A tale about the mapping class group

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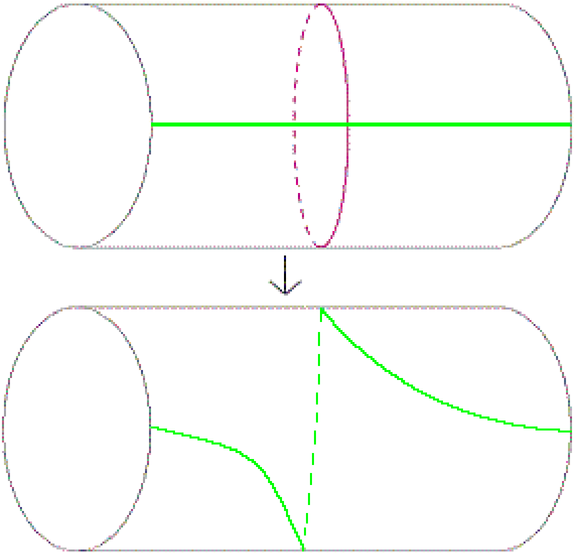
Consider a closed oriented surface S of genus $g \geq 2$. The mapping class group of S is the group $\mathrm{Mod}(S) = \mathrm{Aut}(\pi_1(S))/\mathrm{Inn}(\pi_1(S))$ of outer automorphisms of $\pi_1(S)$. It also equals the group of isotopy classes of homeomorphisms of S (Earle and Eells '69).

The mapping class group is finitely presented (McCool- Zieschang '77) and admits an explicit generating set by Dehn twists about $2g+1$ suitably chosen simple closed curves (Humphries '79, earlier work by Dehn '38 and Lickorich '64).



Definition

- A group is called *linear* if it admits an injective homomorphism into $\mathrm{GL}(n, \mathbb{C})$ for some $n > 0$.
- A group has *property (T)* if every affine isometric action on a Hilbert space has a global fixed point.

Example

A countable group which has a finite index subgroup Λ admitting a surjective homomorphism $\Lambda \to \mathbb{Z}$ does not have property (T).
$\mathrm{SL}(n, \mathbb{Z})$ has property (T) iff $n \geq 3$.

Open questions:

- Is $\mathrm{Mod}(S)$ linear?
- Does $\mathrm{Mod}(S)$ have property (T)?

Known results in low complexity:

- For a once-punctured torus, $\mathrm{Mod}(S) = \mathrm{PSL}(2, \mathbb{Z})$ is virtually free, linear and does not have (T).
- For $g = 2$, $\mathrm{Mod}(S)$ virtually surjects onto F_2 (Taherkhani '00) and hence does not have (T), and it is linear (Budney '05, using Bigelow '01 or Krammer '02).

Known results for the outer automorphism group of F_m:

- $\mathrm{Out}(F_2) = \mathrm{GL}(2, \mathbb{Z})$.
- $\mathrm{Out}(F_3)$ virtually surjects onto $\mathbb{Z}$ (McCool '89).
- For $m \geq 4$, $\mathrm{Out}(F_m)$ is not linear (Formanek-Pocesi '92), and for $m \geq 5$ is has (T) (Kaluba-Kielak-Nowak '21).

Motivating example: $\Gamma = \pi_1(S)$ a surface group. It acts on the *hyperbolic plane* $\mathbb{H}^2$, is linear and does not have (T).

- $\mathbb{H}^2$ is contractible.
- The *hyperbolic metric* g on $\mathbb{H}^2$ is $\mathrm{PSL}(2,\mathbb{R})$-invariant.
- g is a complete Riemannian metric of constant curvature -1.
- A geodesic ray for g starting from a fixed point $x \in \mathbb{H}^2$ is globally minimizing, and two distinct such geodesic rays γ, η diverge linearly: $d(\gamma(t), \eta(t)) \geq 2t - C$.
- $\mathbb{H}^2$ can be compactified by adding a *circle S^1 at infinity*, the space of geodesic rays starting from x. The closure $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup S^1$ is homeomorphic to a closed disk in $\mathbb{C}$.
- The isometric action of $\mathrm{PSL}(2,\mathbb{R})$ extends to an action on $\overline{\mathbb{H}^2}$ by homeomorphisms.

More generally, *word hyperbolic groups* are characterized by the *thin triangle property*: Geodesic triangles are δ-thin for a fixed $\delta > 0$.

- A hyperbolic group admits a proper isometric action on an L^p-space for *some* $p \geq 2$ (Yu '05).

