Fibring of Manifolds and Groups

Dawid Kielak

Manifolds in dimension three

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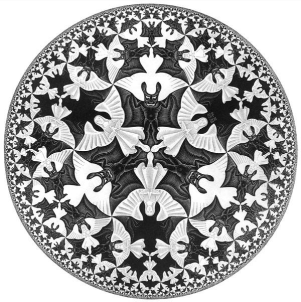
There are eight geometries involved.

Hyperbolic manifolds

The most common among the eight is hyperbolic geometry.

Definition

The quotient $\mathbb{H}^n/\Gamma$ where Γ is a torsion-free lattice in $\mathrm{Isom}^+(\mathbb{H}^n) = \mathrm{SO}(n,1)$ is called a *finite-volume hyperbolic manifold of dimension n*.



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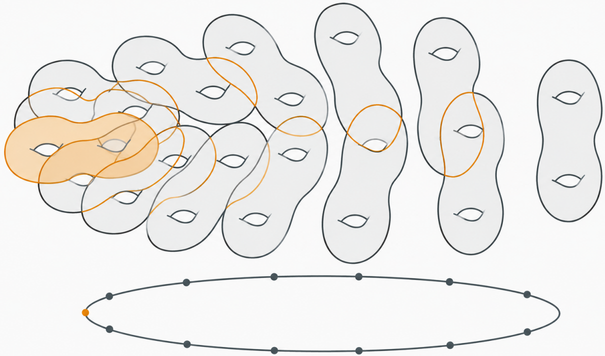
"This dubious-sounding question seems to have a definite chance for a positive answer."

Fibre-bundles over $\mathbb{S}^1$

Definition

A smooth map $f\colon M \to \mathbb{S}^1$ is a *fibre bundle* over $\mathbb{S}^1$ if there exists a smooth manifold F (the fibre) such that $\mathbb{S}^1$ can be covered by open sets U with $f^{-1}(U) \cong F \times U$ and $f\colon F \times U \to U$ being the projection.



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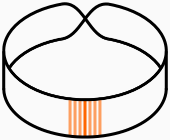
There is an analogous definition in the topological category, M and F can then be any spaces.



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Fundamental groups of the above are virtually RFRS, that is, have finite index subgroups that are residually {poly-$\mathbb{Z}$ *and virtually abelian}.*

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Theorem (Agol 2008)

An irreducible compact 3*-manifold* M *with RFRS fundamental group virtually fibres if and only if* $\chi(M) = 0$*.*

L^2-homology

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Theorem (Agol 2008, Lott–Lück 1995)

A compact 3-manifold M with RFRS fundamental group virtually fibres if and only if $\beta_i^{(2)}(M) = 0$ for all i.

Obstruction to fibring

Theorem (Agol 2008, Lott–Lück 1995)

A compact 3-manifold M with RFRS fundamental group virtually fibres if and only if $\beta_i^{(2)}(M) = 0$ for all i.

Theorem (Lück 1994)

If a topological space X fibres over $\mathbb{S}^1$ with compact fibre then $\beta_i^{(2)}(X) = 0$ for all i.

L^2-homology

Classical point of view

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Algebraic point of view

Instead, for RFRS groups, we have

$$\beta_i^{(2)}(G) = \dim_{\mathcal{D}(G)} \mathrm{H}_i(G; \mathcal{D}(G))$$

where $\mathcal{D}(G)$ is a specific skew-field containing $\mathbb{Q}G$.

Fibring in group theory

Stallings Theorem

Theorem (Stallings 1961)

For a compact 3-manifold M, an epimorphism $\phi\colon \pi_1(M) \to \mathbb{Z}$ is induced by a fibre bundle if and only if the kernel of ϕ is finitely generated.

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A compact 3-manifold M with RFRS fundamental group virtually fibres if and only if $\beta_i^{(2)}(M) = 0$ for all i.

Theorem (K. 2020)

An infinite RFRS group G admits a virtual epimorphism to $\mathbb{Z}$ with a finitely generated kernel if and only if $\beta_1^{(2)}(G) = 0$.

Algebraic fibring

Theorem (K. 2020)

A RFRS group G virtually maps onto $\mathbb{Z}$ with a finitely generated kernel if and only if $\beta_i^{(2)}(G) = 0$ for $i \leqslant 1$.

