$[a, b] = \{x \in X \mid abx = x\}.$$

There is a canonical retraction, or gate map,

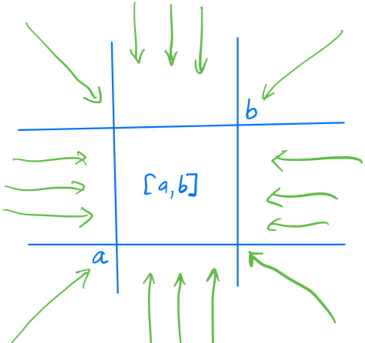
$$p_{ab} : X \longrightarrow [a, b], \qquad x \longmapsto abx.$$

Proof of retraction:

$$ab(abx) = (aba)bx = (aab)bx = abx$$

Some consequences of the axioms

- If $x, y \in [a, b]$, then $[x, y] \subseteq [a, b]$.
- If $x \in [a, b]$, then $[a, x] \cap [x, b] = \{x\}$.



$\mathbb{R}$ has only the usual median

Corollary

The real line has a unique topological median.

Idea.

For any median m on $\mathbb{R}$, the interval $[a, b]_m$ is a retract of $\mathbb{R}$. Hence it is connected and contains a and b.

If $a \leq x \leq b$, then connectedness forces $x \in [a, b]_m$, so

$$axb = x.$$

By symmetry, this determines the median uniquely: it is the usual middle point of three real numbers. $\square$

Exercise

A tree has a unique median – center of a tripod.

Medians via walls/foliations

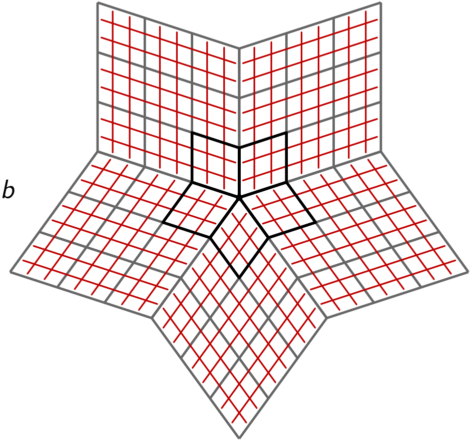
Suppose X has a collection of walls or leaves; each wall separates X into two half-spaces.
Assuming existence and uniqueness we attempt to define:

For $a, b, c \in X$, define $m = m(a, b, c)$ by the majority rule:

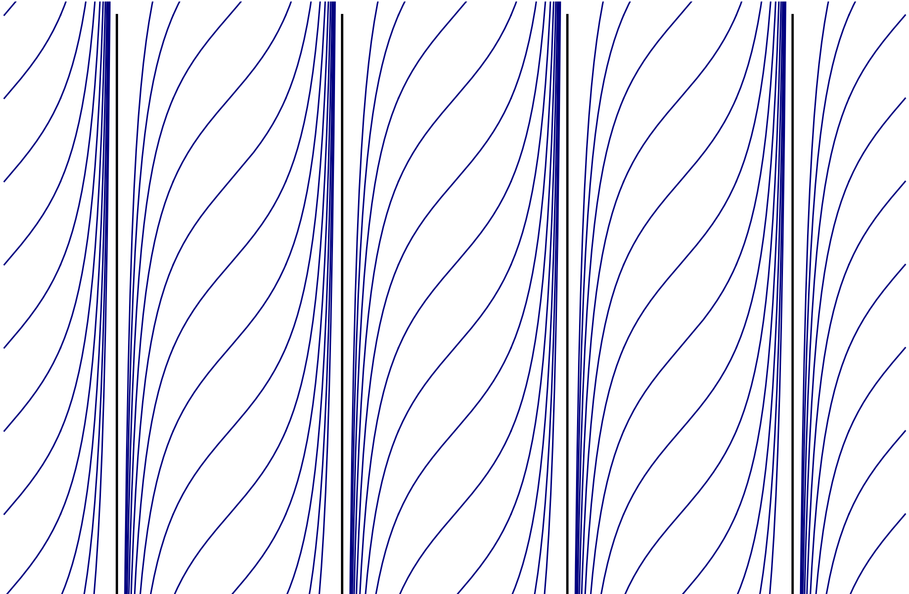
for every wall W, m lies on the same side of W as at least two of a, b, c.