Properties of lattices Γ in simple Lie groups G of rank at least two:

- Γ has property (T) (Kazdhan)
- Every affine isometric action of Γ on a uniformly convex Banach space has a global fixed point (Bader-Furman-Gelander-Monod '07, and de Laat- de la Salle '23).

Thus fixed point properties for actions on suitably chosen Banach spaces distinguish higher rank lattices from hyperbolic groups.

Strategy to show that a hyperbolic group Γ admits a proper isometric action on some L^p-space (Nica 13).

1. Compactify Γ by adding the *Gromov boundary* $\partial\Gamma$: $\Gamma \cup \partial\Gamma$ is a compact space with a left action of Γ by transformations, and $\Gamma \subset (\Gamma \cup \partial\Gamma)$ is open.
2. Construct a Γ-invariant measure ν on $\partial\Gamma \times \partial\Gamma \setminus \Delta$ in the measure class of a product $\mu \times \mu$ so that the action of Γ on $(\partial\Gamma, \mu)$ is measure class preserving and that μ is *Ahlfors regular*: For a *Gromov distance* on $\partial\Gamma$, the volume of a metric ball of radius r equals r^Q up to a uniform multiplicative constant, and $Q \geq 1$ is fixed.
3. Show that for $p > 2Q$ the function $(x, y) \to \mathrm{RN}_\phi(x) - \mathrm{RN}_\phi(y)$ $(\phi \in \Gamma)$ defines an *unbounded cocycle* with values in $L^p(\partial\Gamma \times \partial\Gamma \setminus \Delta, \nu)$, where RN is the Radon Nikodym derivative w.r.t. to μ, yielding an affine isometric action without a global fixed point.

Question: In what way does $\mathrm{Mod}(S)$ resemble a hyperbolic group? Some positive and negative facts:

- Given a pair of disjoint *subsurfaces* S_1, S_2 of S, the mapping class groups $\mathrm{Mod}(S_i)$ of S_i embed into $\mathrm{Mod}(S)$, and their images *commute*: Many elements in $\mathrm{Mod}(S)$ of infinite order have *large centralizers* (containing F_2) $\Rightarrow$ "higher rank".
- $\mathrm{Mod}(S)$ is *acylindrically hyperbolic* (Bowditch '08): It admits an isometric action on a (non-proper) hyperbolic space with some weak discontinuity property $\Rightarrow$ "rank one".
- $\mathrm{Mod}(S)$ is *hierarchically hyperbolic* (Bestvina-Bromberg-Fujiwara '15, building on work of Masur-Minsky '00).
- $\mathrm{Mod}(S)$ admits a proper action on *Teichmüller space*, which can be compactified by attaching the space of *projective measured geodesic laminations*, the sphere S^{6g-7}. There also is a (larger) *horocompactification* (Walsh '19).

Definition

An action of a group Γ by homeomorphisms on a space X is

- *minimal* if every Γ-orbit is dense
- *strongly proximal* if the closure of every Γ-orbit on the space $\mathcal{P}(X)$ of probability measures on X contains a Dirac measure
- topologically free if the fixed point set of any $\gamma \neq \mathrm{Id}$ has empty interior.

Definition

Let Γ be a finitely generated group. A boundary of Γ is compact Γ-space Z with the following properties.

- There exists a topology on $\Gamma \cup Z$ which restricts to the discrete topology on Γ and the given topology on Z and such that $\Gamma \cup Z$ is compact.
- The left action of Γ on itself extends to the Γ-action on Z.

The boundary is *small* if the right action of Γ extends to the trivial action of Γ on Z.

Definition

The dimension of a compact metrizable space X is the smallest number n such that any open cover $\mathcal{U}$ of X admits a refinement $\mathcal{V}$ such that any point in X is contained in at most $n+1$ sets from $\mathcal{V}$.

Definition

A compact, metrizable, finite dimensional, contractible and locally contractible space is an Euclidean retract.

Definition (Bestvina '96, Farrell-Lafont '08)

A $\mathcal{E}\mathcal{Z}$-structure for a torsion free group Γ is a pair $(\overline{X}, Z)$ of spaces s.th. $\overline{X}$ is an Euclidean retract, $X = \overline{X} - Z$ admits a properly discontinuous action of Γ with compact quotient extending to $\overline{X}$, every point in Z has a neighborhood basis in $\overline{X}$ consisting of sets whose intersections with X are contractible, and the Γ-action is $\mathcal{U}$-small for every open cover $\mathcal{U}$ of $\overline{X}$.

