GROUP APPROXIMATION
CHALLENGES, SUCCESSES AND FAILURES

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1 Residually finite groups

Definition

A group Γ is **residually-finite (RF)** if $\exists\{\varphi_n : \Gamma \to \mathrm{Sym}(n)\}$ s.t.

(i) φ_n is a homomorphism, i.e., $\forall x, y \in \Gamma$, $\varphi_n(xy) = \varphi_n(x)\varphi_n(y)$

(ii) $\forall 1 \neq x \in \Gamma$, $\varphi_n(x) \neq 1$ for all $n \gg n(x)$.

- Many groups (e.g., linear, mapping class groups, etc.) are RF and many are not.
- RF give a lot of information, e.g., word problem is decidable.

1 Residually finite groups

General groups (even finitely generated) are wild, e.g., **Tarski monster groups** (Olshanskii): an infinite 2-generated group s.t. every proper subgroup is cyclic of order p (for a large fixed prime p).

But:

Theorem (Zelmanov '89; Restricted Burnside Problem)

If Γ is a finitely generated RF group of a fixed exponent, then Γ is finite.

Theorem (Lubotzky-Mann, '89)

An RF group of **finite (Prüfer) rank** ($\exists m \in \mathbb{N}$ s.t. every f.g. subgroup is generated by m elements) is virtually solvable.

2 Sofic groups

Put on $\mathrm{Sym}(n)$ the (normalized) **Hamming metric**:

$$d_n(\sigma,\tau) = \frac{1}{n}\#\{i \in [n] \mid \sigma(i) \neq \tau(i)\}$$

Definition (Gromov, Weiss)

A group Γ is **sofic** if $\exists\{\varphi_n : \Gamma \to \mathrm{Sym}(n)\}$ s.t.

(i) $\forall x, y \in \Gamma,\ \lim_{n\to\infty} d_n(\varphi_n(xy), \varphi_n(x)\varphi_n(y)) = 0$

(ii) $\forall 1 \neq x \in \Gamma,\ \underline{\lim}_{n\to\infty}\ d_n(\varphi_n(x), \mathrm{id}_n) > 0$

Examples:

1. Residually Finite (RF) groups
2. Amenable groups

2 Sofic groups

A number of open problems for general groups are known to have a positive answer for sofic groups, e.g., Kaplansky's problem.

Theorem (Elek-Szabó)

Let K be a field and Γ a sofic group. If $\alpha, \beta \in K[\Gamma]$ with $\alpha \cdot \beta = 1$, then $\beta \cdot \alpha = 1$.

Open for general groups!

Open Problem (Gromov, Weiss)

Are all (f.g.) groups sofic?

3 Generalizations

Let $\mathcal{G} = (G_n, d_n)$ be a family of groups with bi-invariant metrics d_n.

Definition

A (f.g.) group Γ is $\mathcal{G}$-**approximable** if $\exists \varphi_n : \Gamma \to G_n$ s.t. (asymptotic homomorphisms):

(i) $\forall x, y \in \Gamma$, $d_n(\varphi_n(xy), \varphi_n(x)\varphi_n(y)) \xrightarrow[n\to\infty]{} 0$

(ii) $\forall 1 \neq x \in \Gamma$, $\underline{\lim}_{n\to\infty} d_n(\varphi_n(x), 1_{G_n}) > 0$

Main examples:

(0) $\mathcal{G} = \{\mathrm{Sym}(n), d_n\}$ as before.

3 Generalizations

Let now $G_n = U(n)$ and d_n defined using a norm $\|\cdot\|$ on $M_n(\mathbb{C})$ by:

$$d_n(g,h) = \|g-h\|$$

Where $|A| = (A^*A)^{1/2}$. Many choices for the norm:

1. **Frobenius (L^2-norm):** $\|A\|_F = \left(\operatorname{tr}|A|^2\right)^{1/2}$
2. **Hilbert-Schmidt:** $\|A\| = \left(\frac{1}{n}\operatorname{tr}|A|^2\right)^{1/2}$
3. p-**Schatten:** $\|A\| = (\operatorname{tr}|A|^p)^{1/p}$ $\qquad (1 \le p < \infty$ so $p = 2$ gives Frobenius)
4. **Operator norm:** $\|A\| = \max\{\|Av\| \mid v \in \mathbb{C}^n, \|v\| = 1\}$

3 Generalizations

(0) $\mathcal{G}$-approximable $\equiv$ sofic

(1) Frobenius-approximable

(2) HS-approximable $\equiv$ hyperlinear $\equiv$ Connes' embeddable groups

(3) p-approximable

(4) op-approximable $\equiv$ MF (Matricial Field)

- RF groups are all such.
- Amenable groups also (For MF, it is a highly non-trivial result of Tikuisis-White-Winter (2016)).