- In $\mathbb{R}^n$, hyperplanes parallel to coordinate planes give the usual median.
- In cube complexes, same definition.
- Intervals can be read from half-spaces:

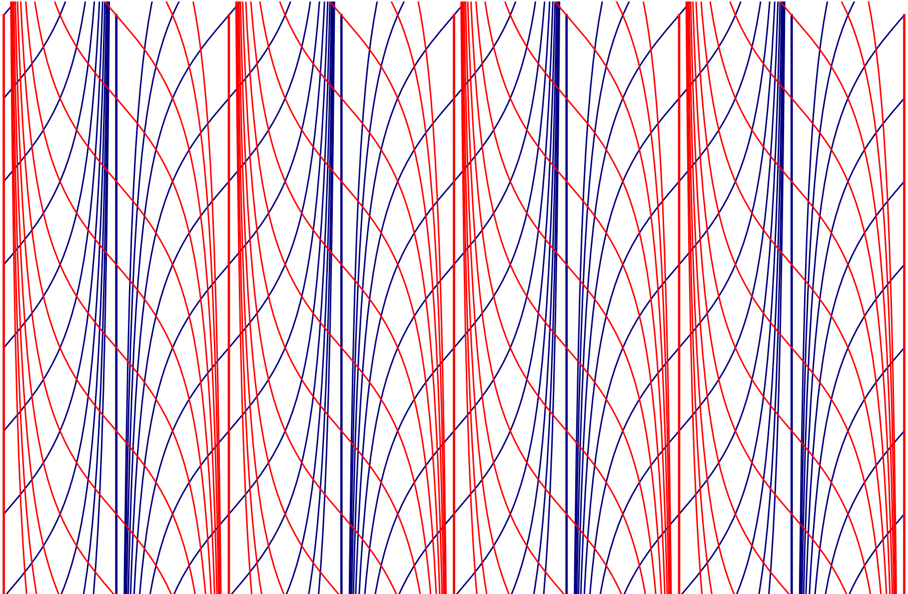
$[a, b] =$ intersection of all halfspaces containing a, b