- The closed disk $\mathbb{H}^2 \cup S^1$ defines a $\mathcal{E}\mathcal{Z}$-structure for a surface group.
- Let Γ be a torsion free hyperbolic group, with Gromov boundary $\partial\Gamma$. The pair $(\mathcal{R}(\Gamma), \partial\Gamma)$ is a $\mathcal{E}\mathcal{Z}$-structure for Γ where $\mathcal{R}(\Gamma)$ is the *Rips complex* (Bestvina-Mess '98).
- Let Γ be a torsion free group which acts properly and cocompactly on a $\mathrm{CAT}(0)$ cube complex X. Then the visual boundary ∂X of X is finite dimensional (Swenson), and $(X, \partial X)$ is an $\mathcal{E}\mathcal{Z}$-structure for Γ (Bestvina '96).

Theorem (Durham-Minsky-Sisto '25)

Hierarchically hyperbolic groups admit $\mathcal{E}\mathcal{Z}$-structures and fulfill the Farrell Jones conjecture.

Theorem (H '25)

The mapping class group Γ admits an explicit $\mathcal{E}\mathcal{Z}$-structure $(\overline{X}, Z)$. The action of Γ on Z is minimal, strongly proximal and topologically free. If $S_0 \subset S$ is an essential subsurface, then the boundary of $\Gamma \cap \mathrm{Mod}(S_0)$ is embedded in Z.

Idea for the construction:

Definition

Teichmüller space $\mathcal{T}(S)$ for S is the space of all *marked hyperbolic metrics* on S. It is diffeomorphic to $\mathbb{R}^{6g-6}$, and $\mathrm{Mod}(S)$ acts properly on $\mathcal{T}(S)$ by change of markings.

Problem: The action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ is not cocompact.

The *systole* of a hyperbolic surface X is a shortest closed geodesic on X. The systole of X may be arbitrarily small.

For small $\epsilon > 0$ define

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \ell_X(\mathrm{sys}(X)) \geq \epsilon\}$$

where ℓ_X denotes the length for the metric X.

Theorem (Ji-Wolpert)

$\mathcal{T}_\epsilon(S)$ is a manifold with corners which is a $\mathrm{Mod}(S)$-equivariant deformation retract of $\mathcal{T}(S)$. $\mathrm{Mod}(S)$ acts on $\mathcal{T}_\epsilon(S)$ properly and cocompactly.

Idea: Compactify $\mathcal{T}_\epsilon(S)$.

The *virtual cohomological dimension* of $\mathrm{Mod}(S)$ equals

$$\mathrm{vcd}(\mathrm{Mod}(S)) = \sup\{n \mid H^n(\Gamma, M) \neq 0 \text{ for some } \Gamma - \text{module } M\}$$

where Γ is a torsion free f.i. subgroup

Fact: $\mathrm{vcd}(\Gamma)$ is the minimal dimension of a $K(\Gamma, 1)$-complex.

Theorem (Harer)

$\mathrm{vcd}(\mathrm{Mod}(S)) = 4g - 5.$

Fact: If $(\overline{X}, Z)$ is a $\mathcal{E}\mathcal{Z}$-structure for Γ then

$$\tilde{H}^q(Z) = H_c^{q+1}(X) = H^{q+1}(\Gamma; \mathbb{Z}\Gamma) \quad \forall q$$

Fact: If Z is a compact space of finite covering dimension, then $\dim(Z) = \dim_{\mathbb{Z}}(Z)$.

Corollary (Gabai, Bestvina-Bromberg 19)

$\dim(\partial \mathcal{CG}(S)) \leq 4g - 6$.

Construction of a measure on Z: Use symbolic dynamics and Morse quasi-geodesics

There is a theory of conformal densities (Patterson Sullivan measures) for horoboundary compactifications of metric spaces with Morse quasi-geodesics (Yang '22) using shadow systems as a replacement for balls.

Using the Morse property, one constructs from a suitable conformal density μ a $\mathrm{Mod}(S)$-invariant measure ν on $Z \times Z \setminus \Delta$ in the measure class of $\mu \times \mu$. The support of ν is a closed invariant subset of $Z \times Z \setminus \Delta$.

The action of $\mathrm{Mod}(S)$ on $(Z \times Z \setminus \Delta, \nu)$ induces a representation of $\mathrm{Mod}(S)$ on the space $\mathcal{L}^p$ of L^p-functions on $Z \times Z \setminus \Delta$.
For large enough p, this action admits an unbounded cocycle ($=$ the group admits an affine isometric action on $\mathcal{L}^p$ with an unbounded orbit).
A proper action of $\mathrm{Mod}(S)$ is obtained by an inductive procedure.