Are there non-$\mathcal{G}$-approximable?

3 Generalizations

Theorem (de Chiffre - Glebsky - Lubotzky - Thom, 2020)

$\exists$ a finitely presented **non-Frobenius** approximable group.

Theorem (Lubotzky - Oppenheim, 2021)

$\exists$ a finitely presented group Γ which is **not** p-**approximable** for any $1 < p < \infty$.

Theorem (Bachner - Dogon - Lubotzky, 2026)

$\exists$ a finitely presented group which is **not** 1-**approximable**.

So, left open: **Sofic?** **Hyperlinear?** **MF?**

- **The novelty:** While previous efforts went to simple (or almost simple) groups like Burger-Mozes, Higman, our groups are **"almost RF"**. Not because they are "less approximated" but because we have methods working for them.

4 Outline of proof

Definition

A group Γ is $\mathcal{G}$-**stable** if for every sequence of asymptotic homomorphisms $\{\varphi_n : \Gamma \to G_n\}$, there exist true homomorphisms $\psi_n : \Gamma \to G_n$ s.t.

$$\forall x \in \Gamma, \quad \lim_{n\to\infty} d_n(\psi_n(x), \varphi_n(x)) = 0$$

i.e., asymptotic homomorphisms can be corrected to real homomorphisms.

Proposition (Glebsky-Rivera, Arzhantseva-Paunescu)

If Γ is finitely generated, $\mathcal{G}$-approximable, and $\mathcal{G}$-stable, then it is RF (residually finite).

Corollary

A f.g. non-RF group which is $\mathcal{G}$-stable is <u>not</u> $\mathcal{G}$-approximable.

Theorem (de Chiffre - Glebsky - Lubotzky - Thom)

Let Γ be a finitely generated group. If $H^2(\Gamma, V) = 0$ for every unitary representation of Γ on any Hilbert space V, then Γ is **Frobenius stable**.

Remark

$$H^1(\Gamma, V) = 0 \quad \forall V \iff (T)$$

Why H^2 is relevant?

$$\begin{tikzcd} & \prod_n U(n) \arrow[d, "\pi"] \\ \Gamma \arrow[ur, "\varphi=(\varphi_n)"] \arrow[ur, "\psi?"'] \arrow[r, "\bar{\varphi}"'] & \prod_n U(n)/\sim \end{tikzcd}$$

- $\bar{\varphi}$ is a **true homomorphism**!

- Stability $\iff$ $\bar{\varphi}$ can be lifted to a true homomorphism ψ.

- $\ker \pi$ is very non-abelian. But as $\|\cdot\|_F$ is submultiplicative, it can be graded by abelian Hilbert spaces.

Where to find groups Γ with $H^2(\Gamma, V) = 0\ \forall V$?

Theorem (Garland, 70's)

If G is a simple p-adic Lie group of rank r and Γ a lattice in $G(\mathbb{Q}_\ell)$, then:

$$H^i(\Gamma, V) = 0 \quad \text{for every } 1 \leq i < r$$

So: If $\mathrm{rk}(G) \geq 3$, we have plenty.

But: They are linear and hence RF.

4 Outline of proof

Deligne (1978) provided lattices $\Gamma \leq \mathrm{Sp}(2g, \mathbb{R})$ with:

$$1 \to Z \to \tilde{\Gamma} \to \Gamma \to 0$$

where Z is finite and central, and $\tilde{\Gamma}$ is **not RF**.

Imitating his method for $\Gamma \leq \mathrm{Sp}(2g, \mathbb{Q}_\ell)$, we get "Deligne-type" $\tilde{\Gamma}$ which are not RF and still $H^2(\tilde{\Gamma}, V) = 0 \; \forall V$, hence **not Frobenius approximated**. ■

- **Lubotzky-Oppenheim** is an extension for any $1 < p < \infty$ by choosing $l \gg p$ and proving $H^2(\Gamma, V) = 0$ for every **suitable Banach space** (that can appear in $\ker \pi$).
- **Bachner-Dogon-Lubotzky** for $p = 1$ uses a somewhat different proof.
- **Bader**^{3} **& Sauer** (work in progress) will give a unified approach to $H^2(\Gamma, V) = 0$ for "most" Banach spaces.