Theorem (S. Fisher 2024)

A RFRS group G virtually maps onto $\mathbb{Z}$ with a kernel of type $\mathtt{FP}_m(\mathbb{Q})$ if and only if $\beta_i^{(2)}(G) = 0$ for all $i \leqslant m$.

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Definition

G is *of type* $\mathtt{FP}_m(\mathbb{Q})$ if G is finitely generated and $\mathrm{H}_i(G; \prod_\infty \mathbb{Q}G) = 0$ for all $i < m$.

Less formally: homology over $\mathbb{Q}G$-modules thinks that G has a classifying space with finite m-skeleton.

Higher dimensions, homologically

Poincaré duality

For a closed n-manifold M we have

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for all i.

More generally, this works for local coefficients.

If M is closed and aspherical, we may replace M by $G = \pi_1(M)$ and get

$$\mathrm{H}_{n-i}(G;A) = \mathrm{H}^i(G;A)$$

for all $\mathbb{Z}G$-modules A.

Poincaré-duality groups

Definition

A group G is a Poincaré-duality group of dimension n if it is of type $\mathtt{FP}_n$ and

$$\mathrm{H}^i(G; A) \cong \mathrm{H}_{n-i}(G; A)$$

for every G-module A.

Poincaré-duality groups

Definition

A group G is a Poincaré-duality group of dimension n over $\mathbb{Q}$ if it is of type $\mathtt{FP}_n(\mathbb{Q})$ and

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for every $\mathbb{Q}G$-module A.

Virtual fibring, homologically

Theorem (S. Fisher–Italiano–K.)

Let G be a RFRS Poincaré-duality group of dimension n over $\mathbb{Q}$. The following are equivalent:

- *There exists a virtual epimorphism onto $\mathbb{Z}$ whose kernel is a Poincaré-duality group of dimension $n-1$ over $\mathbb{Q}$.*
- $\beta_i^{(2)}(G) = 0$ *for all i.*

Back to topology

Farrell Theorem

Theorem (Stallings 1961)

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Theorem (Stallings 1961)

For a compact 3*-manifold* M*, an epimorphism* $\phi\colon \pi_1(M) \to \mathbb{Z}$ *is induced by a fibre bundle if and only if the kernel of* ϕ *is finitely generated.*

Theorem (Farrell 1967)

For a closed aspherical manifold M *of dimension at least* 6 *with* $\mathrm{Wh}(\pi_1(M)) = 0$*, an epimorphism* $\phi\colon \pi_1(M) \to \mathbb{Z}$ *is induced by a fibre bundle if and only if* $\ker \phi$ *admits a finite classifying space.*

Dubious-sounding conjecture

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Theorem (Dodziuk 1979, Gaboriau 2002)

For M as above, $\beta_i^{(2)}(\pi_1 M) = 0$ for all i if and only if M is odd dimensional.

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Theorem (Italiano–Martelli–Migliorini 2023)

There exist finite-volume fibred hyperbolic manifolds in dimension 5.

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Theorem (Avramidi–Okun–Schreve 2024)

There exist closed aspherical Gromov-hyperbolic manifolds in dimension 7 that do not virtually fibre.

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If the dimension is at least 9, almost all known examples are arithmetic, and these divide into two large classes, ‘simplest type’ and ‘non-standard’.

Theorem (Haglund–Wise 2008)

The manifolds of simplest type as well as the Gromov–Piatetski-Shapiro examples are virtually RFRS.

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What we need:

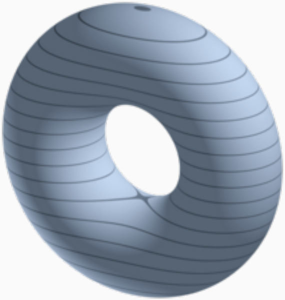
$$1 \to K \to \pi_1(M) \to \mathbb{Z} \to 1$$

with K of type $\mathtt{F}$, equivalently, type $\mathtt{FP}(\mathbb{Z})$ and finitely presented.

Morse functions

Definition

A surjective smooth function $f\colon M \to \mathbb{S}^1$ is *Morse* if its critical points are non-degenerate (the Hessians are non-singular). The *index* of a critical point is the number of negative eigenvalues of the Hessian.



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Every finite-volume hyperbolic manifold M virtually admits a circle-valued Morse function with all critical points of index $\dim M/2$.

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Theorem (Llosa Isenrich–Py [2025])

If M is a closed arithmetic complex-hyperbolic manifold then it virtually admits a Morse function as above.

Thank you!