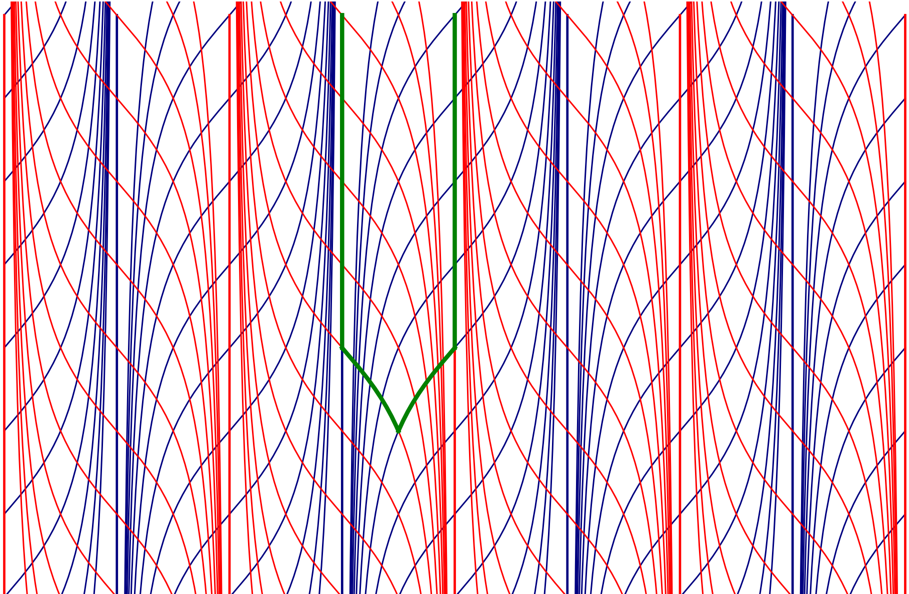
Example: tan foliation



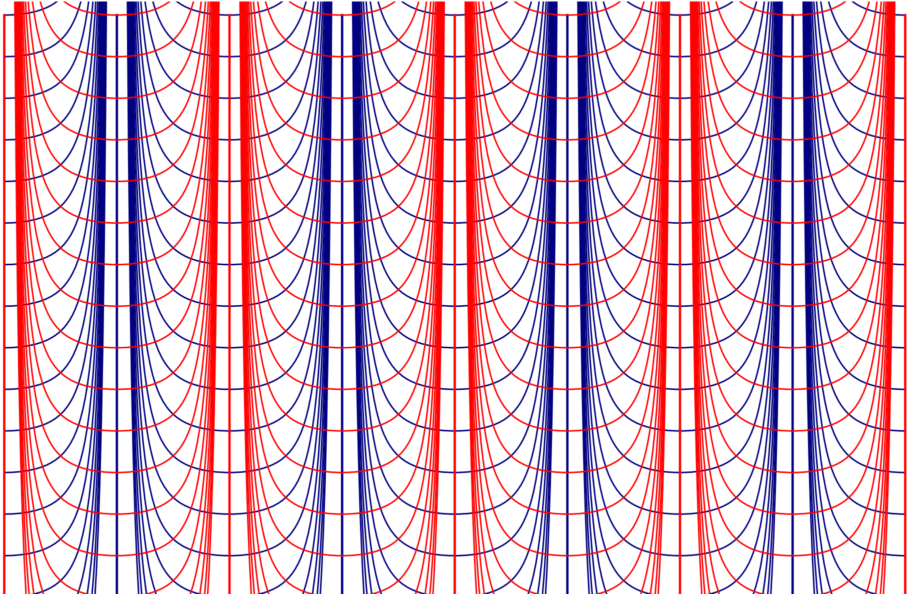
Example: tan foliation



Example: tan foliation



Nonexample: Reeb foliations



Sholander's interval criterion

Instead of starting with a ternary operation, start with intervals.

Sholander's theorem

Suppose an assignment $I(a, b) \subseteq X$ satisfies:

1. $I(a, a) = \{a\}$;
2. if $c, d \in I(a, b)$, then $I(c, d) \subseteq I(a, b)$;
3. for all a, b, c, the triple intersection

$$I(a, b) \cap I(a, c) \cap I(b, c)$$

   consists of exactly one point.

Then

$$\{m(a, b, c)\} = I(a, b) \cap I(a, c) \cap I(b, c)$$

defines a median operation.

Does every space admit a median?

No! There are topological restrictions.

Proposition(Fioravanti)

If X admits a topological median, then

$$\pi_n(X) = 0 \qquad \text{for all } n \geq 1.$$

Proof: Let $f : S^n \to X$ be given, $x \in S^n$ basepoint, $y = f(x)$.

$$\begin{tikzcd}
x \in S^n \arrow[r, "f"] \arrow[d, "\Delta"'] & X \ni y \\
S^n \times S^n \arrow[ur, "{(u,v)\mapsto m(f(u),f(v),y)}"] & S^n \vee S^n \arrow[l] \arrow[u, "y"']
\end{tikzcd}$$

Now homotope Δ into $S^n \vee S^n$.

We restrict to path connected spaces X homeomorphic to closed sets in $\mathbb{R}^n$.

Proposition (Bowditch)

If such X has a median structure, then it is contractible and locally contractible, hence an ER (Euclidean retract).

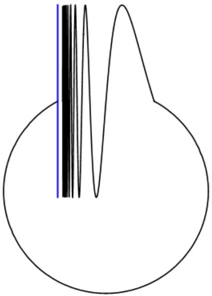
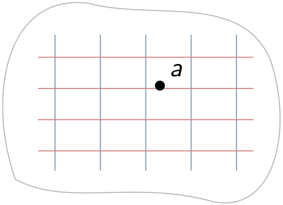


Figure: The Warsaw circle has no median structure

Local cubulation

Local cubulation theorem

Let m be a median on $\mathbb{R}^n$. Then every point $a \in \mathbb{R}^n$ has a neighborhood N homeomorphic to a finite $\mathrm{CAT}(0)$ cube complex, and on N the median agrees with the cubical median.

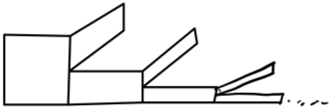


near each point, the median looks cubical

When intervals are compact

Compact intervals implies $\mathrm{CAT}(0)$ exhaustion

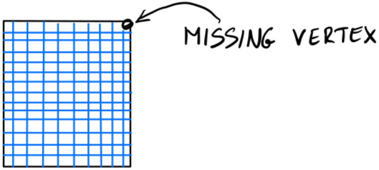
If all intervals are compact, then compact subsets of $\mathbb{R}^n$ are contained in compact neighborhoods that are finite $\mathrm{CAT}(0)$ cube complexes with the induced median.



There is no global cubing.

When intervals are not compact

If there is a noncompact interval then there is a square with a missing vertex, i.e. a median embedding $[0,1]^2 \setminus \{(1,1)\}$ that doesn't extend to $(1,1)$.