Some ideas of proof for the construction of Z:

Guiding principle: The boundary of the direct product of two hyperbolic planes $\mathbb{H}^2 \times \mathbb{H}^2$ is the *join*

$$S^1 * S^1 = \{\sum_i a_i x_i \mid x_i \in S^1, a_i \geq 0, \sum_i a_i = 1\}.$$

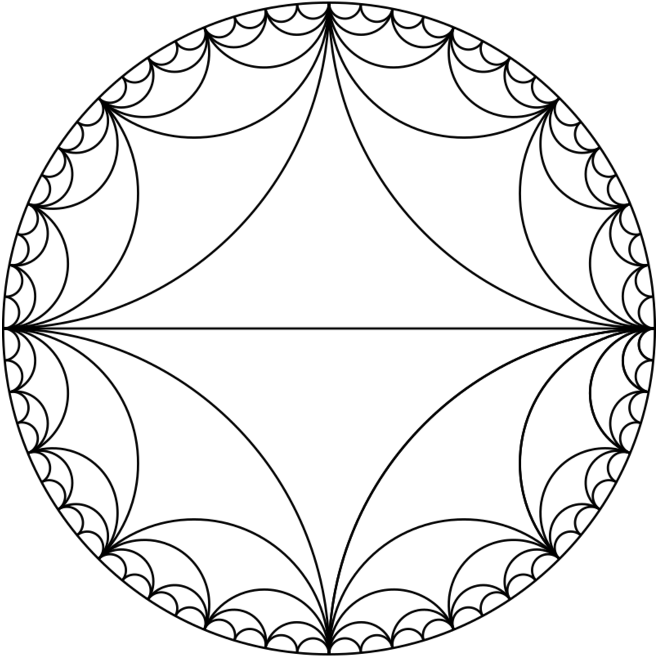
The join of two separable metrizable spaces is separable metrizable.

• If c is a simple multicurve cutting S into components $S_1, \ldots, S_k$ then

$$\mathrm{Mod}(S_1) \times \cdots \times \mathrm{Mod}(S_k)/\mathcal{C} < \mathrm{Mod}(S)$$

as an *undistorted subgroup* (here $\mathcal{C}$ is a subgroup of the center) and hence the boundary of $\mathrm{Mod}(S)$ should contain the join of the boundaries of $\mathrm{Mod}(S_i)$.

• If $\hat{S}$ is a one-holed torus then $\mathrm{Mod}(S) = \mathrm{SL}(2,\mathbb{Z})$ is virtually free and its boundary is the Gromov boundary of $\mathrm{SL}(2,\mathbb{Z})$, a Cantor set.



The Farey tesselation of the hyperbolic plane is a $\mathrm{PSL}(2,\mathbb{Z})$-invariant tree T, and there is a surjective map $\partial T \to S^1 = \mathbb{R} \cup \{\infty\}$ which is one-to-one on $\mathbb{R} \setminus \mathbb{Q}$ and two-to-one on $\mathbb{Q}$.

The Cantor set $\partial T = Z$ can be obtained from S^1 by replacing any rational point by two points.

More open questions:

1. The *asymptotic dimension* of a metric space X is defined to be $\leq n$ if for every $R > 0$ there exists a cover $\mathcal{U}$ of X by sets of uniformly bounded diameter and such that each closed ball of radius R intersects at most $n+1$ sets from $\mathcal{U}$.
   Bestvina-Bromberg-Fujiwara: $\mathrm{asym}(\mathrm{Mod}(S)) < \infty$ (in fact polynomial in g).
   Conj: $\mathrm{asym}(\mathrm{Mod}(S)) = 4g - 5$.
2. Homological stability: For $g_0 \leq g$ and in degree $k \leq 2g_0/3$, the homology $H_k(\mathrm{Mod}(S), \mathbb{Q})$ equals the homology of $\mathrm{Mod}(S_0)$ for a subsurface S_0 of genus $\geq g_0$ (Harer).
   Church-Farb: $H_{4g-5}(\mathrm{Mod}(S), \mathbb{Q}) = 0$
   Is there homological stability with coefficients? Can one describe odd degree nontrivial homology classes?

3. Is it true that for every finite index subgroup Γ of $\mathrm{Mod}(S)$ we have $H^1(\Gamma, \mathbb{Q}) = 0$?
4. Is there a $\mathcal{E}\mathcal{Z}$-structure for $\mathrm{Out}(F_n)$? (Bestvina-Horbez)
5. If Γ is a torsion free *acylindrically hyperbolic group* with a $\mathcal{Z}$-structure Z, can one equivariantly embed the Gromov boundary of the hyperbolic space into Z?