4 Outline of proof

By an old result of Higman ($\exists$ a finitely presented group Δ containing a copy of every finitely presented group), we deduce that this Δ is **not** p-**approximable** for any $1 \leq p \leq \infty$.

- **Sofic & HS** are problematic as the norm is not submultiplicative.

- $p = \infty$ **(i.e., operator norm)** is problematic as we do not know groups with $H^2(\Gamma, B) = 0$ for a Banach space B of that type (e.g., $B = L^\infty(\Gamma)$).

4 Outline of proof

Remark

Hill (2019) showed that whenever Γ is an arithmetic group with the congruence subgroup property (CSP), then there $\exists$ a finite abelian group A and an extension:

$$1 \to A \to \tilde{\Gamma} \to \Gamma \to 1$$

with $\tilde{\Gamma}$ **not RF** (residually finite).

So, there are plenty of groups of Deligne type and, together with the results of B^3S, many groups that are **not** p**-approximable** $\forall\ 1 \leq p < \infty$.

5 Non RF hyperbolic groups?

This is a long-standing problem! Here are a few remarks related to the previous discussion.

Definition

A group Γ is called **hyper-RF** if for every extension

$$1 \to Z \to \tilde{\Gamma} \to \Gamma \to 1$$

with $|Z| < \infty$, $\tilde{\Gamma}$ is also RF.

Claim 1

If every hyperbolic group is RF, then every hyperbolic group is hyper-RF.

5 Non RF hyperbolic groups?

One may try to start with a hyperbolic Γ and to look for a finite cover $\tilde{\Gamma}$ that is not RF.

Theorem (Stover - Toledo, 2022)

Let $G = \mathrm{SU}(n,1)$ and Γ an arithmetic lattice in G of the simple type. If $\tilde{\Gamma}$ is a finite central cover coming from the central cover $\tilde{G}$ of G, then $\tilde{\Gamma}$ is RF.

But what about those not of the simple type?

5 Non RF hyperbolic groups?

Proposition

Let G be a rank one simple Lie group. If $\exists \Gamma \leq G$ a cocompact arithmetic lattice with the CSP, then $\exists$ a non-RF hyperbolic group.

Remark: Maybe relevant for $G = \mathrm{Sp}(n,1)$ or $F_4^{(-20)}$

Proof 1 (Lubotzky, 2011)

Divide Γ by a long random element to get a hyperbolic group Δ. The group Δ has only finitely many finite index normal subgroups since their preimages in Γ are all congruence. But the intersection of infinitely many congruence normal subgroups must be trivial. ∎

Proof 2 (Hill, 2019)

By Hill's Theorem, $\exists \tilde{\Gamma}$ fitting into:

$$1 \to A \to \tilde{\Gamma} \to \Gamma \to 1$$

and $\tilde{\Gamma}$ is not RF and is hyperbolic. ∎

5 Non RF hyperbolic groups?

For a discussion of hyper-RF and some ramifications — see a work in progress joint with **M. Bridson** and **A. Jaikin-Zapirain**.

Final Remark

It is not absurd to look for counterexamples of type $\tilde{\Gamma}$ as those we have here.

Theorem (Kapovich - Wise, 2000)

If $\exists$ a non-RF hyperbolic group, then there exists such a group with <u>no</u> finite index subgroups.

6 The Aldous-Lyons conjecture

- **In probability theory:** Every (unimodular) infinite graph can be (Benjamini-Schramm) approximated by finite graphs.

- **In group theory:** Every invariant random subgroup (IRS) of the free group is a limit of finite index IRS's.

If true, then every group is sofic!

6 The Aldous-Lyons conjecture

Theorem (Bowen - Chapman - Lubotzky - Vidick)

A-L is not true.

The proof is modeled on the famous $\mathrm{MIP}^* = \mathrm{RE}$ paper by Ji, Natarajan, Vidick, Wright, and Yuen in quantum information theory.

This paper resolved in the negative the **Connes Embedding Conjecture**, which is also a conjecture about approximating every von Neumann algebra by finite-dimensional ones.

If it were true, then every group would be hyperlinear.

The counterexamples of both theorems leave open the questions whether every group is **sofic / hyperlinear**.