Local cubulation

Same theorem for polyhedra

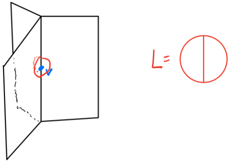
Suppose X is a polyhedron (and ER) so that the link $L = Lk(v, X)$ at any point v satisfies:
For every $y \in L$, the map

$$H_*(L - \{y\}) \longrightarrow H_*(L)$$

is not onto. Then any median on X is locally cubical and if all intervals are compact X is exhausted by finite $\mathrm{CAT}(0)$ cube complexes.

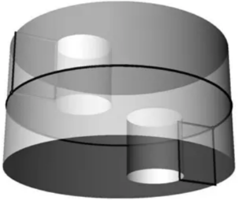
Examples:

- $X = \text{tripod} \times \mathbb{R}$



Local cubulation

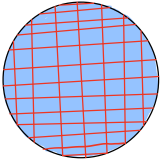
- Bing's house with two rooms does not have a median.



h/t Ken Baker

Links of points are circles and theta graphs, so the link condition holds.

- A round disk in $\mathbb{R}^2$ has induced median that's not locally cubical. The link condition fails.



Web structures

The local cubulation theorems say (for spaces with the link condition):

$$\text{median structure} \quad \Longrightarrow \quad \text{local cubical charts.}$$

The converse question is:

$$\text{compatible local cubical charts} \quad \Longrightarrow \quad \text{global median?}$$

Web structures

Definition: web structure

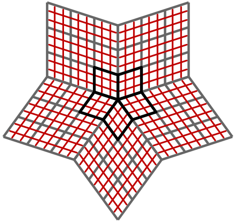
A web structure on X consists of:

- for each $x \in X$, a compact neighborhood N_x identified with a $\mathrm{CAT}(0)$ cube complex C_x;
- cubes of C_x are called local cubes;
- any two local cubes intersect in a collection of subcubes of each.

Each local cube is foliated by subcubes parallel to its cubical directions; these local foliations are required to match on overlaps.

Web structures

Examples:

- A pair of transverse foliations in $\mathbb{R}^2$, e.g. tan foliations or the Reeb example.
- A tiling of the plane by squares with 5 squares around a vertex. We get a web structure, but  not a pair of foliations.
- (Detlef Hardorp) Every closed orientable 3-manifold admits three codimension 1 foliations transverse to each other.

Local to global

Theorem: web structures and medians (in progress)

Let an ER X be equipped with a web structure. There is a median on X agreeing with the local cubulations if and only if the following hold:

1. Leaf space Hausdorff: If a sequence of leaves Hausdorff-converges to a set ℓ, then ℓ is contained in a single leaf.
2. No 3-cubes with missing vertices: Any embedding of $[0,1]^3 \smallsetminus \{(1,1,1)\}$ into X that sends leaves into leaves extends to $[0,1]^3$.

In this case, the median is unique.

Summary

For ERs X satisfying the link condition, there is a natural one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{Median structures} \\ m : X^3 \to X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Web structures on } X \\ \text{with Hausdorff leaf space, and} \\ \text{no 3-cubes with a missing vertex} \end{array} \right\}$$

In addition:

m has compact intervals $\longleftrightarrow$ no squares with a missing vertex

Roller boundary

Compactification theorem

Let X be an ER with a locally cubulable median. Then there is a compactification

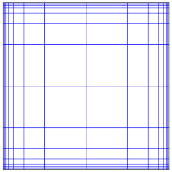
$$X \subseteq \overline{X}$$

such that:

- $\overline{X}$ is an Euclidean retract;
- $\overline{X} \setminus X$ is a Z-set in $\overline{X}$.

Examples:

- For the standard median on $\mathbb{R}^2$, the compactification is a square.
- The Whitehead manifold has no median structure.



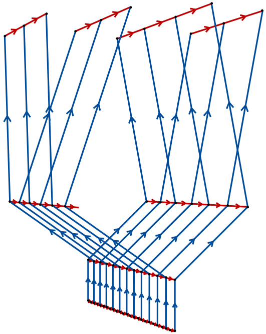
Group actions

We are interested in (proper, cocompact) group actions

$$G \curvearrowright X$$

preserving a median on X (but not preserving a $\mathrm{CAT}(0)$ cubing).
Examples:

- Baumslag–Solitar type examples, e.g. $BS(1,2)$.
- Sol 3-manifolds: $T^2 \hookrightarrow M \to S^1$ with hyperbolic monodromy.
- Hyperbolic 3-manifolds fibering over S^1 with pseudo-Anosov monodromy.



h/t Jim Belk

Some open questions

1. Is there a Tits alternative for groups acting properly on median spaces?
2. Can such groups have Property (T)?
3. Is there a mapping-class-group invariant median on Teichmuller space?